J. Korea Soc. Math. Educ. Ser. B: Pure Appl. Math.
6(1999), no. 1, 17–25

A NOTE ON THE INTEGRATION WITH RESPECT TO FINITELY ADDITIVE SET FUNCTIONS

Bong Jin Kim

Abstract. In this paper, we investigate the properties of the Dunford-Schwartz integral (the integral with respect to a finitely additive measure). Though it is not equivalent to the cylinder integral, we can show that a cylinder probability ν on $(H, \mathcal{C})$ can be extend as a finitely additive probability measure $\hat{\nu}$ on a field $\hat{\mathcal{C}} \supset \mathcal{C}$ which is equivalent to the Dunford-Schwartz integral on $(H, \hat{\mathcal{C}}, \hat{\nu})$.

1. Introduction

The theory of integration with respect to finitely additive measures, namely the Dunford-Schwartz integral, was studied for a long time by several mathematicans (cf. [1], [2]). Indeed, if we let H be an infinite dimensional real separable Hilbert space and let $\mathcal{P} = \mathcal{P}(H)$ be the class of orthogonal projections on H with finite dimensional range. Then for $P \in \mathcal{P}$,

$$\mathcal{C}_P := \{P^{-1}B : B \text{ is a Borel set in the Range of } P\}$$

is a σ-field . And the sets in $\mathcal{C}_P$ are called *cylinder sets* with base P.

Let $\mathcal{C} = \cup \mathcal{C}_P$. Then $\mathcal{C}$ is a field but is not a σ-field .

Definition 1.1. A *cylinder probability* ν on H is a finitely additive nonnegative set function on $\mathcal{C}$ with $\nu(H) = 1$ such that for all P in $\mathcal{P}(H)$, the restriction ν_P of ν to $\mathcal{C}_P$ is countably additive.

Received by the editors December 19, 1998.
1991 *Mathematics Subject Classification.* 28C20.
Key words and phrases. cylinder probability, cylinder integral, finitely additive set function, Dunford-Schwartz integral.
This work was supported by the Daejin University Research Grants in 1998.

We call the integral with respect to a cylinder probability ν on $(H, \mathcal{C})$. as a *cylinder integral* for convenience. The cylinder integral with respect to ν is the same as the one defined in Kallianpur and Karadikar [3].

We consider the Dunford-Schwartz integral (the integral with respect to a finitely additive measure) and investigate the properties of the integral. Though it is not equivalent to the cylinder integral defined above, we can show that a cylinder probability ν on $(H, \mathcal{C})$ can be extend as a finitely additive probability measure $\hat{\nu}$ on a field $\hat{\mathcal{C}} \supset \mathcal{C}$ which is equivalent to the Dunford-Schwartz integral on $(H, \hat{\mathcal{C}}, \hat{\nu})$. And throughout this paper, we shall restrict our attention to the finite measures. Without loss of generality, we can assume that all finitely additive measures are finitely additive probability measures.

2. Integration with respect to a finitely additive measures

Let X be a nonempty set, $\mathcal{S}$ be a field of subsets of X and let μ be a finitely additive probability measure on $(X, \mathcal{S})$.

For any subset A of X, let $\mu^*(A)$ and $\mu_*(A)$ be defined by

$$\mu^*(A) = \inf\{\mu(B) : B \supseteq A, B \in \mathcal{S}\}$$

and

$$\mu_*(A) = \sup\{\mu(C) : C \subseteq A, C \in \mathcal{S}\}.$$

It is easy to see that $\mu_*(A) = 1 - \mu^*(A^c)$, $A \subseteq X$.

