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ON THE $G(F)$-SEQUENCE OF A CW-TRIPLE

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ABSTRACT. We find some conditions under which $G(f)$-sequence of a CW-pair (X, A) is exact. And we also introduce a $G(f)$-sequence for a CW-triple (X, A, B) and examine when the sequence is exact.

1. Introduction

Since Gottlieb [1, 2] introduced and studied the subgroups $G_n(X)$ of homotopy groups $\pi_n(X)$, several authors have studied and generalized the subgroups $G_n(X)$. Especially Kim and Woo [4], and Lee and Woo [6, 7] introduced subgroups $G_n(X, A)$ and $G_n^{Rel}(X, A)$ of $\pi_n(X)$ and $\pi_n(X, A)$ respectively and showed that they form a sequence

$$\cdots \xrightarrow{j_*} G_{n+1}^{Rel}(X,A) \xrightarrow{\partial} G_n(A) \xrightarrow{i_*} G_n(X,A) \xrightarrow{j_*} G_n^{Rel}(X,A) \xrightarrow{\partial} \cdots$$

$$\cdots \xrightarrow{j_*} G_2^{Rel}(X,A) \xrightarrow{\partial} G_1(A) \xrightarrow{i_*} G_1(X,A)$$

where i_*, j_* and ∂ are restrictions of the usual homomorphisms of the homotopy sequence

$$\cdots \xrightarrow{j_*} \pi_{n+1}(X,A) \xrightarrow{\partial} \pi_n(A) \xrightarrow{i_*} \pi_n(X) \longrightarrow \cdots \xrightarrow{j_*} \pi_2(X,A) \xrightarrow{\partial} \pi_1(A) \xrightarrow{i_*} \pi_1(X).$$

Lee [8] and Woo [9] extend the concept of the above G-sequence into the $G(f)$-sequence for any self map $f : (X, A) \to (X, A)$. And the $G(f)$-sequence is shown to be exact when $i : A \to X$ has a left homotopy inverse (cf. [8]) or $i : A \to X$ is homotopic to a constant map (cf. [9]).

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2. Preliminaries

In this paper, all spaces are finite pathwise-connected CW-complexes, all topological pairs are CW-pairs and all subspaces mentioned contain the same base point as their total spaces.

We denote by A^A the subspace of the function space X^A consisting of $f \in X^A$ such that $f(A) \subset A$. Let I^n be the n-dimensional cube, let ∂I^n be its boundary and let J^{n-1} be the union of all $(n-1)$ faces of I^n except for the initial face. We use the same notation ω for the evaluation maps of X^X and X^A into X at the base point x_0 and use $i : A \to X$ as the inclusion map. Let $f : (X, A) \to (X, A)$ be a self-map, $\overline{f} : A \to A$ be the restriction of f and $f_A : A \to X$ be the restriction of f to A.

For a CW-pair (X, A), the evaluation subgroups $G_n(X)$ are defined by

$$\begin{aligned} G_n(X) =& \omega_*(\pi_n(X^X, 1_X)) \\ =& \{[f] \in \pi_n(X) \mid \exists \text{ map } H : X \times I^n \to X \text{ such that} \\ & [H|_{x_0 \times I^n}] = [f] \text{ and } H|_{X \times u} = 1_X \text{ for } u \in \partial I^n\} \end{aligned}$$

for $n \geq 1$. Kim and Woo [4] generalized $G_n(X)$ to $G_n^{f_A}(X, A)$ for any map $f_A : A \to X$. These subgroups are defined by

$$\begin{aligned} G_n^{f_A}(X, A) =& \omega_*(\pi_n(X^A, f_A)) \\ =& \{[h] \in \pi_n(X) \mid \exists \text{ map } H : A \times I^n \to X \text{ such that} \\ & [H|_{x_0 \times I^n}] = [h] \text{ and } H|_{A \times u} = f_A \text{ for } u \in \partial I^n\} \end{aligned}$$

for $n \geq 1$. These groups are called the *generalized evaluation subgroups* of the homotopy groups. In case that $f_A = i : A \to X$ is the inclusion, we denote $G_n^i(X, A)$ by $G_n(X, A)$. And note that $G_n(X) \subset G_n(X, A)$ from the definition.

