J. Korea Soc. Math. Educ. Ser. B: Pure Appl. Math.
6(1999), no. 1, 27–32

VARIATIONAL-TYPE INEQUALITIES FOR SET-VALUED MAPPINGS ON NORMED LINEAR SPACES

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ABSTRACT. In this paper, we show that the existence of the solutions to the variational-type inequalities for set-valued mappings on normed linear spaces using Fan's section theorem.

1. Introduction and preliminaries

Variational inequalities introduced by Hartman and Stampacchia [5] have been extended and generalized in various directions as a powerful tool of current mathematical technology. Recently, Behera and Panda [3] introduced variational-type inequalities for single-valued mappings in Hausdorff topological vector spaces.

In this paper, we extend the existence theorem for variational-type inequalities in [3] to set-valued case. In the proof of our main theorem, we use Fan's section theorem [4], which has been applied to variational inequality problems, complementary problems, game theory, and so on.

First we introduce the following theorem.

Theorem 1.1 (Fan's Section Theorem). *Let K be a nonempty compact convex subset of a Hausdorff topological vector space X. Let A be a subset of $K \times K$ satisfying the following conditions;*

(1) *for each $x \in K$, $(x,x) \in A$,*
(2) *for each fixed $x \in K$, the set $A_x = \{y \in K : (x,y) \in A\}$ is closed in K, and*
(3) *for each fixed $y \in K$, the set $A^y = \{x \in K : (x,y) \notin A\}$ is convex in K.*

Then there exists an $x_0 \in K$ such that $K \times \{x_0\} \subset A$.

Received by the editors December 30, 1998.
1991 *Mathematics Subject Classification.* 49J40.
Key words and phrases. Fan's section theorem, set-valued mapping, variational-type inequality.

Definition 1.1 [2]. Let X and Y be two topological vector spaces and $T : X \to 2^Y$ be a set-valued mapping. T is said to be *upper semicontinuous* (briefly, u.s.c.) at $x_0 \in X$ if for any open neighbourhood N containing $T(x_0)$ there exists a neighbourhood M of x_0 such that $T(M) \subset N$. T is said to be u.s.c. if T is u.s.c. at every point $x \in X$.

Definition 1.2 [6]. Let X and Y be two topological vector spaces and $T : X \to 2^Y$ be a set-valued mapping. T is said to be *closed* at $x \in X$ if for each nets $\{x_\alpha\}$ converging to x and $\{y_\alpha\}$ converging to y such that $y_\alpha \in T(x_\alpha)$ for all α, we have $y \in T(x)$. T is said to be closed if it is closed at every point $x \in X$.

Lemma 1.2 [1]. *Let X and Y be two topological vector spaces and $T : X \to 2^Y$ be a set-valued mapping. The followings hold.*

(1) *If K is a compact subset of X, and T is u.s.c. and compact-valued, then $T(K)$ is compact.*

(2) *If T is u.s.c. and compact-valued, then T is closed.*

Throughout this paper, we denote by $\langle y, x\rangle$ the duality mapping between elements $y \in X^*$ and $x \in X$.

2. Main Results

The following theorem is our main result.

Theorem 2.1. *Let K be a nonempty compact convex subset of a normed linear space X. Assume that $T : K \to 2^{X^*}$ is u.s.c. and compact-valued, $\theta : K \times K \to X$ is a bounded mapping, and $\eta : K \times K \to \mathbb{R}$ is a mapping satisfying the following conditions;*

(1) *for each $x \in K$, there exists $t \in T(x)$ such that $\langle t, \theta(x,x)\rangle + \eta(x,x) = 0$,*

(2) *the mapping*

$$x \longmapsto \langle t, \theta(x,y)\rangle + \eta(y,x)$$

of K into $\mathbb{R}$ is convex for all $y \in K$ and for all $t \in T(y)$,

(3) *for each $x \in K$, the mappings $y \mapsto \theta(x,y)$ and $y \mapsto \eta(y,x)$ are continuous.*