Definition 2.1. A sequence of real-valued functions $\{f_k\}$ on X is *converges in μ-probability* to a function f, written as $f_k \underset{\mu}{\longrightarrow} f$, if for every $\epsilon > 0$,

$$\lim_{k \to \infty} \mu^*(\{x : |f_k(x) - f(x)| > \epsilon\}) = 0.$$

Definition 2.2. A function f of the form

$$f(x) = \sum_{j=1}^{n} a_j \chi_{A_j}(x), \tag{2.1}$$

where $a_j \in \mathbb{R}$, $A_j \in \mathcal{S}$, $k \geq 1$, is called a *$\mathcal{S}$-simple function.*

Let $l^0(X, \mathcal{S}, \mu)$ be the class of all real-valued functions f on X such that there exists a sequence of $\mathcal{S}$-simple functions $\{f_k\}$ such that $f_k \underset{\mu}{\longrightarrow} f$.

Theorem 2.3. *Let $k \geq 1$, $f_1, f_2, \cdots, f_k \in l^0(X, \mathcal{S}, \mu)$ and let $g : \mathbb{R}^k \to \mathbb{R}$ be a continuous function. Then*

$$g(f_1, \cdots, f_k) \in l^0(X, \mathcal{S}, \mu).$$

Proof. Let $\epsilon > 0$ and $\delta > 0$ be given and let $N_1 \in \mathbb{N}$ be such that $n_i \geqslant N_1$ with $n_i \in \mathbb{N}$ implies

$$\mu^*(\{x \in X : |f_i(x) - f_{i,n_i}(x)| > 1\}) < \frac{\epsilon}{8k}, \quad 1 \leqslant i \leqslant k,$$

where f_{i,n_i} be a $\mathcal{S}$-simple functions. Let M be such that

$$\mu^*(\{x \in X : |f_{i,n_i}(x)| > M - 1\}) < \frac{\epsilon}{8k}, \quad 1 \leqslant i \leqslant k.$$

Then

$$\begin{aligned}
&\mu^*(\{x \in X : |f_{i,n_i}(x)| > M, \text{ for some } i,\ 1 \leqslant i \leqslant k\} \\
&= \mu^*(\bigcup_{i=1}^{k}\{x \in X : |f_{i,n_i}(x)| > M\}) \\
&\leqslant \sum_{i=1}^{k} \mu^*(\{x \in X : |f_{i,n_i}(x)| > M\}) \\
&\leqslant \sum_{i=1}^{k} \mu^*(\{x \in X : |f_{i,n_i}(x)| > |f_i(x)| + 1\} \\
&\quad + \sum_{i=1}^{k} \mu^*(\{x \in X : |f_i(x)| > M - 1\}) \\
&\leqslant \sum_{i=1}^{k} \mu^*(\{x \in X : |f_{i,n_i}(x) - f_i(x)| > 1\}) \\
&\quad + \sum_{i=1}^{k} \mu^*(\{x \in X : |f_i(x)| > M - 1\}) < \frac{\epsilon}{8} + \frac{\epsilon}{8} = \frac{\epsilon}{4}.
\end{aligned} \tag{2.2}$$

And let $K = \{\vec{x} = (x_1, \cdots, x_k) \in \mathbb{R}^k : |x_i| \leqslant M, \quad 1 \leqslant i \leqslant k\}$. Then K is compact and hence g is uniformly continuous on K.

Thus there exists $\delta_1 > 0$ such that

$$\vec{x} = (x_1, \cdots, x_k) \in \mathbb{R}^k, \quad \vec{x'} = (x'_1, \cdots, x'_k) \in \mathbb{R}^k, \quad |x_i - x'_i| < \delta_1, \quad 1 \leqslant i \leqslant k$$

implies

$$g(\vec{x}) - g(\vec{x'})| < \delta.$$

Thus

$$\begin{aligned} &\{x \in X : |g \circ (f_1, \cdots, f_k)(x) - g \circ (f_{1,n_1}, \cdots, f_{k,n_k})(x)| > \delta\} \\ &\quad \subseteq \{x \in X : (f_1, \cdots, f_k)(x) \notin K\} \cup \{x \in X : (f_{1,n_1}, \cdots, f_{k,n_k})(x) \notin K\} \\ &\quad \cup (\bigcup_{i=1}^{k} \{x \in X : |(f_i - f_{i,n_i})(x)| > \delta_1\}). \end{aligned} \tag{2.3}$$

