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MODEL FOR THE CONTAMINATION OF CONFINED AQUIFERS BY POLLUTANTS

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ABSTRACT. This paper studies the problem of an infinite confined aquifer which at time $t < 0$ is assumed motionless. At time $t = 0$ crude oil seeps into the aquifer, thereby contaminating the valuable drinking water. Since the crude oil and water are im-miscible, the problem is posed as a one-dimensional two-phase unsteady moving boundary problem. A similarity solution is developed in which the moving front parameter is obtained by Newton-Ralphson iteration. A numerical scheme, involving the front tracking method, is devised employing the fourth order Runge-Kutta method. Comparison of the exact and numerical schemes shows an error of only 3%. Thus the developed numerical scheme is quite accurate in tackling more realistic problems where exact solutions are not possible.

1. Introduction

The Niger Delta has been an area of intensive petroleum exploration activities (see Etu-Efeotor and Odigi [1]). The ground water in the Niger Delta and some other part of Nigeria where pipeline carrying crude or refired oil passes through is usually effected by pollution as a result of outburst or leakage of oil into the aquifer.

The Niger Delta consists mainly of freshwater swamps, mangrove swamps, sand beaches and estuaries. There is abundant surface and ground water in the area. The non-deltaic regions of Nigeria with pipelines carrying oil consists of freshwater streams or rivers and surficial-soil materials. But in general, pipelines carrying oil are usually constructed along the above mentioned morphologic features. Aquifer horizons are shallow in the delta but fairly deepseated in the non-deltaic areas.

Drilling for petroleum and the transportation of oil by pipelines have been done under high degree of technical competence, efficient regulatory and preventive measures. However, despite these measures employed, several cases of oil spillage have

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been reported in the Niger Delta. The effects of contamination vary from slight in some locations, to more serious and inconveniencing degree in others. The possible consequences associated with oil leakage from pipeline are grave. When there is an oil leakage or spillage, the sourrounding medium which is either soil or water is first contaminated and later penetrates into the aquifer.

Apart from water supply problems which are easily identifiable (see Etu-Efeotor and Odigi [1]), ground water contamination by petroleum is a serious hydrological problem in both the Niger Delta and other parts of Nigeria. This problem has received little attention. Therefore the paper discusses a model which would enhance the predicability of the behaviour of pollutants in aquifers.

2. Mathematical Formulation

The differential equation that governs the variation of hydralic head in an aquifer is (see, for example, Todd [2])

$$\frac{S}{Kb}\frac{\partial h'}{\partial t'} = \frac{\partial^2 h'}{\partial x'^2} + \frac{\partial^2 h'}{\partial y'^2} + \frac{\partial^2 h'}{\partial z'^2} \tag{1}$$

where(x', y', z') is the cartesian coordinate system and t' is time. K is the permeability, S is the compressibility factor or specific storage while b is the breadth of the aquifer. In one dimensional steady situation, equation (1) reduces to the simple solution

$$h' = C_1 x' + C_2. \tag{2}$$

In ground water hydrology, it is usual to take the constant C_2 as zero and determine C_1 from the Darcy velocity equation

$$u' = K\frac{dh'}{dx'}.$$

Therefore

$$h' = \frac{u'x'}{K} \tag{3}$$

where u' is the velocity of the water in the aquifer.

In this problem, we take $C_1 = 0$ so that the water in the aquifer under a constant head is motionless. This is approximately true since u is usually quite small.

The mathematical statement of the problem consists of an infinitely long aquifer which at time $t' < 0$ is motionless. At time $t' = 0$, pollutant, which in this case is crude oil, seeps into the aquifer. If lengths are made dimensionless by the aquifer breadth b and time by nb/K_1, where n is the porosity, then denoting the water and crude oil phases by superscript/subscript 1 and 2 respectively, the governing equations become

$$\sigma\frac{\partial h^{(1)}}{\partial t} = \frac{\partial^2 h^{(1)}}{\partial x^2}, \quad K\sigma\frac{\partial h^{(2)}}{\partial t} = \frac{\partial^2 h^{(2)}}{\partial x^2} \tag{4}$$

