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A WEAK CONVERGENCE IN A σ-COMPLETE ABSTRACT M SPACE

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ABSTRACT. In this note we give a characterization on weak convergence of bounded linear functionals in σ-complete abstract M spaces.

1. Introduction

There are many cases when abstract Banach lattices can be represented as concrete lattices of functions. Such representation theorems are very useful e.g., they facilitate the application of many results of measure theory to the study of Banach lattices. The well known theorems in this direction are those of S. Kakutani on the concrete representation of the so-called abstract L_p and M spaces.

Motivated from the Rainwater Theorem [1], we are encouraged to investigate some properties of bounded linear functionals on σ-complete abstract M spaces.

We present first some basic facts which we shall use through this note and then give the main results concerning weak convergence in bounded linear functionals on M spaces.

A Banach lattice (BL, for short) X for which $\|x+y\| = \max(\|x\|, \|y\|)$, whenever $x, y \in X$ and $x \wedge y = 0$, is called an *abstract M space.* If X is an abstract M space, then $\|x \vee y\| = \max(\|x\|, \|y\|)$ whenever $x, y \geq 0$. A BL X is called an *abstract L_1 space* if $\|x + y\| = \|x\| + \|y\|$ whenever $x, y \in X$ and $x \wedge y = 0$.

The dual X^* of a BL X is also a BL provided that its positive cone is defined by $x^* \geq 0$ in X^* if and only if $x^*(x) \geq 0$, for every $x \geq 0$ in X. For any $x^*, y^* \in X^*$ and every $x \geq 0$ in X, we have

$$(x^* \vee y^*)(x) = \sup\{x^*(u) + y^*(x-u) : 0 \leq u \leq x\}$$

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and

$$(x^* \wedge y^*)(x) = \inf\{x^*(u) + y^*(x-u) : 0 \le u \le x\}.$$

For a BL X, if X is an abstract M space, then X^* is an abstract L_1 space and if X is an abstract L_1 space, then X^* is an abstract M space (cf. [5]). Also, if X is a BL, then X^* is a space of regular functionals [2]. Obviously, for a BL X and $x^* \in X^*$, we have [3]

$$x^*(x) = \sup\{|x^*(y)| : |y| \le x\}.$$

Every BL X has the so-called decomposition property: if x_1, x_2 and y are positive elements in X and $y \le x_1 + x_2$ then there are $0 \le y_1 \le x_1$ and $0 \le y_2 \le x_2$ such that $y = y_1 + y_2$. Since every $x^* \in X^*$ can be decomposed as a difference of two non-negative elements, it follows that every norm bounded monotone sequence $\{x_n\}_{n=1}^{\infty}$ in X is weak Cauchy. If, in addition, $x_n \to x$ weakly for some $x \in X$ then $||x_n - x|| \to 0$ as $n \to \infty$. This is a consequence of the fact that weak convergence to x implies the existence of convex combinations of the x_n's which tend strongly to x.

For a subset Y of a BL X, we define

$$Y^{\perp} = \{x \in X : |x| \wedge |y| = 0 \ \text{ for } \ y \in Y\}, \ x^{\perp} = \{x\}^{\perp}.$$

If $x \in X = Y + Y^{\perp}$, then x has a unique decomposition $x = y + z$ with $y \in Y$, $z \in Y^{\perp}$. In such a case, we write $x|_Y = y$ and $x^*|_Y(x) = x^*(y)$ for $x^* \in X^*$.

For a Banach space X, we always denote by $B(X)$ and $S(X)$ the unit ball and the unit sphere of X respectively. The set of all extreme points of $B(X)$ is denoted by $\partial B(X)$.

For more detail about Banach lattices, also see Lindenstrauss-Tzafriri [5] and Yosida [8].

2. Main Results

Definition 1. A BL X is said to be *σ-complete* if every order bounded set (sequence) $\{x_n\}$ in X has a $\sup\{x_\alpha\}_{\alpha \in A}$, and a BL X is said to be *bounded σ-complete*, provided that any norm bounded and order monotone sequence in X is order convergent.