Now, let $N_2 \in \mathbb{N}$ be such that $n_i \geqslant N_2$ implies

$$\mu^*\{x \in X : |f_i(x) - f_{i,n_i}(x)| > \delta_1\} < \frac{\epsilon}{2k}. \tag{2.4}$$

Thus, from (2.2), (2.3) and (2.4), we have, for $n_i \geqslant \max(N_1, N_2)$, $1 \leqslant i \leqslant k$,

$$\begin{aligned} &\mu^*(\{x \in X : |g \circ (f_1, \cdots, f_k)(x) - g \circ (f_{1,n_1}, \cdots, f_{k,n_k})(x)| > \delta\}) \\ &\quad \leqslant \mu^*(\{x \in X : (f_1, \cdots, f_k)(x) \notin K\}) \\ &\quad\quad + \mu^*(\{x \in X : (f_{1,n_1}, \cdots, f_{k,n_k})(x) \notin K\}) \\ &\quad\quad + \sum_{i=1}^{k} \mu^*(\{x \in X : |(f_i - f_{i,n_i})(x)| > \delta_1)\}) \\ &\quad < \frac{\epsilon}{4} + \frac{\epsilon}{4} + \sum_{i=1}^{k} \frac{\epsilon}{2k} = \epsilon. \end{aligned}$$

Thus $g(f_1, \cdots, f_k) \in l^0(X, \mathcal{S}, \mu)$. □

Clearly, $l^0(X, \mathcal{S}, \mu)$ is a vector space of functions.

For a simple function $f : X \to \mathbb{R}$ given by (2.1), we define its *Dunford-Schwartz integral with respect to* μ by

$$\int^* f(x) d\mu := \sum_{j=1}^{n} a_j \mu(A_j).$$

Remark 1. If f be a $\mathcal{S}$-simple function, $a \in \mathbb{R}^+$ and $|f| \leqslant M$, then we have $\int^* f d\mu \leqslant a + M\mu(\{x : |f(x)| \geqslant a\})$. Thus for $f \in l^0(X, \mathcal{S}, \mu)$, $|f| \leqslant M$, there exists a sequence $\{f_k\}$ of $\mathcal{S}$-simple functions such that $f_k \underset{\mu}{\to} f$ furthermore $\{\int^* f_k d\mu\}$ is a Cauchy sequence of real numbers.

Definition 2.5. Let $f \in l^0(X, \mathcal{S}, \mu)$, $|f| \leqslant M$. Let $\{f_k\}$ be any sequence of $\mathcal{S}$ -simple functions such that $|f_k| \leqslant M$ and $f_k \underset{\mu}{\rightarrow} f$. Then we define $\int^* f d\mu$ by

$$\int^* f d\mu := \lim_{k\to\infty} \int^* f_k d\mu. \tag{2.6}$$

Let $f \in l^0(X, \mathcal{S}, \mu)$ be positive. Then we define

$$\int^* f d\mu := \lim_{k\to\infty} \int^* (f \wedge k) d\mu \tag{2.7}$$

and for f such that $\int^* |f| d\mu < \infty$,

$$\int^* f d\mu := \int^* f^+ d\mu - \int^* f^- d\mu \tag{2.8}$$

where $f^+ = f \vee 0$, $f^- = -(f \wedge 0)$.

By the above Remark 1, Definition 2.5 is well-defined, that is, the Dunford-Schwartz integral does not depends on the choice of $\{f_k\}$.

We now introduce completion of a finitely additive probability space $(X, \mathcal{S}, \mu)$. Let

$$\hat{\mathcal{S}} = \{A \subseteq X : \mu^*(A) = \mu_*(A)\} \tag{2.9}$$

and let $\bar{\mu} : \hat{\mathcal{S}} \rightarrow [0, 1]$ be defined by

$$\bar{\mu}(A) = \mu^*(A), \quad A \in \hat{\mathcal{S}}. \tag{2.10}$$

Then clearly, $\mathcal{S} \subseteq \hat{\mathcal{S}}$. And we can see that $\hat{\mathcal{S}}$ is a field and that $\bar{\mu}$ is a finitely additive measure on $(X, \hat{\mathcal{S}})$.