In [8], the group $G_n^{Rel}(f)$ is defined by

$$\begin{aligned} G_n^{Rel}(f) =& \omega_*(\pi_n(X^A, A^A, f_A)) \\ =& \{[h] \in \pi_n(X, A) \mid \exists \text{ map } H \colon (X \times I^n, A \times \partial I^n) \to (X, A) \text{ such that} \\ & [H_{\,x_0 \times I^n}] = [h] \text{ and } H|_{X \times u} = f \text{ for } u \in J^{n-1}\} \end{aligned}$$

for $n \geq 2$. In case that $f = 1_X : X \to X$ is the identity map, $G_n^{Rel}(f)$ is denoted by $G_n^{Rel}(X, A)$.

3. The $G(f)$-sequence of a CW-pair

Let $f : (X, A) \to (X, A)$ be a self-map and let $\bar{f} : A \to A$ and $f_A : A \to X$ be its restrictions.

Theorem 3.1 [8,9]. *Let $f : (X, A) \to (X, A)$ be a self-map. Then $G_n^{\bar{f}}(A, A)$, $G_n^{f_A}(X, A)$ and $G_n^{Rel}(f)$ form a chain complex.*

$$\cdots \longrightarrow G_n^{\bar{f}}(A, A) \xrightarrow{i_*} G_n^{f_A}(X, A) \xrightarrow{j_*} G_n^{Rel}(f) \xrightarrow{\partial} G_{n-1}^{\bar{f}}(A, A) \longrightarrow \cdots$$

$$\cdots \xrightarrow{j_*} G_2^{Rel}(f) \xrightarrow{\partial} G_1^{\bar{f}}(A, A) \xrightarrow{i_*} G_1^{f_A}(X, A).$$

Proof. The inclusion map $\underline{i} : A^A \to X^A$ and the evaluation map $\omega : (X^A, A^A) \to (X, A)$ give rise to the following commutative diagram:

$$\begin{array}{ccc} A^A & \xrightarrow{\underline{i}} & X^A \\ \downarrow{\scriptstyle \omega} & & \downarrow{\scriptstyle \omega} \\ A & \xrightarrow{i} & X. \end{array}$$

And this diagram induces the following commutative diagram:

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{\underline{i}_*} & \pi_n(X^A, f_A) & \xrightarrow{\underline{j}_*} & \pi_n(X^A, A^A, f_A) & \xrightarrow{\partial} & \pi_{n-1}(A^A, \bar{f}) & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle \omega_*} & & \downarrow{\scriptstyle \omega_*} & & \downarrow{\scriptstyle \omega_*} & & \\ \cdots & \xrightarrow{i_*} & G_n^{f_A}(X, A) & \xrightarrow{j_*} & G_n^{Rel}(f) & \xrightarrow{\partial} & G_{n-1}^{\bar{f}}(A, A) & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle inc} & & \downarrow{\scriptstyle inc} & & \downarrow{\scriptstyle inc} & & \\ \cdots & \xrightarrow{i_*} & \pi_n(X) & \xrightarrow{j_*} & \pi_n(X, A) & \xrightarrow{\partial} & \pi_{n-1}(A) & \longrightarrow & \cdots. \end{array}$$

Since the ω_*'s are epimorphisms, we have

$$\begin{aligned} j_* G_n^{f_A}(X, A) = j_* \omega_* \pi_n(X^A, f_A) &= \omega_* \underline{j}_* \pi_n(X^A, f_A) \\ &\subset \omega_* \pi_n(X^A, A^A, f_A) = G_n^{Rel}(f). \end{aligned}$$

Similarly the remains can be checked and, since the botton row is a chain complex, the middle sequence is a chain complex. ☐

The above middle sequence is called the *G(f)-sequence* of a CW-pair (X, A). And note that even though the top and bottom rows are exact, the $G(f)$-sequence is not necessarily exact [7, Theorem 3.4].