Obviously, every bounded σ-complete BL is σ-complete, but the converse is not true in general. For example, consider c_0. c_0 is σ-complete but not bounded σ-complete. Moreover, the space $C(K)$ of all continuous functions on a compact Hausdorff topological space K is σ-complete if and only if K is basically disconnected [5].

Definition 2. A BL X is said to be σ-order continuous if, for every downward directed set (sequence) $\{x_\alpha\}_{\alpha\in A}$ in X with $\bigwedge_{\alpha\in A} x_\alpha = 0$, $\lim_\alpha ||x_\alpha|| = 0$.

A simple example of a σ -order continuous BL, which is not order continuous, is the subspace of $l_\infty(\Gamma)$ spanned by $c_0(\Gamma)$ and the function identically equal to one, where Γ is an uncountable set. Typical examples of order complete BL, which are not σ-order continuous, are l_∞ and $L_\infty(0,1)$ [5].

Lemma 3 [4]. *Let an abstract M space X be σ-complete and $x^* \in S(X^*)$. Then $x^* \in \partial B(X^*)$ if and only if $x^*(x)x^*(y) = 0$ for all $x, y \in X$ such that $x \wedge y = 0$.*

Lemma 4. *Let X be a σ-complete lattice. Then for any $x_1, x_2, \cdots, x_m \in X$, X can be decomposed into m many pairwise orthogonal subspaces, i.e., $X = Y_1 + Y_2 + \cdots Y_m$ satisfying $(x_n - \bigwedge_{1\leq m}(x_i))|_{Y_n} = 0,\ 1 \leq n \leq m$.*

Proof. Since for any $x, y, z \in X$, $(x-z)\wedge(y-z) = x\wedge y - z$ [7], putting $z = x \wedge y$, we have

$$(x - x\wedge y)\perp(y - x\wedge y). \qquad (*)$$

Put $x' = \bigwedge_{1\leq n\leq m}(x_n)$ and $Y_1 = (x_1 - x')^\perp$. Then we have $X = Y_1 + Y_1^\perp$ [5]. Letting $x = x_1,\ y = \bigwedge_{2\leq n\leq m}(x_n)$ in $(*)$, we have

$$(x_1 - x')|_{Y_1} = 0, \quad (\bigwedge_{2\leq n\leq m}(x_n) - x')|_{Y_1^\perp} = 0.$$

Put $Y_2 = \{x \in Y_1^\perp : x\perp(x_2 - x')|_{Y_1^\perp}\}$. Then we also obtain $Y_1^\perp = Y_2 + Y_2^\perp \cap Y_1^\perp$. Now replace x and y by $x_2|_{Y_1^\perp}$ and $\bigwedge_{3\leq n\leq m}(x_n)$, respectively, in $(*)$. We can have

$$(x_2 - x')|_{Y_2} = 0, \quad (\bigwedge_{3\leq n\leq m}(x_n) - x')|_{Y_2^\perp} = 0.$$

Inductively, we can have pairwise orthogonal subspaces

$$Y_1, Y_2, \cdots, Y_{m-1}, Y_m = Y_{m-1}^\perp \cap Y_{m-2}^\perp$$

of X such that $X = Y_1 + Y_2 + \cdots + Y_m$ and $(x_n - x')|_{Y_n} = 0$ for all $n \leq m$. □

We now state the main results concerning weak convergence and weak compactness in a BL X.

Theorem 5. *Let an abstract M space X be a σ-complete. Then $x_n \to 0$ weakly in X if and only if $\{x_n\}$ is bounded and $\lim_{m\to\infty} \| \bigwedge_{i\leq m}(|x_{n_i}|)\| = 0$ for all subsequences $\{x_{n_i}\}$ in $\{x_n\}$.*

Proof. Assume first that $x_n \to 0$ weakly in X. If the conclusion fails, then there exist a constant $\epsilon > 0$ and a subsequence of $\{x_n\}$, denoted by $\{x_n\}$, satisfying $\| \bigwedge_{n\leq m}(|x_n|)\| > 2\epsilon$ for all $m \geq 1$.

