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ON THE w-DERIVED SET

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ABSTRACT. We introduce the notion of the w-derived set and w-dense, and investigate some of their properties.

1. Introduction

We define the notion of the w-derived set which is more general than that of the derived set, and examine the relation between the derived set and the w-derived set. And we investigate some properties of the w-derived set.

Also, we introduce the notion of w-dense, and study its property.

2. w-derived set

We denote by A' and cl A the derived set and the closure of the set A, respectively.

Definition 2.1. Let X be a space. For a subset A of X, the *w-derived set* A'_w of A is defined by

$$A'_w = \{x \in X \mid (A - \{x\}) \cap \operatorname{cl} U \neq \emptyset \text{ for all neighborhoods } U \text{ of } x\}.$$

It is obvious that $A' \subset A'_w$, but as the following example illustrates, there exists a subset A of a space X such that $A' \neq A'_w$.

Example 2.2. Consider the topology $\tau = \{\emptyset, \{1\}, X\}$ on $X = \{0, 1\}$. Let $A = \{0\}$. Then $A' = \emptyset$ and $A'_w = \{1\}$. Therefore, $A' \neq A'_w$.

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Theorem 2.3. *Let X be a T_1-space and let V be an open subset of X. Then $V' = V'_w$.*

Proof. Let $x \in V'_w$. Then for any neighborhood U of x, $(V - \{x\}) \cap \mathrm{cl}\, U \neq \emptyset$. Take $y \in (V - \{x\}) \cap \mathrm{cl}\, U$. Since $y \in \mathrm{cl}\, U$ and $(V - \{x\})$ is a neighborhood of y, $(V - \{x\}) \cap U \neq \emptyset$. Thus $x \in V'$, so $V'_w \subset V'$. Since $V' \subset V'_w$, $V' = V'_w$. □

Theorem 2.4. *Let A and B be subsets of a space X. Then the followings hold.*

(1) *If $A \subset B$, then $A'_w \subset B'_w$.*
(2) $(A \cup B)'_w = A'_w \cup B'_w$.

Proof. (1) Let $x \in A'_w$ and let V be any neighborhood of x. Then $(A - \{x\}) \cap \mathrm{cl}\, V \neq \emptyset$. Since $(A - \{x\}) \cap \mathrm{cl}\, V \subset (B - \{x\}) \cap \mathrm{cl}\, V$, $(B - \{x\}) \cap \mathrm{cl}\, V \neq \emptyset$. Thus $x \in B'_w$, so $A'_w \subset B'_w$.

(2) Suppose $x \notin A'_w \cup B'_w$. Then $x \notin A'_w$ and $x \notin B'_w$. Therefore there exist neighborhoods U and V of x such that $(A - \{x\}) \cap \mathrm{cl}\, U = \emptyset$ and $(B - \{x\}) \cap \mathrm{cl}\, V = \emptyset$. Now $U \cap V$ is a neighborhood of x and $(A \cup B - \{x\}) \cap \mathrm{cl}\,(U \cap V) = \emptyset$. Therefore $x \notin (A \cup B)'_w$, so $(A \cup B)'_w \subset A'_w \cup B'_w$. Since $A \subset A \cup B$ and $B \subset A \cup B$, by (1) $A'_w \subset (A \cup B)'_w$ and $B'_w \subset (A \cup B)'_w$. Therefore $A'_w \cup B'_w \subset (A \cup B)'_w$. Hence $(A \cup B)'_w = A'_w \cup B'_w$. □

Definition 2.5. Let X be a space and let A be a subset of X. The *w-closure* $\mathrm{cl}_w(A)$ of A is defined by

$$\mathrm{cl}_w(A) = \{x \in X \mid A \cap \mathrm{cl}\, U \neq \emptyset \text{ for all neighborhoods } U \text{ of } x\}.$$

It is clear that $A \subset \mathrm{cl}\, A \subset \mathrm{cl}_w(A)$.

Theorem 2.6 [2]. *For any open subset U of X, $\mathrm{cl}\, U = \mathrm{cl}_w(U)$.*

Theorem 2.7. *For subsets A and B of a space X, the followings hold.*

(1) $\mathrm{cl}_w \emptyset = \emptyset$.
(2) $A \subset \mathrm{cl}_w(A)$.
(3) $\mathrm{cl}_w(A) \subset \mathrm{cl}_w(B)$ *whenever* $A \subset B$.
(4) $\mathrm{cl}_w(A \cup B) = \mathrm{cl}_w(A) \cup \mathrm{cl}_w(B)$.
(5) $\mathrm{cl}_w(A \cap B) \subset \mathrm{cl}_w(A) \cap \mathrm{cl}_w(B)$.

Proof. (1) Since $\emptyset$ is an open set and $\mathrm{cl}\, \emptyset = \emptyset$, by Theorem 2.6, $\mathrm{cl}\, \emptyset = \mathrm{cl}_w \emptyset$. Therefore $\mathrm{cl}_w \emptyset = \emptyset$.

(2) Since $A \subset \operatorname{cl} A \subset \operatorname{cl}_w(A)$, $A \subset \operatorname{cl}_w(A)$.

(3) Let $x \in \operatorname{cl}_w(A)$ and let U be any neighborhood of x. Then $A \cap \operatorname{cl} U \neq \emptyset$. Since $A \subset B$, $B \cap \operatorname{cl} U \neq \emptyset$. Therefore $x \in \operatorname{cl}_w(B)$, so $\operatorname{cl}_w(A) \subset \operatorname{cl}_w(B)$.

