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A NOTE ON PASCAL'S MATRIX

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Abstract. We can get the Pascal's matrix of order n by taking the first n rows of Pascal's triangle and filling in with 0's on the right. In this paper we obtain some well known combinatorial identities and a factorization of the Stirling matrix from the Pascal's matrix.

1. Introduction

The numbers $\binom{n}{k}$ are the so-called *binomial coefficients* which count the number of k-combinations of a set of n elements. They have many fascinating properties and satisfy a number of interesting identities. Moreover, the binomial coefficients are open displayed in an array known as *Pascal's triangle.* Each entry in the triangle, other than those equal to 1 occurring on the left side and hypotenuse, is obtained by adding together two entries in the row above: the one directly above and the one immediately to the left.

We define the *Pascal's matrix* $P = [p_{ij}]$ of order n by taking the first n rows of Pascal's triangle and filling in with 0's on the right (cf. Call and Velleman [5]). That is,

$$p_{ij} = \begin{cases} \binom{i-1}{j-1} & \text{if } i \geq j \\ 0 & \text{otherwise.} \end{cases}$$

Thus we have

$$P = \begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & O & \\ 1 & 2 & 1 & & & \\ 1 & 3 & 3 & 1 & & \\ \vdots & \vdots & \vdots & \ddots & \ddots & \\ 1 & n-1 & \cdots & \cdots & n-1 & 1 \end{bmatrix}.$$

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In [3], it is shown that the Pascal's matrix P can be factorized by following:

$$P = G_n G_{n-1} \cdots G_1$$

where for each $k = 1, \dots, n$,

$$G_k = \begin{bmatrix} I_{n-k} & O^T \\ O & T_k \end{bmatrix}$$

where $T_k = [t_{ij}]$ is the lower triangular matrix of order k defined by

$$t_{ij} = \begin{cases} 1 & \text{if } i \geq j \\ 0 & \text{otherwise.} \end{cases}$$

Cleary the determinant of P is 1, and the inverse of P is obtained (cf. Brawer and Pirovino [3]). In fact, $P^{-1} = [p'_{ij}]$ where

$$p'_{ij} = \begin{cases} (-1)^{i-j}\binom{i-1}{j-1} & \text{if } i \geq j \\ 0 & \text{otherwise.} \end{cases}$$

Thus P^{-1} is the same as P except that the minus signs appear at (i,j)-positions with $i - j = 1 \pmod 2$.

In this paper, we obtain some well known combinatorial identities and a factorization of the Stirling matrix from the Pascal's matrix.

2. Results

We consider an ordinary chessboard which is divided into $(n-1)^2$ squares in n rows and n columns. Let c_{ij} be the number of paths with the length $i + j - 2$ from $(1,1)$-position to (i,j)-position. Then it is easy to show that

$$c_{ij} = \frac{(i+j-2)!}{(i-1)!(j-1)!} = \binom{i+j-2}{j-1}. \tag{2.1}$$

For each c_{ij} $(1 \leq i,\ j \leq n)$ in (2.1), define $C_n = [c_{ij}]$ to be the matrix of order n. For example,

$$C_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}, \quad C_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix}.$$

In the following theorem, by a combinatorial argument we show that C_n can be expressed by the Pascal's matrix.

Theorem 2.1. *Let $C_n = [c_{ij}]$ be the matrix of order n with entries in (2.1). Then $C_n = PP^T$.*

Proof. We may assume that $i \le j$. Then we can write each path from $(1,1)$-position to (i,j)-position on the $n \times n$ chessboard as a sequence of the form

$$x_1R,\ y_1D,\ x_2R,\ y_2D,\ \cdots,\ x_kR,\ y_kD$$

where R denotes "right", D denotes "down" and, for some $1 \le k \le j$,

$$x_1 + x_2 + \cdots + x_k = j - 1 \quad (x_1 \ge 0;\ x_2, \cdots, x_k > 0), \tag{2.2}$$

$$y_1 + y_2 + \cdots + y_k = i - 1 \quad (y_1, \cdots, y_{k-1} > 0;\ y_k \ge 0). \tag{2.3}$$

For a fixed k with $1 \le k \le j$, clearly the number of such sequences is n_1n_2 where n_1 and n_2 are the number of solutions to (2.2) and (2.3) respectively.