Then there exists an $x_0 \in K$ and $t_0 \in T(x_0)$ such that for any $y \in K$

$$\langle t_0, \theta(y,x_0)\rangle + \eta(x_0,y) \geq 0.$$

Proof. Let

$$A := \{(x,y) \in K \times K : \text{there exists } t \in T(y) \text{ such that } \langle t, \theta(x,y)\rangle + \eta(y,x) \geq 0\},$$

then it is easily shown that $(x,x) \in A$. For each fixed $x \in K$,

$$\begin{aligned} A_x &:= \{y \in K : (x,y) \in A\} \\ &= \{y \in K : \text{there exists } t \in T(y) \text{ such that } \langle t, \theta(x,y)\rangle + \eta(y,x) \geq 0\} \end{aligned}$$

is closed. Indeed, if we let $\{y_\lambda\}$ be a net in A_x such that $y_\lambda \to y_0$ then, since $y_\lambda \in A_x$, there exists $t_\lambda \in T(y_\lambda)$ such that $\langle t_\lambda, \theta(x,y_\lambda)\rangle + \eta(y_\lambda, x) \geq 0$.

Since $T(K)$ is compact, by (1) of Lemma 1.2, there exists $t_0 \in T(K)$ such that $t_\lambda \to t_0$. Since T is closed by (2) of Lemma 1.2, $t_0 \in T(y_0)$. By condition (3), we have

$$\begin{aligned} &\big|\langle t_\lambda, \theta(x,y_\lambda)\rangle + \eta(y_\lambda,x) - \big(\langle t_0, \theta(x,y_0)\rangle + \eta(y_0,x)\big)\big| \\ &\quad \leq \big|\langle t_\lambda, \theta(x,y_\lambda)\rangle - \langle t_0, \theta(x,y_0)\rangle\big| + \big|\eta(y_\lambda,x) - \eta(y_0,x)\big| \\ &\quad \leq \big|\langle t_\lambda - t_0, \theta(x,y_\lambda)\rangle\big| + \big|\langle t_0, \theta(x,y_\lambda) - \theta(x,y_0)\rangle\big| + \big|\eta(y_\lambda,x) - \eta(y_0,x)\big| \\ &\quad \leq \big\|t_\lambda - t_0\big\|\big\|\theta(x,y_\lambda)\big\| + \big\|t_0\big\|\big\|\theta(x,y_\lambda) - \theta(x,y_0)\big\| + \big|\eta(y_\lambda,x) - \eta(y_0,x)\big| \\ &\quad \to 0 \text{ as } \lambda \to \infty. \end{aligned}$$

Consequently, there exists $t_0 \in T(y_0)$ such that $\langle t_0, \theta(x,y_0)\rangle + \eta(y_0,x) \geq 0$.

Hence $y_0 \in A_x$ and A_x is closed.

On the other hand, for each fixed $y \in K$,

$$\begin{aligned} A^y &:= \{x \in K : (x,y) \notin A\} \\ &= \{x \in K : \text{for all } t \in T(y), \langle t, \theta(x,y)\rangle + \eta(y,x) < 0\} \end{aligned}$$

is convex. In fact, let $x_1, x_2 \in A^y, \alpha \in (0,1)$ and $z = \alpha x_1 + (1-\alpha)x_2$, then for all $t \in T(y)$,

$$\begin{aligned} &\langle t, \theta(z,y)\rangle + \eta(y,z) \\ &\quad = \langle t, \theta(\alpha x_1 + (1-\alpha)x_2, y)\rangle + \eta(y, \alpha x_1 + (1-\alpha)x_2) \\ &\quad \leq \alpha\big[\langle t, \theta(x_1,y)\rangle + \eta(y,x_1)\big] + (1-\alpha)\big[\langle t, \theta(x_2,y)\rangle + \eta(y,x_2)\big] \\ &\quad < 0, \end{aligned}$$

hence $z \in A^y$. By Theorem 1.1, there exists an $x_0 \in K$ such that $K \times \{x_0\} \subset A$. This implies that there exists an $x_0 \in K$ such that for all $y \in K$ there exists $t_0 \in T(x_0)$ such that $\langle t_0, \theta(y, x_0)\rangle + \eta(x_0, y) \geq 0$.

Remark 2.2. Applying Theorem 2.1 in [3] to normed linear spaces, we obtain a special case of Theorem 2.1.

In Theorem 2.1, we considered K is a nonempty compact convex subset of a normed linear space X. But in the following theorem, we don't assume that K is compact.