(4) By (3), $\operatorname{cl}_w(A) \subset \operatorname{cl}_w(A \cup B)$ and $\operatorname{cl}_w(B) \subset \operatorname{cl}_w(A \cup B)$. Hence $\operatorname{cl}_w(A) \cup \operatorname{cl}_w(B) \subset \operatorname{cl}_w(A \cup B)$. Now suppose $x \notin \operatorname{cl}_w(A) \cup \operatorname{cl}_w(B)$. Then $x \notin \operatorname{cl}_w(A)$ and $x \notin \operatorname{cl}_w(B)$. Therefore there are neighborhoods U and V of x such that $A \cap \operatorname{cl} U = \emptyset$ and $B \cap \operatorname{cl} V = \emptyset$. Now $U \cap V$ is a neighborhood of x and $(A \cup B) \cap \operatorname{cl}(U \cap V) = \emptyset$. Thus $x \notin \operatorname{cl}_w(A \cup B)$, so $\operatorname{cl}_w(A \cup B) \subset \operatorname{cl}_w(A) \cup \operatorname{cl}_w(B)$. Hence $\operatorname{cl}_w(A \cup B) = \operatorname{cl}_w(A) \cup \operatorname{cl}_w(B)$.

(5) Since $A \cap B \subset A$ and $A \cap B \subset B$, by (3) $\operatorname{cl}_w(A \cap B) \subset \operatorname{cl}_w(A)$ and $\operatorname{cl}_w(A \cap B) \subset \operatorname{cl}_w(B)$. Therefore $\operatorname{cl}_w(A \cap B) \subset \operatorname{cl}_w(A) \cap \operatorname{cl}_w(B)$. □

In the following example, we show that there exist subsets A and B of a space X such that $\operatorname{cl}_w(A \cap B) \neq \operatorname{cl}_w(A) \cap \operatorname{cl}_w(B)$.

Example 2.8. Let τ be a topology $\{\emptyset, \{1\}, X\}$ on $X = \{0, 1\}$, and let $A = \{0\}$ and $B = \{1\}$. Then $A \cap B = \emptyset$, so $\operatorname{cl}_w(A \cap B) = \emptyset$. However, $\operatorname{cl}_w(A) = \operatorname{cl}_w(B) = X$, so $\operatorname{cl}_w(A) \cap \operatorname{cl}_w(B) = X$. Therefore $\operatorname{cl}_w(A \cap B) \neq \operatorname{cl}_w(A) \cap \operatorname{cl}_w(B)$.

The following result is a consequence of Theorem 2.7.

Corollary 2.9. *For subsets A and B of a space X,*

$$X - \operatorname{cl}_w(A \cup B) = (X - \operatorname{cl}_w(A)) \cap (X - \operatorname{cl}_w(B)).$$

Theorem 2.10. *Let A be a subset of a space X. Then $\operatorname{cl}_w(A) = A \cup A'_w$.*

Proof. Let $x \in \operatorname{cl}_w(A)$ and let U be any neighborhood of x. Then $A \cap \operatorname{cl} U \neq \emptyset$. If $x \notin A$, then $(A - \{x\}) \cap \operatorname{cl} U \neq \emptyset$ and hence $x \in A'_w$. If $x \in A$, we are through. Thus $\operatorname{cl}_w(A) \subset A \cup A'_w$. On the other hand, since $A \subset \operatorname{cl}_w(A)$ and $A'_w \subset \operatorname{cl}_w(A)$, we obtain $A \cup A'_w \subset \operatorname{cl}_w(A)$. □

Theorem 2.11. *A space X is regular if and only if for each subset A of X, $A' = A'_w$.*

Proof. For each subset A of X, suppose $A' = A'_w$. Let $x \in X$ and let U be a neighborhood of x. Since $x \notin X - U$ and $X - U = \operatorname{cl}(X - U) = (X - U) \cup (X - U)' = (X - U) \cup (X - U)'_w$, we obtain $X - U = \operatorname{cl}_w(X - U)$ from Theorem 2.10, so $x \notin \operatorname{cl}_w(X - U)$. Therefore there is a neighborhood V of x such that $(X - U) \cap \operatorname{cl} V = \emptyset$, so $\operatorname{cl} V \subset X - (X - U) = U$. Hence X is regular.

Conversely, suppose X is regular. Let $x \in A'_w$. Then for any neighborhood U of x, there exists a neighborhood V of x such that $\mathrm{cl}\, V \subset U$. Since $(A - \{x\}) \cap \mathrm{cl}\, V \neq \emptyset$, $(A - \{x\}) \cap U \neq \emptyset$. Thus $x \in A'$, so $A'_w \subset A'$. Since $A' \subset A'_w$, $A' = A'_w$. □

Definition 2.12. A subset A of a space X is *w-dense* in X provided $\mathrm{cl}_w(A) = X$.

If a subset A of a space X is dense in X, then A is w-dense in X. However, in the Example 2.8, since $\mathrm{cl}\, A = \{0\}$ and $\mathrm{cl}_w(A) = X$, the converse does not hold generally, but in every regular space it holds.

Theorem 2.13 [2]. *A space X is regular if and only if for any subset A of X, we have* $\mathrm{cl}\, A = \mathrm{cl}_w(A)$.

As a consequence of Theorem 2.13, we obtain the following corollary.

Corollary 2.14. *Let A be a subset of a regular space X. Then a set A is w-dense in X if and only if A is dense in X.*

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