We claim that

$$n_1 = \binom{j-1}{k-1} \quad \text{and} \quad n_2 = \binom{i-1}{k-1}.$$

To show $n_1 = \binom{j-1}{k-1}$, we choose $k-1$ elements of the numbers $1, 2, \cdots, j-1$. Then there is a 1-1 correspondence between solutions to (2.2) and $(k-1)$-subsets of $\{1, 2, \cdots, j-1\}$. Namely, if $\{x_1, x_2, \cdots, x_k\}$ is a solution to (2.2), then $\{x_1, x_1 + x_2, \cdots, x_1 + x_2 + \cdots + x_{k-1}\}$ is a $(k-1)$-subset of $\{1, 2, \cdots, j-1\}$. Also, if $\{z_1, z_2, \cdots, z_{k-1}\}$ is a $(k-1)$-subset of $\{1, 2, \cdots, j-1\}$ with $0 \le z_1 < z_2 < \cdots < z_{k-1}$, then

$$x_1 = z_1,\ x_2 = z_2 - z_1,\ \cdots,\ x_{k-1} = z_{k-1} - z_{k-2},\ x_k = j - 1 - z_{k-1}$$

solves (2.2) also. Thus $n_1 = \binom{j-1}{k-1}$.

Similarly, we can show that $n_2 = \binom{i-1}{k-1}$. Hence, if we note that $\binom{n}{r} = 0$ for $r > n$, then the number c_{ij} of paths from (1.1)-position to (i,j)-position is

$$c_{ij} = \sum_{k=1}^{j} \binom{i-1}{k-1}\binom{j-1}{k-1} = \sum_{k=1}^{n} \binom{i-1}{k-1}\binom{j-1}{k-1} = \sum_{k=1}^{n} p_{ik}p_{jk} = (PP^T)_{ij} \tag{2.4}$$

where $(PP^T)_{ij}$ is the (i,j) entry of the matrix PP^T. Therefore $C_n = PP^T$, which completes the proof. □

Note that it is known that PP^T is the Cholesky factorization of C_n [3].

In particular, if $i = j$ in (2.4) then we can establish the following identity from (2.1):

$$\sum_{t=0}^{n} \binom{n}{t}^2 = \binom{2n}{n}. \tag{2.5}$$

More generally, the following *Vandermonde Convolution* can be derived from Theorem 2.1.

Corollary 2.2.

$$\sum_{t=0}^{n} \binom{m}{t}\binom{n}{n-t} = \binom{m+n}{n}.$$

Proof. . From (2.1) and (2.4), we get

$$\sum_{k=1}^{j} \binom{i-1}{k-1}\binom{j-1}{k-1} = \binom{i+j-2}{j-1}.$$

Thus if we take $i-1=m$, $j-1=n$ and $k-1=t$ then Vandermonde convolution follows immediately from $\binom{n}{t} = \binom{n}{n-t}$ and $\binom{n}{n+1} = 0$. □

Vandermonde convolution can be extended as following:

$$\binom{n-n_t}{n_1,\ n_2,\ \cdots,\ n_{t-1}} \sum_{k=0}^{n_t} \binom{n_t}{k}\binom{n-n_t}{k} = \binom{n}{n_1,\ n_2,\ \cdots,\ n_t} \tag{2.6}$$

where $n_1 + n_2 + \cdots + n_t = n$ and $\binom{n}{n_1,\ n_2,\ \cdots,\ n_t} = \frac{n!}{n_1!n_2!\cdots n_t!}$.

Note that $\det C_n = 1$ and $A^{-1} = (P^{-1})^T P^{-1}$.

Next, we consider a famous counting problem which is called *Stirling number.* Let $S(n,k)$ denote the Stirling number for integers n and k with $1 \le k \le n$. Then the number $S(n,k)$ counts the number of partitions of a set X of n elements into k indistinguishable boxes in which no box is empty.