Lemma 3.2 [4]. *If $i : A \to X$ is an inclusion, then $G_n(X) \subset G_n^i(X, A) \subset G_n^{f_A}(X, A)$, for any map $f : (X, A) \to (X, A)$.*

Proof. $G_n(X) \subset G_n^i(X, A)$ is clear from the definition.

Let $[h] \in \pi_n(X)$ be the arbitrary element of $G_n^i(X, A)$. Then there is a homotopy $H : A \times I^n \to X$ such that $[H|_{x_0 \times I^n}] = [h]$ and $H|_{A \times u} = i$ for $u \in \partial I^n$. If we define a homotopy $F : A \times I^n \to X$ by $F(a, u) = H(f(a), u)$, then $[F|_{x_0 \times I^n}] = [h]$ and $F|_{A \times u} = f_A$ for $u \in \partial I^n$. Thus $[h] \in G_n^{f_A}(X, A)$. ☐

A CW-pair (X, A) is called an *H-pair* if X and A are H-spaces and the restriction of the product in X to $A \times A$ is homotopic to the product in A.

The $G(f)$-sequence of a CW-pair (X, A) is known to be exact (cf. [8, 9]) if (X, A) satisfies one of the following:

(1) $i : A \to X$ has a left homotopy inverse.

(2) $i : A \to X$ is homotopic to a constant map.

Theorem 3.3 [9]. *Let $f : (X, A) \to (X, A)$ be a self-map. If $f_A : A \to X$ is homotopic to a constant map, then $G_n^{f_A}(X, A) = \pi_n(X)$ for $n \geq 1$.*

Theorem 3.4 [2, 7]. *If (X, A) is an H-pair, then $G_n(X) = G_n(X, A) = \pi_n(X)$.*

Proof. For a given map $h : (I^n, \partial I^n) \to (X, x_0)$, define $F : X \times I^n \to X$ by $F(x, u) = x \cdot h(u)$, where $x \cdot h(u)$ denotes x multiplied by $h(u)$. Since

$$F(x_0, u) = x_0 \cdot h(u) = h(u) \text{ for all } u \in I^n$$

and

$$F(x, v) = x \cdot h(v) = x \cdot x_0 = x \text{ for all } v \in \partial I^n,$$

the existence of F implies that $[h] \in G_n(X)$ and we obtain that $G_n(X) = \pi_n(X)$. And obviously we have $G_n(X) = G_n(X, A) = \pi_n(X)$. ☐

Theorem 3.5. *If (X, A) is an H-pair and $f : (X, A) \to (X, A)$ is any map, then $G_n^{Rel}(f) = \pi_n(X, A)$.*

Proof. Let $h : (I^n, I^{n-1}, J^{n-1}) \to (X, A, x_0)$ be a map and let $k : (I^n, I^{n-1} \times 0) \to (I^n, J^{n-1})$ be a homeomorphism in (cf. [4, p. 81]), which satisfies $k^2 = 1_{I^n}$.

Define $H : (X \times I^{n-1} \times 0) \cup (A \times I^{n-1} \times I) \to X$ by

$$\begin{aligned} H(x, u, 0) &= x && \text{for } (x, u, 0) \in X \times I^{n-1} \times 0, \\ H(a, u, t) &= a \cdot hk(u, t) && \text{for } (a, u, t) \in A \times I^{n-1} \times I. \end{aligned}$$