Lemma 2.3 [7]. *The convex hull of a finite family of compact, convex subsets of a Hausdorff topological vector space is compact.*

Theorem 2.4. *Let K be a nonempty convex subset of a normed linear space X. Assume that $T : K \to 2^{X^*}$ is u.s.c. and compact-valued, $\theta : K \times K \to X$ is a bounded mapping, and $\eta : K \times K \to \mathbb{R}$ is a mapping satisfying the following conditions;*

(1) *for each $x \in K$, there exists $t \in T(x)$ such that $\langle t, \theta(x, x)\rangle + \eta(x, x) = 0$,*
(2) *the mapping*

$$x \longmapsto \langle t, \theta(x, y)\rangle + \eta(y, x)$$

of K into $\mathbb{R}$ is convex for all $y \in K$ and for all $t \in T(y)$,
(3) *for each $x \in K$, the mappings $y \mapsto \theta(x, y)$ and $y \mapsto \eta(y, x)$ are continuous, and*
(4) *there exists a nonempty compact convex subset D of K and $u \in D$ such that for all $x \in K \setminus D$ there exists $t \in T(x)$ such that*

$$\langle t, \theta(u, x)\rangle + \eta(x, u) < 0.$$

Then there exists an $x_0 \in D$ and $t_0 \in T(x_0)$ such that for any $y \in K$

$$\langle t_0, \theta(y, x_0)\rangle + \eta(x_0, y) \geq 0. \tag{2.1}$$

Proof. For each $x \in K$, let

$$B_x := \{y \in D : \text{there exists } t \in T(y) \text{ such that } \langle t, \theta(x, y)\rangle + \eta(y, x) \geq 0\},$$

then it is easily shown that B_x is nonempty. And for each $x \in K$, let

$$C_x := \{y \in K : \text{there exists } t \in T(y) \text{ such that } \langle t, \theta(x, y)\rangle + \eta(y, x) \geq 0\},$$

then we can show that C_x is closed with the same method in the proof of Theorem 2.1. Since D is closed in X, $B_x = D \cap C_x$ is a closed subset of D. It is clear that (2.1) has a solution if $\bigcap_{x \in K} D_x \neq \emptyset$. For this, it is sufficient to prove that the family $\{B_x : x \in K\}$ has the finite intersection property. Let $x_1, x_2, \cdots, x_n$ be arbitrary finite elements of K and let $D_h = co(D \cup \{x_1, x_2, \cdots, x_n\})$, where co denotes the convex hull. Then D_h is a compact convex subset of K by Lemma 2.3. By Theorem 2.1, there exists an $x_0 \in D_h$ such that for all $y \in D_h$ there exists $t_0 \in T(x_0)$ such that

$$\big\langle t_0, \theta(y, x_0)\big\rangle + \eta(x_0, y) \geq 0. \tag{2.2}$$

It can be shown that $x_0 \in D$. In fact, if $x_0 \notin D$, then by (4), there exists a $u \in D$ such that for such $x_0 \in K \setminus D$ there exists $t_0 \in T(x_0)$

$$\big\langle t_0, \theta(u, x_0)\big\rangle + \eta(x_0, u) < 0,$$

which is a contradiction to (2,2), when $u = y$. Thus $x_0 \in D$. In particular, $x_0 \in C_{x_i}$ for all x_i. In fact, if $x_0 \notin C_{x_i}$ for some x_i, then for all $t \in T(x_0)$,

$$\big\langle t, \theta(x_i, x_0)\big\rangle + \eta(x_0, x_i) < 0. \tag{2.3}$$

But since $x_i \in D_h$, from Theorem 2.1, we can choose $w \in T(x_0)$ such that

$$\big\langle w, \theta(x_i, x_0)\big\rangle + \eta(x_0, x_i) \geq 0,$$

which contradict (2.3). Hence $x_0 \in B_{x_i}$ for $i = 1, 2, \cdots, n$. Therefore

$$\bigcap_{i=1}^{n} B_{x_i} \neq \emptyset.$$

Hence the family $\{B_x : x \in K\}$ has the finite intersection property, so there exists $y \in D$ such that for each $x \in K$ there exists $t \in T(y)$ such that

$$\big\langle t, \theta(x, y)\big\rangle + \eta(y, x) \geq 0.$$

Consequently, there exists an $x_0 \in D$ such that for all $y \in K$ there exists $t_0 \in T(x_0)$ such that

$$\big\langle t_0, \theta(y, x_0)\big\rangle + \eta(x_0, y) \geq 0.$$

Remark 2.5. Applying Theorem 2.2 in [5] to normed linear spaces, we obtain a special case of Theorem 2.4.

Acknowledgment.

The author thanks to the referee for his/her valuable comments.

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