⊠연구논문

Nonparametric Reliability Estimation in Strength-Stress Model: B-Spline Approach*

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Abstract

In this paper we develope a new nonparametric estimator of the reliability in strength-stress model. This estimator is constructed using the maximum likelihood estimate of cumulative failure rate in the class of distributions which have piecewise linear failure rate functions between each pair of observations. Large sample properties of our estimator are examined. The proposed estimator is compared with previously known estimator by Monte Carlo study.

1. Introduction

The strength-stress model has been widely used in a variety of areas including testing the reliability of the item or design procedures. This model was first introduced in 1950's and can be found on various applications in civil, aerospace engineering etc. In strength-stress model, let X be the strength of the unit and Y the stress placed on the unit by the operating environment. Suppose X and Y are two random variables with cumulative distribution functions (cdfs) $F(x)$ and $G(y)$ respectively. Then the reliability, denoted by R, of a system is the probability that its strength exceeds the stress. That is,

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$$R = P(X > Y) = \int_0^\infty G(t)\,dF(t) = \int_0^\infty \overline{F}(t)\,dG(t) \tag{1.1}$$

where $\overline{F}(t) = 1 - F(t)$.

Parametric analyses are found in most literature: Church and Harris(1970) obtained the confidence interval for R under the assumption that X and Y are independently normally distributed and the distribution of Y is known. Beg(1980) derived estimator of R for exponential family. Sathe and Shah(1981) derived minimum variance unbiased estimator for R when X and Y are independently exponentially distributed random variables.

We consider estimation of the reliability, R, neither the distribution of strength nor stress is known. The nonparametric estimation of R is very useful in practice and a few nonparametric estimation procedure have been suggested in the literature. Let the data consists of a random sample of size m of strengths $X_1, \cdots, X_m$ from F and an independent random sample of size n of stress $Y_1, \cdots, Y_n$ from G. Birnbaum(1956) show that the Wilcoxon-Mann-Whitney statistic could be used as an estimator of R as follows:

$$\hat{R} = \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} U_{ij} \quad \text{where} \quad U_{ij} = \begin{cases} 1 & , X_i > Y_j \\ 0 & , X_i \le Y_j. \end{cases}$$

We can express $\hat{R}$ as

$$\hat{R} = \int_0^\infty (1 - F_m(y))\,dG_n(y)$$

where $F_m(y)$ and $G_n(y)$ are the empirical distributions of the X's and Y's, respectively. Birnbaum and McCarty(1958) derived distribution-free upper confidence bound on R, which is based on independent samples of X and Y. Govindarajulu(1968) discussed the estimation of R using the Kolmogorov-Smirnov statistic when one of the distribution is known.

For censored observations Delong and Sen(1981) dealt with the estimation of R based on progressively truncated version of the Wilcoxon-Mann-Whitney statistics. They considered the stochastic processes related to some generalized U-statistics under progressive right censoring for prediction purposes. McNichols and Padgett(1988) considered the situation in which censoring is performed to the strength of the item by some prespecified time.

In this paper, we propose a nonparametric estimator of R based on a complete sample. This estimator is constructed using the estimator $\hat{\Lambda}(\cdot)$, where $\hat{\Lambda}(\cdot)$ is derived as the maximum likelihood estimate of cumulative failure rate $\Lambda(\cdot)$ in the class of distributions which have piecewise linear failure rate functions between each pair of strength(stress) observations. We derive the asymptotic properties of the our estimator. Monte Carlo simulation are conducted to investigate the performance of our new nonparametric estimator. In Section 2 we discuss nonparametric estimation procedure for R and derive some properties of the estimator. To see the finite sample performance of the estimator we give some simulation results in Section 3.

2. Estimation of the Reliability

In this section we define new nonparametric estimator of reliability in strength-stress model. To define our estimator we use the maximum likelihood estimate of cumulative failure rate in the class of distributions which have piecewise linear failure rate functions between each pair of strength(stress) observations. More precisely, if we define $X'_{(1)} < X'_{(2)} < \cdots < X'_{(K_1)}$ to be the different sorted values of $X_1, X_2, \cdots, X_m$, then the estimator of the cumulative failure rate is

$$\hat{\Lambda}(x) = \sum_{j=1}^{K_1} \frac{m_j \int_0^x B_j(u)du}{\sum_{i=1}^{m} \int_0^{x_i} B_j(u)du} \tag{2.1}$$

where m_j is the number of strengths equal $X'_{(j)}$ and B_j is a linear B-spline. For more detail, see Na, et al.(1998). With this estimator (2.1), the corresponding estimator of the reliability function of X is defined as

$$\hat{\bar{F}}(x) = \exp\left(- \sum_{j=1}^{K_1} \frac{m_j \int_0^x B_j(u)du}{\sum_{i=1}^{m} \int_0^{x_i} B_j(u)du} \right). \tag{2.2}$$