Since $H(a, u, 0) = a \cdot hk(u, 0) = a \cdot x_0 = a$ for $a \in A$ and $u \in I^{n-1}$, H is well-defined continuous map. By the absolute homotopy extension property, there is an extension $F : X \times I^n \to X$ of H. The map F satisfies that

$$\begin{aligned} F(x, u, 0) &= x, && \text{for } u \in I^{n-1} \\ F(a, u, t) &= a \cdot hk(u, t) \in A && \text{for } (u, t) \in J^{n-1} \\ F(x_0, v) &= x_0 \cdot hk(v), && \text{for } v \in I^n. \end{aligned}$$

Let $\overline{F} = F \circ (f \times k) : X \times I^n \to X$. Then

$$\begin{aligned} \overline{F}(a, v) =& F \circ (f \times k)(a, v) = F(f(a), k(v)) = f(a) \cdot hk^2(v) \\ =& f(a) \cdot h(v) \in A, \qquad \text{for } a \in A, \ \ v \in \partial I^n, \end{aligned}$$

that is, $\overline{F}(A \times \partial I^n) \subset A$. And we have

$$\begin{aligned} \overline{F}(x_0, v) =& F \circ (f \times k)(x_0, v) = F(x_0, k(v)) = x_0 \cdot hk^2(v) \\ =& x_0 \cdot h(v) = h(v), \qquad \text{for all } v \in I^n, \\ \overline{F}(x, v) =& F \circ (f \times k)(x, v) = F(f(x), k(v)) = f(x) \cdot hk^2(v) \\ =& f(x) \cdot h(v) = f(x) \cdot x_0 = f(x), \qquad \text{for all } \ v \in J^{n-1}. \end{aligned}$$

This means that $[h] \in G_n^{Rel}(f)$. □

Theorem 3.6. *If a CW-pair (X, A) is an H-pair, then the $G(f)$-sequence of a pair (X, A) is exact.*

Proof. From Theorem 3.4, $G_n^{\overline{f}}(A, A) = \pi_n(A)$ and $G_n^{f_A}(X, A) = \pi_n(X)$. And by Theorem 3.5, $G_n^{Rel}(f) = \pi_n(X, A)$. Thus the $G(f)$-sequence is just the homotopy sequence of the pair (X, A) and so it is exact. □

4. The $G(f)$-sequence of a CW-triple

A triple (X, A, B) is called a CW-triple if (X, A), (X, B) and (A, B) are CW-pairs. For a triple (X, A, B), we know that there is a homotopy sequence

$$\cdots \xrightarrow{\partial_*} \pi_n(A,B) \xrightarrow{i_*} \pi_n(X,B) \xrightarrow{j_*} \pi_n(X,A) \xrightarrow{\partial_*} \pi_{n-1}(A,B) \longrightarrow \cdots$$

which is exact. And note that the boundary operator $\partial_* : \pi_n(X, A) \to \pi_{n-1}(A, B)$ is defined as the composite $\pi_n(X, A) \xrightarrow{\partial} \pi_{n-1}(A) \xrightarrow{j_*} \pi_{n-1}(A, B)$.

Unlike the case for CW-pair, the relative evaluation subgroups $G_n^{Rel}(A, B)$, $G_n^{Rel}(X, B)$ and $G_n^{Rel}(X, A)$ do not form a chain complex in general.

But in [7], the authors showed that the following.

Proposition 4.1 [7, Lemma 3.16]. *Let (X, A, B) be a CW-triple. If A is a G-space, that is, $G_n(A) = \pi_n(A)$ for all n, and if the inclusion $i : A \to X$ is homotopic to a constant map, then we have*

$$\begin{aligned} i_*(G_n^{Rel}(A,B)) &\subset G_n^{Rel}(X,B) \quad \text{for } n \geq 1, \\ j_*(G_n^{Rel}(X,B)) &\subset G_n^{Rel}(X,A) \quad \text{for } n \geq 1, \\ \partial_*(G_n^{Rel}(X,A)) &\subset G_{n-1}^{Rel}(A,B) \quad \text{for } n \geq 2. \end{aligned}$$

Now we generalize the G-sequence of a CW-pair (X,A) to the $G(f)$-sequence of a CW-triple (X, A, B).

