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ON SUB-KAC ALGEBRAS AND SUBGROUPS

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ABSTRACT. Let $\mathbb{K}_a(G)$ (resp. $\mathbb{K}_s(G)$) be the abelian (resp. symmetric) Kac algebra for a locally compact group G. We show that there exists a one-to-one correspondence between the subgroups of G and the sub-Kac algebras of $\mathbb{K}_a(G)$ (resp. $\mathbb{K}_s(G)$). Moreover we obtain similar correspondences between the subgroups of G and the reduced Kac algebras of $\mathbb{K}_a(G)$ (resp. $\mathbb{K}_s(G)$).

1. Introduction

The theory of Kac algebra is one of the most important topics in operator algebra theory and recently much effort has been made to develope it (cf. [DeC1], [E], [EN], [ES1], [KP], [M], [S]). It is widely acknowledged that the category of locally compact groups is identified with a subcategory of Kac algebras. Typical examples of von Neumann algebras come from groups and so Kac algebras related to groups are also interesting (cf. [DeC2], [ILP], [TT],[W],[Y]).

Let $\mathbb{H} = (M, \Gamma, \kappa)$ be a co-involutive Hopf-von Neumann algebra and φ a Haar weight on $\mathbb{H}$. We shall say that the quadruple $\mathbb{K} = (M, \Gamma, \kappa, \varphi)$ is a Kac algebra. For any locally compact group G, $\mathbb{K}_a(G) = (L^\infty(G), \Gamma_a, \kappa_a, \varphi_a)$ is the abelian Kac algebra and $\mathbb{K}_s(G) = (L(G), \Gamma_s, \kappa_s, \varphi_s)$ is the symmetric Kac algebra. It is interesting that there exist a one-to-one correspondence between the sub-Kac algebras of $\mathbb{K}_a(G)$ and $\mathbb{K}_s(G)$.

The purpose of this paper is to give a complete relation between the subgroups of G and the sub-Kac algebras of either $\mathbb{K}_a(G)$ or $\mathbb{K}_s(G)$, which allow us to obtain a similar relation between the subgroups of G and the reduced Kac algebras of either $\mathbb{K}_a(G)$ or $\mathbb{K}_s(G)$.

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For any open and closed subgroup H of G, note that the group von Neumann algebra $L(H)$ can be identified with a subalgebra $\tilde{M}$ of $L(G)$. We will show that $\tilde{\mathbb{K}}= (\tilde{M},\ \tilde{\Gamma},\ \tilde{\kappa},\ \tilde{\varphi})$ is a sub-Kac algebra of $\mathbb{K}_s(G)$, which is isomorphic to $\mathbb{K}_s(H)$. Conversely for any sub-Kac algebra $\tilde{\mathbb{K}}$ of $\mathbb{K}_s(G)$, there exists an open and closed subgroup H of G with $\tilde{\mathbb{K}} \cong \mathbb{K}_s(H)$. So, we will give a one-to-one corresponce between open and closed subgroups and sub-Kac algebras. Similarly, for the abelian Kac algebra $\mathbb{K}_a(G)$, we will give a one-to-one correspondence between normal and compact subgroups K of G and sub-Kac algebras $\tilde{\mathbb{K}}$ of $\mathbb{K}_a(G)$ with $\tilde{\mathbb{K}} \cong \mathbb{K}_a(G/K)$.

On the other hand, by taking the duality of Kac algebra, we know that the dual Kac algebra of a reduced Kac algebra of $\mathbb{K}$ becomes a sub-Kac algebra of $\hat{\mathbb{K}}$ and the dual Kac algebra of a sub-Kac algebra of $\mathbb{K}$ becomes a reduced Kac algebra of $\hat{\mathbb{K}}$. So as an application of our result to reduced Kac algebras of $\mathbb{K}_a(G)$ (resp. $\mathbb{K}_s(G)$), we obtain a one-to-one correspondence between the subgroups of G and the reduced Kac algebras $\mathbb{K}_a(G)$ (resp. $\mathbb{K}_s(G)$).

2. Preliminaries

In this section, we introduce notations and briefly review fundamental results which will be necessary for our discussion. In order to fix notations, we first describe the notion of Kac algebras defined by Enock and Schwartz [ES2]. For the general theory of Kac algebras and for the definition of notations, we refer to [ES2].