Now, for $A \in \hat{\mathcal{S}}$ with $\bar{\mu}(A) = 0$ and $B \subset A$,

$$1 = \mu_*(B) + \mu^*(B^c) \leqslant \mu_*(A) + \mu^*(B^c) = \mu^*(B^c) \leqslant 1$$

and

$$1 = \mu^*(B) + \mu_*(B^c) \geqslant \mu^*(B) + \mu_*(A^c) = \mu^*(B) + 1.$$

Thus $\mu^*(B) = \mu_*(B) = 0$. That is, $B \in \hat{\mathcal{S}}$ so that $(X, \hat{\mathcal{S}}, \bar{\mu})$ is complete.

$(X, \hat{\mathcal{S}}, \bar{\mu})$ is called the *completion* of $(X, \mathcal{S}, \mu)$.

We have the followings.

Lemma 2.6. $(\bar{\mu})^* = \mu^*$ *and* $(\bar{\mu})_* = \mu_*$.

Proof. For any $A \subseteq X$

$$\begin{aligned}(\bar{\mu})^*(A) &= \inf\{\bar{\mu}(B) : B \in \hat{\mathcal{S}}, B \supseteq A\} \\ &= \inf\{\mu^*(B) : B \in \hat{\mathcal{S}}, B \supseteq A\} \\ &\geq \mu^*(A)\end{aligned}$$

and

$$\begin{aligned}(\bar{\mu})^*(A) &= \inf\{\bar{\mu}(B) : B \in \hat{\mathcal{S}}, B \supseteq A\} \\ &\leq \inf\{\mu^*(B) : B \in \mathcal{S}, B \supseteq A\} = \mu^*(A).\end{aligned}$$

Thus $(\bar{\mu})^* = \mu^*$. Similarly $(\bar{\mu})_* = \mu_*$. □

Proposition 2.7. $l^0(X, \hat{\mathcal{S}}, \bar{\mu}) = l^0(X, \mathcal{S}, \mu)$.

Proof. If $f \in l^0(X, \mathcal{S}, \mu)$ and $\epsilon > 0$ be given, then there exists a sequence $\{f_k\}$ of $\mathcal{S}$-simple functions such that $f_k \underset{\mu}{\longrightarrow} f$. Since $\{f_k\}$ also a sequence of $\hat{\mathcal{S}}$-simple functions and by Lemma 2.6, we have, for $f_k \underset{\bar{\mu}}{\longrightarrow} f$

$$(\bar{\mu})^*(\{x : |f_k(x) - f(x)| > \epsilon\}) = \mu^*(\{x : |f_k(x) - f(x)| > \epsilon\}) \to 0$$

as $k \to \infty$. Thus $f \in l^0(X, \hat{\mathcal{S}}, \bar{\mu})$ and $\epsilon > 0$ be given, then there exists a sequence $\{f_k\}$ of $\hat{\mathcal{S}}$-simple functions, say, $f_k(x) = \sum_{j=1}^n a_{k_j}\chi_{E_{k_j}}$, where $E_{k_j} \in \hat{\mathcal{S}}$, such that $f_k \underset{\bar{\mu}}{\longrightarrow} f$. Then $E_{k_j} = \tilde{E}_{k_j} \cup N_j$, where $N_j \subset \tilde{N}_j \in \mathcal{N}$, the set of μ null sets.