Similarly, The estimator of the distribution function of Y is defined as

$$\widehat{G}(y)=1-\exp\left(-\sum_{j=1}^{K_2}\frac{n_j\int_0^{y}B_j(u)du}{\sum_{i=1}^{n}\int_0^{y_i}B_j(u)du}\right) \tag{2.3}$$

where n_j is the number of stress equal $Y'_{(j)}$ and $Y'_{(1)}<Y'_{(2)}<\cdots<Y'_{(K_2)}$ to be the different sorted values of $Y_1,\cdots,Y_n$.

Now we propose an estimator of R in (1.1)

$$\widehat{R}=\int_0^{\infty}\widehat{\overline{F}}(t)\,d\widehat{G}(t) \tag{2.4}$$

where $\widehat{\overline{F}}(t)$ and $\widehat{G}(t)$ are defined in (2.2) and (2.3), respectively.
We shall prove consistency and normality of the proposed estimator.

Theorem 2.1 Let F and G be continuous distributions. Then $\widehat{R}$ is a consistent estimator of R as $m,n\to\infty$.

Proof. Using (1.1) and (2.4) write

$$\begin{aligned}|\widehat{R}-R| &= \left|\int_0^{\infty}\widehat{\overline{F}}(t)d\widehat{G}(t)-\int_0^{\infty}\overline{F}(t)dG(t)\right| \\ &\le M_{1n}+M_{2n}\end{aligned}$$

where $M_{1n}=\left|\int_0^{\infty}(\widehat{\overline{F}}(t)-\overline{F}(t))d\widehat{G}(t)\right|$ and $M_{2n}=\left|\int_0^{\infty}\overline{F}(t)(d\widehat{G}(t)-dG(t))\right|$.
We will show that M_{1n} and M_{2n} converge in probability to 0 as $m,n\to\infty$.

$$\begin{aligned}M_{1n} &\le \int_0^{\infty}|\widehat{\overline{F}}(t)-\overline{F}(t)|\,d\widehat{G}(t) \\ &\le \sup_t|\widehat{\overline{F}}(t)-\overline{F}(t)|\int_0^{\infty}d\widehat{G}(t).\end{aligned}$$

Since $\sqrt{m}\sup|\widehat{\overline{F}}(t)-\overline{F}(t)|$ is bounded in probability (see Na, et al.(1998)),

M_{1n} converges in probability to 0. And by Helly's theorem, M_{2n} converges in probability to 0. Therefore we have that $\hat{R}$ converges in probability to R as $m, n \to \infty$. □

In order to study the limiting distribution of $\hat{R}$ in terms of the joint limiting distribution of $\hat{\bar{F}}$ and $\hat{G}$, we define $\Lambda_{1,m}(t)$ and $\Lambda_{2,n}(t)$ by $\Lambda_{1,m}(t) = \sqrt{m}(\hat{\bar{F}}(t) - \bar{F}(t))$ and $\Lambda_{2,n}(t) = \sqrt{n}(\hat{G}(t) - G(t))$, respectively. Then we have the expansion $\sqrt{N}(\hat{R} - R) = \sqrt{\frac{N}{m}} A_N + \sqrt{\frac{N}{n}} B_N + \sqrt{\frac{N}{m}} R_N$ where $N = m + n$,

$$
\begin{aligned}
A_N &= \int_0^\infty \Lambda_{1,m}(t)\, dG(t), \\
B_N &= \int_0^\infty \Lambda_{2,n}(t)\, dF(t) \qquad (2.5) \\
R_N &= \int_0^\infty \Lambda_{1,m}(t)\, d(\hat{G}(t) - G(t)).
\end{aligned}
$$

We will show that A_N and B_N converge weakly to A and B respectively defined by $A = \int_0^\infty \Lambda_1(t)\, dG(t)$ and $B = \int_0^\infty \Lambda_2(t)\, dF(t)$ where $\Lambda_1(t)$ is a Gaussian process with mean 0 and covariance

$$\mathrm{Cov}(\Lambda_1(t_1), \Lambda_1(t_2)) = \bar{F}(t_1) F(t_2) \text{ for } t_1 \le t_2$$

and $\Lambda_2(t)$ is a Gaussian process with mean 0 and covariance

$$\mathrm{Cov}(\Lambda_2(t_1), \Lambda_2(t_2)) = \bar{G}(t_1) G(t_2) \text{ for } t_1 \le t_2.$$

Likewise we will show that R_N converges in probability to 0. This will establish the following theorem.