We first define $y_1^1 = x_1^+$ and $y_2^1 = x_1^-$. Suppose that $\{y_s^k : s \leq 2^k, k \leq m\}$ have already been defined. Then we put $y_{2s-1}^{m+1} = y_s^m \wedge x_{m+1}^+$ and $y_{2s}^{m+1} = y_s^m \wedge x_{m+1}^-$. Continuing by induction, we have $\{y_i^m\}$ satisfying $y_i^m \wedge y_j^m = 0$ for all $m \geq 1$ and all $i, j \leq 2^m$ with $i \neq j$, and moreover, for any $k \leq m$, we have either $x_k^+ \wedge y_s^m = 0$ or $x_k^- \wedge y_s^m = 0$ for every $s = 1, 2, \cdots, 2^m$.

Hence, if we pick $j \leq 2^m$ such that $z_m = y_j^m$ satisfies $\|z_m\| = \max_{j\leq 2^m} \|y_j^m\|$, then

$$\|z_m\| = \| \sum_{i\leq 2^m} y_i^m \| = \| \bigwedge_{n\leq 2^m} (x_n)\| > 2\epsilon.$$

Now, we pick $x_m^* \in S(X^*)$ such that $x_m^*(z_m) = \|z_m\|$. Since $z_m \geq 0$ and X^* is an abstract L_1 space, we have $x_m^* \geq 0$. By [1], $\{x_m^*\}$ has a w^*-cluster $x^* \in B(X^*)$. For all fixed $n \geq 1$, we can find some $m \geq n$ such that $|x^*(x_n) - x_m^*(x_n)| < \epsilon$. Let $Z_m = z_m^\perp$ and $Y_m = Z_m^\perp$. Then $X = Y_m + Z_m$ by [5]. Consequently, $\|x_m^*\| = \|x_m^*|_{Y_m}\| + \|x_m^*|_{Z_m}\|$ by [4], and moreover, from the fact

$$1 \geq \|x_m^*|_{Y_m}\| \geq x_m^*(\frac{z_m}{\|z_m\|}) = 1,$$

we have $\|x_m^*|_{Z_m}\| = 0$. By the choice of z_m, $m \geq n$ implies that $x_n^+ \wedge z_m = 0$ or $x_n^- \wedge z_m = 0$, hence we may assume $x_n^+ \wedge z_m = 0$. Thus, $x_n^-|_{Y_m} \geq z_m|_{Y_m}$, and so

$$\begin{aligned} |x^*(x_n)| &\geq |x_m^*(x_n)| - |x^*(x_n) - x_m^*(x_n)| \\ &> |x_m^*(x_n)| - \epsilon = |x_m^*|_{Y_m}(x_n)| - \epsilon \\ &\geq x_m^*|_{Y_m}(z_m) - \epsilon = x_m^*(z_m) - \epsilon \\ &= \|z_m\| - \epsilon > \epsilon, \end{aligned}$$

which contradicts the hypothesis that x_n converges weakly to 0. Conversely, suppose that x_n does not converge weakly to 0 in X. Then by the Rainwater Theorem [1], there exist some $x^* \in \partial B(X^*)$, $\epsilon > 0$, and a subsequence of $\{x_n\}$, again denoted by $\{x_n\}$, with $x^*(x_n) > \epsilon$ for all $n \geq 1$. Since by Lemma 3, it follows that ${x^*}^+ = 0$ or

${x^*}^- = 0$ and $x^*(x_n) = {x^*}^+(x_n^+) + {x^*}^-(x_n^-) - {x^*}^-(x_n^+) - {x^*}^+(x_n^-)$. Hence, without loss of generality, we may assume $x^* \geq 0$ and $x_n \geq 0$ for all $n \geq 1$.