such that $\rho h^{(2)} - h^{(1)} = (\rho - 1)X$, and

$$\left.\begin{aligned} \frac{dX}{dt} &= -\frac{\partial h^{(1)}}{\partial x} \\ \frac{dX}{dt} &= -\frac{\rho}{\mu}\frac{\partial h^{(2)}}{\partial x} \end{aligned}\right\} \quad (x = X(t), \ t > 0) \tag{5a}$$

with initial and boundary conditions

$$\left.\begin{aligned} h^{(1)} &= H_1 \quad \text{as } x \to \infty \\ h^{(2)} &= H_2 \quad \text{on } x = 0 \end{aligned}\right\} \quad (t > 0) \tag{5b}$$

and

$$\begin{cases} t = 0: \quad h^{(1)} = H_1, \quad h^{(2)} = \widetilde{H}_2 \\ X(t) = 0. \end{cases} \tag{6}$$

H_1, H_2 and $\widetilde{H}_2$ are constant heads. $X(t)$ is the unknown moving front between the im-miscible crude oil and water. Equations in (5a) are the kinematic and dynamic conditions at the moving front boundary as given in Liggett and Liu [3]. The variables $\rho=\rho_2/\rho_1$, $\mu=\mu_2/\mu_1$ and $k=k_2/k_1$ represent the density, viscosity and permeability ratios while $\sigma = S/n$ is a non-dimensional parameter.

3. Similarity Solutions

We introduce the similarity variable

$$\eta = \frac{x}{2}\sqrt{\frac{\sigma}{t}} \tag{7}$$

in which case (4) becomes

$$\frac{d^2h^{(1)}}{d\eta^2} + 2\eta\frac{dh^{(1)}}{d\eta} = 0, \quad \frac{d^2h^{(2)}}{d\eta^2} + 2K\eta\frac{dh^{(2)}}{d\eta} = 0. \tag{8}$$

The solutions to equation (8) satisfying (5b) are

$$h^{(1)} = H_1 - C_1 \operatorname{erfc}(\eta), \quad h^{(2)} = H_2 + C_2 \operatorname{erfc}(K^{1/2}\eta). \tag{9}$$

Now with

$$X = \lambda\sigma^{1/2}t^{1/2}, \quad \lambda = \text{constant}, \tag{10}$$

the boundary conditions at the moving front (5a), with help of (9), reduce to

$$\lambda = -C^{(1)}e^{-1/2\sigma^2\lambda^3}, \tag{11a}$$

$$\lambda = -C^{(2)}\frac{\rho}{\mu}e^{-1/4K\sigma^2\lambda^2} \tag{11b}$$

and

$$\rho\left[H_2 + C^{(2)} \operatorname{erf}(\frac{1}{2}K^{1/2}\sigma\lambda)\right] - H_1 + C^{(1)} \operatorname{erf}(\frac{1}{2}\sigma\lambda) = (\rho-1)\sigma^{1/2}\lambda t^{1/2}. \tag{11c}$$

Combining equations (11a), (11b) and (11c) we get

$$\rho\left[H_2 - \frac{\mu}{\rho}\lambda e^{1/4K\sigma^2\lambda^2} \operatorname{erf}(\frac{1}{2}K^{1/2}\sigma\lambda)\right] - H_1 - \lambda e^{1/4\sigma^2\lambda^2} \operatorname{erfc}(\frac{1}{2}\sigma\lambda)$$
$$= (\rho-1)\sigma^{1/2}\lambda t^{1/2} \tag{12}$$

Equation (12) determines λ as a function of time. But λ is assumed constant, therefore there may exist an asymptotic λ for large time and this is the desired value of λ. Computationally, for each chosen time step, λ is computed from (12) by the Newton-Ralphson method. The asymptotic value of λ may subsequently be obtained. Which value of λ is now used to compute $C^{(1)}$ and $C^{(2)}$ from (11a) and (11b). This similarity solution is compartible with the initial conditions (6) if

$$\widetilde{H}_2 = H_2 + C^{(2)}.$$

Some numerical results of the computation are entered in Table 1 below for $\mu = 2, \rho = 0.8, H_1 = 1, H_2 = 2$ and $k = 1$.

Thus the front moves faster with increase in σ. However the change is small and $\sigma^{1/2}\lambda$ is of order 0(1) and $x \sim t^{1/2}$.