Let $\tilde{f}_k(x) = \sum_{j=1}^n a_{k_j}\chi_{\tilde{E}_{k_j} \cap \tilde{N}_j}$. Then $\{\tilde{f}_k\}$ is a sequence of simple functions and

$$\begin{aligned}&(\bar{\mu})^*(\{x : |f_k(x) - f(x)| > \epsilon\}) \\ &= \mu^*(\{x : |f_k(x) - f(x)| > \epsilon\}) \\ &\geqslant \mu^*(\{x : |\tilde{f}_k(x) - f(x)| > \epsilon\}).\end{aligned}$$

Thus $\tilde{f}_k \underset{\mu}{\longrightarrow} f$, that is, $f \in l^0(X, \mathcal{S}, \mu)$. □

Proposition 2.8. *Let $(X, \hat{\mathcal{S}}, \bar{\mu})$ be a complete finitely additive probability space and let $f \in l^0(X, \hat{\mathcal{S}}, \bar{\mu})$, then there exists a countably additive probability measure λ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that*

$$\int g\, d\lambda = \int^* g(f)\, d\bar{\mu}$$

for all $g \in C_b(\mathbb{R})$, where $C_b(\mathbb{R})$ is the set of all bounded continuous functions.

Proof. Since $f \in l^0(X, \hat{\mathcal{S}}, \bar{\mu})$, given $\epsilon > 0$, we can choose a simple function f_0 such that $\bar{\mu}^*(\{x : |f(x) - f_0(x)| \geq 1\}) < \epsilon$. Thus

$$\bar{\mu}^*(\{x : |f(x)| \geq M\}) < \epsilon$$

where $M = \sup_{x \in X} |f_0(x)| + 1$.

By Theorem 2.3, $g(f) \in l^0(X, \hat{\mathcal{S}}, \bar{\mu})$ for $g \in C_b(\mathbb{R})$.

Let $T : C_b(\mathbb{R}) \to \mathbb{R}$ be defined by $T(g) = \int^* g(f) d\bar{\mu}$. Then T is a positive linear functional. Now, if $g_k \in C_b(\mathbb{R})$, $g_k \downarrow 0$, then by Daniell's theorem $g_k \to 0$ uniformly on $[-M, M]$. Let $k_0 \in \mathbb{N}$ such that $|g_k(x)| < \epsilon$ for $k \geq k_0$ and $|x| \leq M$. Hence

$$\bar{\mu}^*(\{x : |g_k(f(x))| > \epsilon\}) < \epsilon$$

since $\{x : |g_k(f(x))| > \epsilon\} \subseteq \{x : |f(x)| > M\}$ for $k \geq k_0$. Therefore $g_k(f) \underset{\bar{\mu}}{\to} 0$ and $T(g_k) \to 0$ as $k \to \infty$. Thus by Daniell's representation theorem (cf. [1]), there exists a countably additive measure λ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that $T(g) = \int g d\lambda$ for every $g \in C_b(\mathbb{R})$. □

Remark 2. (1) For $f \in l^0(X, \hat{\mathcal{S}}, \bar{\mu})$ and $n \in \mathbb{N}$, let $N_f^n := \{a \in \mathbb{R} : \lambda(\{a\}) > \frac{1}{n}\}$. Then N_f^n is a finite set for λ. Therefore $N_f := \{a \in \mathbb{R} : \lambda(\{a\}) > 0\} = \cup_n N_f^n$ is countable subset of $\mathbb{R}$. Thus for $a \notin N_f$, $\lambda(\{a\}) = 0$, and $\{x \in X : f(x) \leq a\} \in \hat{\mathcal{S}}$ by proposition 2.8. That is, if there exists a countable subset N_f of $\mathbb{R}$ such that for all $a \notin N_f$, $f^{-1}(-\infty, a] \in \hat{\mathcal{S}}$ (μ-completion of $\mathcal{S}$).