Example. Let $X = \{a,b,c,d\}$ then we get the partitions for each $k = 1,2,3,4$:

$k=1: X;$

$k=2: [\{a\},\ \{b,c,d\}],\ [\{b\},\ \{a,c,d\}],\ [\{c\},\ \{a,b,d\}],\ [\{d\},\ \{a,b,c\}],$
$[\{a,b\},\ \{c,d\}],\ [\{a,c\},\ \{b,d\}],\ [\{a,d\},\ \{b,c\}];$

$k=3: [\{a\},\ \{b\},\ \{c,d\}],\ [\{a\},\ \{c\},\ \{b,d\}],\ [\{a\},\ \{d\},\ \{b,c\}],\ [\{c\},\ \{d\},\ \{a,b\}],$
$[\{b\},\ \{d\},\ \{a,c\}],\ [\{b\},\ \{c\},\ \{a,d\}];$

$k=4: [\{a\},\ \{b\},\ \{c\},\ \{d\}].$

Thus we have

$$S(4,k) = \begin{cases} 1 & \text{if } k = 1 \\ 7 & \text{if } k = 2 \\ 6 & \text{if } k = 3 \\ 1 & \text{if } k = 4. \end{cases}$$

It is well known [1] that the Stirling numbers $S(n,k)$ have a Pascal-like recurrence relation as following:

$$S(n,k) = \begin{cases} 1 & \text{if } k = 1 \\ 1 & \text{if } k = n \\ S(n-1,k-1) + kS(n-1,k) & \text{if } 2 \le k \le n-1. \end{cases}$$

As we did for the Pascal's triangle we can obtain a Pascal-like matrix S_n of order n for these Stirling numbers $S(n,k)$.

Define $S_n = [s_{ij}]$ to be the matrix of order n where

$$s_{ij} = \begin{cases} S(i,j) & \text{if } i \ge j \\ 0 & \text{otherwise.} \end{cases}$$

Thus for i and j with $i \ge j$, each entry s_{ij} in the matrix S_n, other than initial values, is obtained by multiplying the entry in the row directly above it by j and adding the result to the entry immediately to its left in the row directly above it. We call S_n the *Stirling matrix* of order n. For example,

$$S_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 7 & 6 & 1 \end{bmatrix}, \quad S_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 \\ 1 & 7 & 6 & 1 & 0 \\ 1 & 15 & 25 & 10 & 1 \end{bmatrix}.$$

By a simple computation, we can easily show the following lemma.

Lemma 2.3. *Let S_n be the Stirling matrix of order n and let P be the Pascal's matrix of order n. Then $S_n = P([1] \oplus S_{n-1})$ where $\oplus$ denotes a direct sum.*

Corollary 2.4. *Let $S(n,k)$ be a Stirling number. Then*

$$S(n,k) = \sum_{r=1}^{n-1} \binom{n-1}{r} S(r,k-1), \quad (k \ne 1). \tag{2.7}$$

Proof.. Let $S_n = [s_{ij}]$ and $P = [p_{ij}]$. Then from Lemma 2.3, if $k \neq 1$, we get

$$S(n,k) = s_{nk} = \sum_{r=1}^{n-1} p_{n\,r+1} s_{r\,k-1} = \sum_{r=1}^{n-1} \binom{n-1}{r} S(r, k-1),$$

which completes the proof. □

For the Pascal's matrix P_k of order k, $1 \leq k \leq n$, define

$$\bar{P}_k = \begin{bmatrix} I_{n-k} & O^T \\ O & P_k \end{bmatrix}$$

to be the matrix of order n. Thus $\bar{P}_n := P$ and $\bar{P}_1$ is the identity matrix of order n.

Corollary 2.5. *Let S_n be the Stirling matrix of order n. Then S_n can be factorized by the $\bar{P}_k$'s:*

$$S_n = \bar{P}_n \, \bar{P}_{n-1} \, \cdots \, \bar{P}_2 \, \bar{P}_1. \tag{2.8}$$

Proof. If we apply (2.7) recursively we obtain (2.8). □

Remark. Note that

$$S_n^{-1} = \bar{P}_1^{-1} \, \bar{P}_2^{-1} \, \cdots \, \bar{P}_{n-1}^{-1} \, \bar{P}_n^{-1}.$$

Example.

$$S_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 \\ 1 & 7 & 6 & 1 & 0 \\ 1 & 15 & 25 & 10 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 1 & 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$S_5^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & -1 & 3 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ -1 & 3 & -3 & 1 & 0 \\ 1 & -4 & 6 & -4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 2 & -3 & 1 & 0 & 0 \\ -6 & 11 & -6 & 1 & 0 \\ 24 & -50 & 35 & -10 & 1 \end{bmatrix}.$$

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