Theorem 2.2 Suppose F and G are continuous distributions. Let $\lambda = \lim_{m\to\infty\, n\to\infty} \frac{m}{N}$ and $0 < \lambda < 1$. Suppose that

$$\int_0^\infty \{\bar{F}(t)F(t)\}^{1/2}dG(t) < \infty \tag{2.6}$$

and

$$\int_0^\infty \{\bar{G}(t)G(t)\}^{1/2}dF(t) < \infty \tag{2.7}$$

hold. Then

$$\sqrt{N}(\hat{R}-R) \xrightarrow{d} N(0, \sigma_1^2/\lambda + \sigma_2^2/(1-\lambda)) \quad \text{as} \quad m, n \to \infty$$

where

$$\begin{aligned} \sigma_1^2 &= \int_0^\infty G^2(t)dF(t) - \left(\int_0^\infty G(t)dF(t)\right)^2 \quad \text{and} \\ \sigma_2^2 &= \int_0^\infty F^2(t)dG(t) - \left(\int_0^\infty F(t)dG(t)\right)^2. \end{aligned} \tag{2.8}$$

Proof. To prove the limiting normality, it suffices to examine the convergence mentioned above. We use the fact, proved by Na, et al.(1998), that $\Lambda_{1,m}(t)$ for $0 \le t < X_{(m)}$ converges weakly to $\{\Lambda_1(t),\ 0 \le t \le \tau\}$. Similarly, $\Lambda_{2,n}(t)$ converges weakly to $\Lambda_2(t)$. Next note that $\int_0^\infty \Lambda_1(t)dG(t)$ is a proper random variable since

$$\begin{aligned} E\int_0^\infty |\Lambda_1(t)|dG(t) &= \int_0^\infty E|\Lambda_1(t)|dG(t) \le \int_0^\infty [E\Lambda_1^2(t)]^{1/2}dG(t) \\ &= \int_0^\infty \{\bar{F}(t)F(t)\}^{1/2}dG(t) < \infty \end{aligned}$$

by condition (2.6). Similarly, $\int_0^\infty \Lambda_2(t)dF(t)$ is a proper random variable by condition (2.7). By the continuous mapping theorem(Billingsly(1968), p.30), the leading terms in (2.5) A_N and B_N converges weakly to A and B, respectively. Turning to the remainder term, since $\sup|\Lambda_{1,m}(t)|$ is bounded in probability, R_N converges in probability to 0. And by Slusky's Theorem $\sqrt{N}(\hat{R}-R)$ converges weakly to $\frac{1}{\sqrt{\lambda}}\int_0^\infty \Lambda_1(t)\,dG(t)+\frac{1}{\sqrt{1-\lambda}}\int_0^\infty \Lambda_2(t)\,dF(t)$. By the theory of stochastic integration (see Chapter 5 in Cram'er and Leadbetter(1967)), we can obtain that the limiting random variable is normal with mean 0 and the variance given by (2.8). □

3. Simulation Study

To see the finite sample performance of our proposed estimator we carry out the following Monte Carlo experiment. The simulation is performed on the subroutine FORTRAN of the package IMSL.

The strength random numbers are generated from the Weibull distribution $W(\alpha_1, \beta)$, i.e.

$$F(t)=1-\exp\{-(\frac{t}{\alpha_1})^\beta\},\quad t\geq 0,\ \alpha_1>0,\ \beta>0.$$

The stress random numbers are generated from the Weibull distribution $W(\alpha_2, \beta)$, i.e.

$$G(t)=1-\exp\left(-(\frac{t}{\alpha_2})^\beta\right),\quad t\geq 0,\ \alpha_2>0,\ \beta>0.$$

In this case, the exact reliability is given by $R=\Pr[X>Y]=\alpha_1^\beta/(\alpha_1^\beta+\alpha_2^\beta)$. From each of a number of specified distributions, chosen so as to have a variaty of shapes, we generate 1000 samples of given size. For each sample we estimate R according to our new procedure in section 2. In addition, we also estimate R using the Birnbaum's(1956) estimator. The Bias, Variance(VAR), and Mean Squared Error(MSE) of our estimator are compared with those of the Birnbaum's (1956) estimator.