Table 1. Some numerical results

σ	λ	$\sigma^{1/2}\lambda$
0.01	10.1357	1.01...
0.05	4.8028	1.07...
0.10	3.6909	1.17...

4. Numerical Scheme

Since an analytical solution is possible for this *ad hoc* problem, it is pertinent to compare these exact solutions with results obtained from a numerical process that is applicable to realistic practical problems in one- and two-dimensions. The methodology we adopt is the front tracking method that has been quite successful in melting and solidification problems. In this case we introduce the variable

$$\zeta = \frac{x}{X(t)} \tag{13}$$

in which case (4) and (5) become

$$\left.\begin{aligned} \frac{1}{X^2}\frac{\partial^2 h^{(1)}}{\partial\zeta^2} + \sigma\frac{\zeta}{X}\frac{dX}{dt}\frac{\partial h^{(1)}}{\partial\zeta} &= \sigma\frac{\partial h^{(1)}}{\partial t} \\ \frac{1}{X^2}\frac{\partial^2 h^{(2)}}{\partial\zeta^2} + K\sigma\frac{X}{\zeta}\frac{dX}{t}\frac{\partial h^{(2)}}{\partial\zeta} &= K\sigma\frac{\partial h^{(2)}}{\partial t} \end{aligned}\right\} \tag{14}$$

such that $\rho h^{(2)} - h^{(1)} = (\rho - 1)X$, and

$$\left.\begin{aligned} \frac{dX}{dt} &= -\frac{1}{X}\frac{\partial h^{(1)}}{\partial\zeta} \\ \frac{dX}{dt} &= -\frac{\rho}{\mu}\frac{1}{X}\frac{\partial h^{(2)}}{\partial\zeta} \end{aligned}\right\} \quad (\zeta = 1, \ \ t > 0) \tag{15a,b}$$

with initial and boundary conditions

$$\left.\begin{aligned} h^{(1)} &= H_1 \quad \text{as } \zeta \to \infty \\ h^{(2)} &= H_2 \quad \text{on } \zeta = 0 \end{aligned}\right\} \quad (t > 0) \tag{15c}$$

and

$$\begin{cases} t = 0: \quad h^{(1)} = H_1, \quad h^{(2)} = H_2 \\ X(t) = 0. \end{cases} \tag{16}$$

and we shall take $H_2 = \widetilde{H}_2 + C$ for comparison.

To integrate (14)-(16), we put

$$h^{(1)}(t) = h^{(1)}(j\Delta\zeta, t), \quad h^{(2)} = h^{(2)}(j\Delta\zeta, t)$$

then replacing second derivatives by central differences and first derivatives by forward/backward differences, we get

$$\sigma \frac{dh_j^{(1)}}{dt} = \sigma \frac{1}{X} \frac{dX}{dt} \zeta_j \frac{h_j^{(1)} - h_{j-1}^{(1)}}{\Delta\zeta} + \frac{1}{X^2} \frac{h_{j+1}^{(1)} - 2h_j^{(1)} + h_{j-1}^{(1)}}{\Delta\zeta^2} \tag{17}$$

where $h_0^{(1)} = H_1$; $h_j^{(1)}(0) = H_1$, $j = 1, 2, \cdots$; $\zeta_0 = 1$, $\zeta_j = \infty$ and

$$K\sigma \frac{dh_j^{(2)}}{dt} = K\sigma \frac{1}{X} \frac{dX}{dt} \zeta_j \frac{h_j^{(2)} - h_{j-1}^{(2)}}{\Delta\zeta} + \frac{1}{X^2} \frac{h_{j+1}^{(2)} - 2h_j^{(2)} + h_{j-1}^{(2)}}{\Delta\zeta^2} \tag{18}$$

where $h_0^{(2)} = H_2$; $h_j^{(2)}(0) = H_2 + C$, $j = 1, 2, \cdots$; $\zeta_0 = 0$, $\zeta_j = 1$. We also get

$$\frac{dX}{dt} = -\frac{1}{X} \frac{h_1^{(1)} - h_0^{(1)}}{\Delta\zeta} \tag{19a}$$