For a self map $f : (X, A, B) \to (X, A, B)$, let

$$\begin{aligned} f_{XA} &: (X, A) \to (X, A), \\ f_{AB} &: (A, B) \to (A, B), \\ f_{XB} &: (X, B) \to (X, B) \end{aligned}$$

denote the maps which are induced by the given map $f : (X, A, B) \to (X, A, B)$.

Theorem 4.2. *Let (X, A, B) be a CW-triple. If A is a G-space, and the inclusion map $i : A \to X$ and $f_A : A \longrightarrow X$ are both homotopic to a constant map, then*

$$\cdots \xrightarrow{j_*} G_{n+1}^{Rel}(f_{XA}) \xrightarrow{\partial_*} G_n^{Rel}(f_{AB}) \xrightarrow{i_*} G_n^{Rel}(f_{XB}) \xrightarrow{j_*} G_n^{Rel}(f_{XA}) \xrightarrow{\partial_*} \cdots$$

$$\cdots \xrightarrow{\partial_*} G_1^{Rel}(f_{AB}) \xrightarrow{i_*} G_1^{Rel}(f_{XB}) \xrightarrow{j_*} G_1^{Rel}(f_{XA}).$$

is a chain complex.

Proof. Consider the following commutative diagram

$$\begin{array}{ccc}(A^B, B^B, f_{AB}) & \xrightarrow{\underline{i}} & (X^B, B^B, f_{XB}) \\ \downarrow{\scriptstyle\omega} & & \downarrow{\scriptstyle\omega} \\ (A, B, x_0) & \xrightarrow{i} & (X, B, x_0).\end{array}$$

The above diagram induces a commutative diagram

$$\begin{array}{ccc}\pi_n(A^B, B^B, f_{AB}) & \xrightarrow{\underline{i}_*} & \pi_n(X^B, B^B, f_{XB}) \\ \downarrow{\scriptstyle\omega_*} & & \downarrow{\scriptstyle\omega_*} \\ \pi_n(A, B, x_0) & \xrightarrow{i_*} & \pi_n(X, B, x_0).\end{array}$$

From the above diagram we have

$$\begin{aligned} i_*(G_n^{Rel}(f_{AB})) &= i_*\omega_*(\pi_n(A^B, B^B, f_{AB})) \\ &= \omega_*\underline{i}_*(\pi_n(A^B, B^B, f_{AB})) \subset G_n^{Rel}(f_{XB}). \end{aligned}$$

Next we show that $j_*(G_n^{Rel}(f_{XB})) \subset G_n^{Rel}(f_{XA})$.

Since A is a G-space, $G_n(A) = G_n^{\overline{f}}(A, A) = \pi_n(A)$. And since $i : A \to X$ is homotopic to a constant map, (X, A) has the exact $G(f)$-sequence and since $f_A : A \to X$ is also homotopic to a constant map, we get $G_n^{f_A}(X, A) = \pi_n(X)$ by Theorem 3.3. Thus we have the following commutative diagram:

$$\begin{array}{ccccccccccc}\cdots \longrightarrow & G_n^{\overline{f}}(A, A) & \xrightarrow{i_*} & G_n^{f_A}(X, A) & \xrightarrow{j_*} & G_n^{Rel}(f_{XA}) & \xrightarrow{\partial} & G_{n-1}^{\overline{f}}(A, A) & \longrightarrow \cdots \\ & \| & & \| & & \downarrow{\scriptstyle inc.} & & \| & \\ \cdots \longrightarrow & \pi_n(A) & \xrightarrow{i_*} & \pi_n(X) & \xrightarrow{j_*} & \pi_n(X, A) & \xrightarrow{\partial} & \pi_{n-1}(A) & \longrightarrow \cdots .\end{array}$$

By Theorem 3.5, we obtain that $\pi_n(X, A) = G_n^{Rel}(f_{XA})$. And so we obtain that

$$j_*(G_n^{Rel}(f_{XB})) \subset j_*(\pi_n(X, B)) \subset \pi_n(X, A) = G_n^{Rel}(f_{XA}).$$

Finally we show that $\partial_*(G_n^{Rel}(f_{XA})) \subset G_{n-1}^{Rel}(f_{AB})$.