Now, we choose $m \geq 1$ such that $\| \bigwedge_{n \leq m}(x_n)\| < \epsilon$. Then by Lemma 4, X can be decomposed into the direct sum of pairwise orthogonal subspaces $Y_1, Y_2, \cdots, Y_m$ such that $x_n|_{Y_n} = x'|_{Y_n}$ for all $n \leq m$, where $x' = \bigwedge_{n \leq m}(x_n)$. Thus by Lemma 3, there exists some $n \leq m$ such that $x^* = x^*|_{Y_n}$ which leads to a contradiction that $\epsilon < x^*(x_n) = x^*(x_n|_{Y_n}) = x^*(x'|_{Y_n}) \leq \|x^*\|\|x'\| < \epsilon$. □

Theorem 6. *Let an abstract M space X be a dual σ-complete. Then a bounded subset W of X is weakly compact if and only if*

$$\sup \lim_{m \to \infty} \inf \| \bigwedge_{n \leq m} (|x_n - x|)\| = 0 \ \textit{for} \ \ \{x_n\} \subset W, \ x \in K$$

where $K = K(x_n)$ is the set of sequentially w^*-clusters of $\{x_n\}$ and, as usual, we denote

$$\inf\{a : a \in Y\} = +\infty \ \ \textit{for} \ \ Y = \emptyset.$$

Proof. Suppose first that W is a weakly compact subset of X. Then for any sequence $\{x_n\}$ in W we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ weakly convergent to some point x in X, and then obviously $x \in K = K(x_n)$. Hence, by Theorem 5, we have

$$\begin{aligned} 0 &= \lim_{m \to \infty} \| \bigwedge_{i \leq m} (|x_{n_i} - x|)\| \\ &\geq \lim_{m \to \infty} \| \bigwedge_{n \leq m} (|x_n - x|)\| \\ &\geq \lim_{m \to \infty} \inf \| \bigwedge_{n \leq m} (|x_n - y|)\| \geq 0 \end{aligned}$$

for each $y \in K$.

Conversely, assume now the given condition holds. Then for any sequence $\{x_n\}$ in W, $K = K(x_n) \neq 0$ and hence, $\{x_n\}$ contains a subsequence, denoted by $\{x_n\}$, w^*-convergent to some point $x \in K$. Therefore, for any subsequence $\{x_{n_i}\}$ of this subsequence, it follows that $K' = K(x_{n_i}) = \{x\}$ implies

$$\lim_{m \to \infty} \| \bigwedge_{i \leq m} (|x_{n_i} - x|)\| = \lim_{m \to \infty} \inf \| \bigwedge_{i \leq m} (|x_n - y|)\| = 0, \ y \in K'.$$

Thus, by Theorem 5, x_n converges weakly to x. □

Remark. Substituting X with L_∞ or l_∞ in Theorem 6. we can obtain assertion of weak compactness for those spaces without any difficulties [8].

REFERENCES

1. J. Diestel, *Sequences and Series in Banach Spaces*, Springer-Verlag, Berlin and New York, 1984.
2. L. V. Kantorvich and G. P. Akilov, *Functional Analysis*, Pergamon Press, Oxford, 1982.
3. H. E. Lacey, *The Isometric Theory of Classical Banach Spaces*, Springer-Verlag, Berlin and New York, 1974.
4. J. N. Lee, *Extremal Structure of* $B(X^*)$, J. Korea Soc. Math. Educ. Ser. B Pure Appl. Math. **5** (1998), 95-100.
5. J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces II*, Springer-Verlag, Berlin and New York, 1979.
6. W. A. J. Luxemburg and A. C. Zaanen, *Riesz Spaces I*, North-Holland, Amsterdam and London, 1971.
7. A. Wilansky, *Modern Methods in Topological Vector Spaces*, McGraw-Hill, New York, 1987.
8. K. Yosida, *Functional Analysis*, Springer-Verlag, Berlin and New York, 1980.
9. A. C. Zaanen, *Riesz Spaces II*, North-Holland, Amsterdam, New York and Oxford, 1983.

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