<Table 3.1> shows that the results of the simulation with 1000 replications when $\beta=1$ (i.e. exponential distribution), $\alpha_1=1$ and $\alpha_2=1,\ 3/7,\ 1/9$ for $m/N=1/4$, 2/4 and 3/4. <Table 3.2> shows that the results of the simulation with 1000 replications when $\beta=2$, $\alpha_1=1$ and $\alpha_2=1,\ \sqrt{3/7},\ 1/3$ for $m/N=1/4, 2/4$ and 3/4. In tables the ratio of MSE is defined as follows;

$$\text{Ratio of MSE} = \frac{\text{MSE of the Birnbaum's Estimator}}{\text{MSE of the proposed Estimator}}.$$

< Table 3.1 > Results of the simulation from X-W(1, 1) and Y-W(α_2, 1)

α_2 (R)	m/N	(m,n)	Birnbaum's			Proposed			Ratio of MSE
			BIAS	VAR	MSE	BIAS	VAR	MSE	
1 (0.5)	1/4	(5,15)	-.0015	.0230	.0230	-.0029	.0215	.0215	1.0705
		(10,30)	.0015	.0075	.0075	.0017	.0074	.0074	1.0126
		(25,75)	-.0073	.0043	.0044	-.0070	.0043	.0044	1.0060
	1/2	(10,10)	-.0053	.0183	.0184	-.0060	.0177	.0177	1.0355
		(30,30)	-.0032	.0056	.0056	-.0030	.0055	.0056	1.0091
		(50,50)	-.0031	.0033	.0033	-.0030	.0033	.0033	1.0050
	3/4	(15, 5)	.0024	.0261	.0261	.0023	.0255	.0255	1.0210
		(45,15)	-.0075	.0075	.0075	-.0073	.0075	.0075	1.0022
		(75,25)	.0014	.0045	.0045	.0015	.0045	.0045	1.0007
3/7 (0.7)	1/4	(5,15)	.0074	.0207	.0207	.0050	.0175	.0175	1.1809
		(10,30)	.0032	.0070	.0070	.0032	.0067	.0067	1.0429
		(25,75)	.0046	.0040	.0040	.0045	.0039	.0039	1.0210
	1/2	(10,10)	-.0007	.0146	.0146	-.0002	.0137	.0137	1.0656
		(30,30)	-.0055	.0045	.0045	-.0055	.0044	.0045	1.0114
		(50,50)	.0014	.0028	.0028	.0014	.0028	.0028	1.0055
	3/4	(15, 5)	-.0012	.0168	.0168	-.0010	.0162	.0162	1.0390
		(45,15)	.0037	.0055	.0055	.0037	.0055	.0055	1.0050
		(75,25)	.0001	.0031	.0031	.0001	.0030	.0030	1.0025
1/9 (0.9)	1/4	(5,15)	.0013	.0100	.0100	-.0056	.0060	.0061	1.6509
		(10,30)	-.0015	.0031	.0031	-.0028	.0026	.0026	1.1896
		(25,75)	-.0015	.0020	.0020	-.0020	.0018	.0018	1.0811
	1/2	(10,10)	.0004	.0053	.0053	-.0014	.0042	.0042	1.2710
		(30,30)	-.0012	.0017	.0017	-.0017	.0016	.0016	1.0617
		(50,50)	.0007	.0010	.0010	.0004	.0010	.0010	1.0317
	3/4	(15, 5)	-.0022	.0053	.0053	-.0034	.0046	.0046	1.1631
		(45,15)	.0000	.0015	.0015	-.0004	.0015	.0015	1.0324
		(75,25)	.0004	.0009	.0009	.0003	.0009	.0009	1.0139

< Table 3.2 > Results of the simulation from X~W(1, 2) and Y~W(α_2, 2)