Note that $G_{n-1}^{\overline{f}}(A, A) \subset G_{n-1}^{f_B}(A, B)$, $\partial(G_n^{Rel}(f_{XA})) \subset G_{n-1}^{\overline{f}}(A, A)$ for $\overline{f} : A \to A$ and $f_B : B \to A$, the restrictions of the self-map $f_{AB} : (A, B) \to (A, B)$. And also note that the boundary operator $\partial_* : \pi_n(X, A) \to \pi_{n-1}(A, B)$ is the composite $\pi_n(X, A) \xrightarrow{\partial} \pi_{n-1}(A) \xrightarrow{j_*} \pi_{n-1}(A, B)$. Then

$$\partial_*(G_n^{Rel}(f_{XA})) = j_*\partial(G_n^{Rel}(f_{XA})) \subset j_*(G_{n-1}^{\overline{f}}(A, A)) \subset G_{n-1}^{Rel}(f_{AB}). \quad \square$$

The chain complex obtained in the above theorem is called the *G(f)-sequence* of a CW-triple (X, A, B).

For a given self-map $f : (X, A, B) \to (X, A, B)$, the ω-homology

$$H_*^{t\omega}(f) = \{H_{n+1}^{t\omega}(f_{XA}), H_n^{t\omega}(f_{AB}) H_n^{t\omega}(f_{XB})\} \quad \text{for } n \geq 1$$

of (X, A, B) is defined to be

$$\begin{aligned} H_{n+1}^{t\omega}(f_{XA}) &= \text{Ker } \partial_*^{n+1} / \text{Im } j_*^{n+1} \\ H_n^{t\omega}(f_{AB}) &= \text{Ker } i_*^n / \text{Im } \partial_*^{n+1} \\ H_n^{t\omega}(f_{XB}) &= \text{Ker } j_*^n / \text{Im } i_*^n \end{aligned}$$

for $n \geq 1$.

The ω -homology groups $H_*^{t\omega}(f) = \{H_{n+1}^{t\omega}(f_{XA}), H_n^{t\omega}(f_{AB}), H_n^{t\omega}(f_{XB})\}$ for ≥ 1 and $H_2^{t\omega}(f_{XA})$ are commutative groups and $H_1^{t\omega}(f_{AB})$, $H_1^{t\omega}(f_{XB})$ are quotient sets.

Theorem 4.3. *If (X, A, B) is a CW-triple such that (X, A) and (A, B) are H-pairs and $f : (X, A, B) \to (X, A, B)$ is a self map, then the ω-homology $H_*^{t\omega}(f)$ for (X, A, B) is trivial.*

Proof. First note that (X, B) is an H-pair. Then, by the Theorem 3.3,

$$\pi_n(X, A) = G_n^{Rel}(f_{XA}), \pi_n(X, B) = G_n^{Rel}(f_{XB}) \text{ and } \pi_n(A, B) = G_n^{Rel}(f_{AB}).$$

Thus the $G(f)$-sequence of a CW-triple (X, A, B) is just the homotopy sequence

$$\cdots \longrightarrow \pi_n(A, B) \xrightarrow{i_*} \pi_n(X, B) \xrightarrow{j_*} \pi_n(X, A) \xrightarrow{\partial_*} \pi_{n-1}(A, B) \longrightarrow \cdots$$

which is exact and the ω-homology $H_*^{t\omega}(f)$ of a CW-triple (X, A, B) is trivial. $\square$

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