where

$$\rho h_j^{(2)} - h_0^{(1)} = (\rho - 1)X, \tag{19b}$$

and

$$-\frac{dX}{dt} = -\frac{\rho}{\mu} \frac{1}{X} \frac{h_j^{(2)} - h_{j-1}^{(2)}}{\Delta\zeta} \tag{19c}$$

where

$$h_j^{(2)}(0) = 0, \; h_0^{(1)}(0) = H_1. \tag{20}$$

Combining equations (19a), (19b) and (19c) gives

$$\rho \frac{dh_j^{(2)}}{dt} - \frac{dh_0^{(1)}}{dt} = (\rho - 1) \frac{1}{X} \frac{h_1^{(1)} - h_0^{(1)}}{\Delta\zeta} \tag{21}$$

$$\frac{dh_j^{(2)}}{dt} + \frac{\rho}{\mu} \frac{dh_0^{(1)}}{dt} = \frac{dh_{j-1}^{(2)}}{dt} + \frac{\rho}{\mu} \frac{dh_1^{(1)}}{dt} \tag{22}$$

Equations (17), (18), (19a), (20), (21), (22) constitute $2j+1$ equations for the unknowns x; $h_0^{(1)}, h_1^{(2)}, \cdots, h_j^{(2)}$, which equations constitute an initial value problem. These are solved by a fourth order Runge-Kutta method.

In Figure 1 the analytical and numerical solutions for $X(t)$ are depicted graphically. Numerical solution underestimates $X(t)$ by less than 3% at the highest difference point.

The curve included in Figure 1 presents a graphical solution to the problem of pollution of ground water by petroleum contaminants.

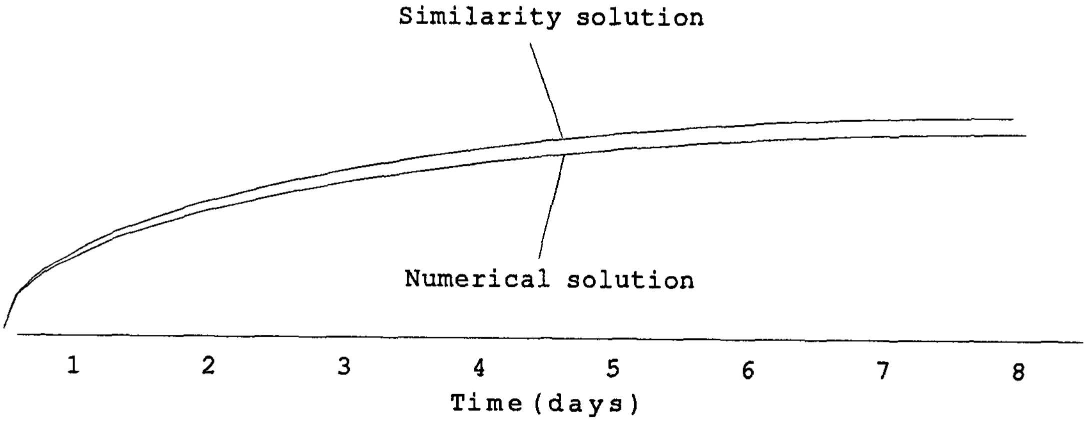

Figure 1. Timing of the moving pollutant front

We have examined this problem using both analytical and numerical schemes. Figure 1 illustrates the rate of change of $X(t)$ which is the created pollutant front into the aquifer. When $0 \leq t \leq 1$, the value $X(t)$ increases proportionately with time. At time $t > 1$, the value $X(t)$ steadily slows down. However, some numerical results have been computerized to show that the $X(t)$ front moves faster with increase in σ. But this change is noted to be relatively small. The computerized results are shown in Table 1 in §3.

One of the practical areas of interest for this application is the problem of flow of oil pollutant into aquifer. And also from a practical point of view, the implication of the study is that the moving front of pollutants in a given aquifer is slow with time but faster at the initial time of contact between the oil and water. This is on the assumption that the slope of the aquifer is small.

Comparing the results obtained from the analytical and numerical schemes the errors estimated is less than 3%. Additional numerical and experimental work is needed to establish the suitability of a combination of oil front tracking and finite difference quadrature methods.

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