α_2 (R)	m/N	(m,n)	Birnbaum's			Proposed			Ratio of MSE
			BIAS	VAR	MSE	BIAS	VAR	MSE	
1 (0.5)	1/4	(5,15)	-.0078	.0252	.0252	-.0249	.0221	.0227	1.1111
		(10,30)	-.0005	.0074	.0074	-.0034	.0072	.0072	1.0211
		(25,75)	-.0027	.0044	.0044	-.0038	.0044	.0044	1.0090
	1/2	(10,10)	-.0022	.0169	.0169	-.0080	.0159	.0160	1.0569
		(30,30)	.0002	.0053	.0053	-.0007	.0053	.0053	1.0050
		(50,50)	-.0021	.0034	.0034	-.0024	.0034	.0034	1.0053
	3/4	(15, 5)	-.0022	.0237	.0237	-.0053	.0232	.0232	1.0219
		(45,15)	-.0042	.0072	.0073	-.0046	.0072	.0072	1.0064
		(75,25)	-.0008	.0043	.0043	-.0009	.0043	.0043	1.0015
$\sqrt{3/7}$ (0.7)	1/4	(5,15)	-.0011	.0198	.0199	-.0279	.0153	.0161	1.2334
		(10,30)	-.0002	.0072	.0072	-.0050	.0068	.0068	1.0612
		(25,75)	.0027	.0042	.0042	.0004	.0041	.0041	1.0338
	1/2	(10,10)	.0055	.0147	.0148	-.0041	.0130	.0131	1.1297
		(30,30)	-.0016	.0045	.0045	-.0032	.0044	.0044	1.0208
		(50,50)	-.0024	.0027	.0027	-.0030	.0027	.0027	1.0082
	3/4	(15, 5)	.0025	.0163	.0163	-.0022	.0152	.0152	1.0721
		(45,15)	-.0027	.0051	.0051	-.0035	.0050	.0050	1.0126
		(75,25)	.0009	.0031	.0031	.0005	.0030	.0031	1.0046
1/3 (0.9)	1/4	(5,15)	.0011	.0094	.0094	-.0426	.0050	.0068	1.3756
		(10,30)	.0005	.0031	.0031	-.0110	.0025	.0026	1.2090
		(25,75)	-.0011	.0019	.0019	-.0065	.0016	.0017	1.1224
	1/2	(10,10)	.0017	.0052	.0052	-.0173	.0036	.0039	1.3090
		(30,30)	-.0005	.0017	.0017	-.0045	.0015	.0015	1.1042
		(50,50)	.0002	.0010	.0010	-.0018	.0009	.0009	1.0490
	3/4	(15, 5)	-.0010	.0051	.0051	-.0119	.0041	.0042	1.1999
		(45,15)	.0010	.0016	.0016	-.0013	.0015	.0015	1.0674
		(75,25)	.0004	.0010	.0010	-.0005	.0009	.0009	1.0300

4. Concluding Remark

In this paper we develope a new nonparametric estimator of the reliability in strength-stress model based on the maximum likelihood estimate of cumulative failure rate in the class of distributions which have piecewise linear failure rate functions between each pair of observations. We investigate the large sample properties of our estimator and conduct Monte Carlo simulation to see the performance of the proposed estimator.

From the simulation results, we notice that our new estimator seems to produce less MSE than the MSE of Birnbaum's(1956) estimator. Particularly, Our estimator produces much less MSE than that of Birnbaum's(1956) estimator when m/N is small or R is large.

An interesting subject for further study is estimation of the reliability R when the strength variable or stress variable is censored and comparing previously known estimators based on a randomly censored data.

References

[1] Beg, M. A.(1980), "Estimation of Pr{Y<X} for exponential-family," *IEEE Transactions on Reliability*, Vol. R-29, No. 2, pp. 158-159.

[2] Billinglsly, P.(1968), *Convergence of Probability Measures*, Wiley, New York.

[3] Birnbaum, Z. W.(1956), "On a use of the Mann-Whitney-statistic," *Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability*, Vol. 1, pp. 13-17.

[4] Birnbaum, Z. W. and McCarty, R. C.(1958), "A distribution-free upper confidence bound for Pr{Y<X}, based on independent samples of X and Y," *Ann. Math. Stat.*, Vol. 29, pp. 558-562.

[5] Church, J. D. and Harris, B.(1970), "The estimation of Reliability from Stress-Strength Relationships," *Technometrics*, Vol. 12, No.1, pp. 49-54.

[6] Cram'er, H. and Leadbetter, M. R.(1967), *Stationary and Related Stochastic Processes*, John Wiley & Sons, New York.

[7] Delong, E. R. and Sen, P. K.(1981), "Estimation of Pr{X>Y} based on Progressively Truncated versions of the Wilcoxon-Mann-Whitney statistics," *Commun. Statist.-Theor. Math.* Vol. 10, pp. 963-981.

[8] Govindarajulu, Z.(1968), "Distribution-free confidence bounds for Pr{X<Y}," *Ann. Inst. Statist. Math.*, Vol. 20, pp. 229-238.

[9] McNichols, D. T. and Padgett, W. J.(1968), "Inference for step-stress accelerated life tests under arbitrary right-censorship," *Journal of Statistical Planning and Inference*, Vol. 20, pp. 169-179.

[10] Na, M. H., Park, S. H. and Kim, J. J.(1998), "Smooth Nonparametric Estimation of Mean Residual Life," *Proceedings of the 12th Asia Quality Management Symposium* held in Seoul, Korea, pp. 571-579.

[11] Sathe, Y. S. and Shah, S. P.(1981), "On estimating Pr{X>Y} for the exponential distribution," *Commun. Statist.-Thror, Math*, Vol. 10, pp. 39-47.