(2) If we let $X = H$ be a infinite dimensional separable real Hilbert space, and A be a self adjoint Hilbert Schmidt operator on H such that the range of A is infinite dimensional. Then we can see that (although not proved here) $f(h) = ||Ah||^2$ is cylinder integrable but is not Dunford-Schwartz integrable. That is, the cylinder integral and the Dunford-Schwartz integral is not equivalent. But we can also see that a cylinder probability ν on $(H, \mathcal{C})$ can be extened as a finitely additive probability measure $\hat{\nu}$ on a field $\hat{\mathcal{C}} \supset \mathcal{C}$ such that for $f \in l^0(H, \hat{\mathcal{C}}, \hat{\nu})$ such that

$$\int^* f d\hat{\nu} = \int f d\nu$$

where $\int f d\nu$ is the cylinder integral as in [3].

Thus we can carry out several problems that we deal with not only the cylinder probability ν on $(H, \mathcal{C})$ but also the extension $\hat{\nu}$ on $(H, \hat{\mathcal{C}})$ of the finitely additive measure ν on $(H, \mathcal{C})$.

(3) In classical measure theory (cf. [3]), if $(X, \mathcal{S}, \mu)$ is a measure space and f is a nonnegative integrable function, then the Lebesgue integral

$$\nu(A) = \int_A f d\mu, \quad A \in \mathcal{S},$$

defines a measure ν on $(X, \mathcal{S})$ that is *absolutely continuous with respect to* μ in the sense that, for all $A \in \mathcal{S}$,

$$\mu(A) = 0 \Rightarrow \nu(A) = 0.$$

When ν is finite, the absolute continuity can also be defined as, for all $\{A_n\} \subset \mathcal{S}$,

$$\mu(A_n) \to 0 \Rightarrow \nu(A_n) \to 0,$$

or, alternatively, as;

for any $\epsilon > 0$, there exists $\delta > 0$ such that $\nu(A) < \epsilon$ whenever $A \in \mathcal{S}$ and $\mu(A) < \delta$.

But if ν and μ be finitely additive measures, the above definition does not imply the existence of a Radon-Nikodym derivatives. This can be demonstrated by the following example.

Example. Let X be a countable infinite set and

$$\mathcal{S} = \{A \subset X : A \quad \text{or} \quad A^c \text{ is finite }\}.$$

Then $\mathcal{S}$ is a field. Now if we define finitely additive measures ν and μ by

$$\nu(A) = \begin{cases} 0, & \text{if } A \text{ is finite} \\ 1, & \text{if } A^c \text{ is finite.} \end{cases}$$

and

$$\mu(A) = \begin{cases} n, & \text{if } A \text{ is a finite set with } n \text{ elements} \\ \infty, & \text{if } A^c \text{ is finite.} \end{cases}$$

Then, clearly ν is absolutely continuous with respect to μ in the above sense. Now, if f is a Radon-Nikodym derivative of ν with respect to μ. That is,

$$\nu(A) = \int_A^* f d\mu, \quad A \in \mathcal{S}.$$

Then

$$\nu(\{x\}) = \int_{\{x\}}^* f d\mu, \quad x \in X.$$

Thus $f(x) = 0$ for all $x \in X$. But $\nu(X) = 1$ and $\int_X f d\mu = 0$.

Since finitely additive measures do not possess, in general, the property of countable additivity, the concept of absolute continuity of countably additive measures mighty be generalized for finitely additive measures in multiple ways.

REFERENCES

1. Ash, R. B., *Real analysis and probability*, Academic Press, New York, 1972.
2. Dunford, N. and Schwartz, J. T., *Linear operators, I*, Wiley Interscience, New York, 1958.
3. Kallianpur, G. and Karandikar, R. L., *A finitely additive white noise approach to nonlinear filtering*, Appl. Math. Optim. **10** (1983), 159-185.
4. ———, *White noise theory of prediction, filtering and smoothing*, Stochastics Monographs, vol. 3, Gordon & Breach Sciences Publishers, New York, 1988.
5. Karandikar, R. L., *On the Feynman-Kac formula and its applications to filtering theory*, Appl. Math. Optim. **16** (1987), 263-276.

DEPARTMENT OF MATHEMATICS, DAEJIN UNIVERSITY, POCHEON, GYEONGGI-DO 487-711, KOREA.

E-mail address: bjkim@road.daejin.ac.kr