A *Kac algebra* $\mathbb{K}$ is a quadruple $(M,\ \Gamma,\ \kappa,\ \varphi)$ in which

(1) $\mathbb{H} = (M,\ \Gamma,\ \kappa)$ is a co-involutive Hopf-von Neumann algebra,

(2) φ is a *Haar weight* (faithful normal semifinite weight) on M,

(3) $(id \otimes \varphi)\Gamma(x) = \varphi(x)1,\ \ x \in M_+$;
$(id \otimes \varphi)((1 \otimes y^*)\Gamma(x)) = \kappa((id \otimes \varphi)(\Gamma(y)^*(1 \otimes x))),\ \ x, y \in \mathcal{N}_\varphi$, and

(4) $\sigma_t^\varphi \circ \kappa = \kappa \circ \sigma_{-t}^\varphi,\ \ t \in (-\infty, \infty)$.

In this case, M is represented standardly on the Hilbert space $\mathcal{H}_\varphi$ obtained from the Haar weight φ.

For every Kac algebra $\mathbb{K} = (M,\ \Gamma,\ \kappa,\ \varphi)$, there canonically exists its dual Kac algebra $\hat{\mathbb{K}} = (\hat{M},\ \hat{\Gamma},\ \hat{\kappa},\ \hat{\varphi})$ and whose underlying von Neumann algebra $\hat{M}$ is also represented standardly on the space $\mathcal{H}_\varphi$.

For a projection R in the center of M with $\Gamma(R) \geq R \otimes R$ and $\kappa(R) = R$, we shall denote by $\mathbb{K}_R$ the quadruple $(M_R, \Gamma_R, \kappa_R, \varphi_R)$ where

(1) M_R is the usual reduced algebra with the canonical surjection $r : M \to M_R$,
(2) $\Gamma_R(r(x)) = (r \otimes r)\Gamma(x)$, for all x in M,
(3) $\kappa_R(r(x)) = r(\kappa(x))$, for all x in M, and
(4) the weight φ_R on M_R is the restriction of φ on M_R.

Here we consider a von Neumann subalgebra $\tilde{M}$ of M such that

(1) $\Gamma(\tilde{M}) \subset \tilde{M} \otimes \tilde{M}$,
(2) $\kappa(\tilde{M}) = \tilde{M}$,
(3) $\sigma_t^{\phi}(\tilde{M}) = \tilde{M}$, $t \in \mathbb{R}$, and
(4) the restriction $\tilde{\phi} = \phi|_{\tilde{M}^+}$ is a semifinite weight.

We shall denote by $\tilde{\mathbb{K}}$ the quadruple $(\tilde{M}, \tilde{\Gamma}, \tilde{\kappa}, \tilde{\varphi})$ where $\tilde{\Gamma}$ and $\tilde{\kappa}$ are respectively the restriction of Γ and κ on $\tilde{M}$. Then it is well known that $\mathbb{K}_R$ (resp. $\tilde{\mathbb{K}}$) is a Kac algebra, called a *reduced Kac algebra* (resp. a *sub-Kac algebra*) of $\mathbb{K}$.

For two Kac algebras $\mathbb{K}_i = (M_i, \Gamma_i, \kappa_i, \varphi_i)$, $i = 1, 2$, we shall say that $\mathbb{K}_1$ and $\mathbb{K}_2$ are *isomorphic* (i.e. $\mathbb{K}_1 \cong \mathbb{K}_2$) if there exist an $\mathbb{H}$-isomorphism $u : (M_1, \Gamma_1, \kappa_1) \to (M_2, \Gamma_2, \kappa_2)$ and $\alpha > 0$ such that $\varphi_2 \circ u = \alpha\varphi_1$. Note that an $\mathbb{H}$-isomorphism u is an unital normal isomorphism from M_1 to M_2 such that $\Gamma_2 u = (u \otimes u)\Gamma_1$ and $\kappa_2 u = u\kappa_1$.

For a sub-Kac algebra $\tilde{\mathbb{K}}$=$(\tilde{M}, \tilde{\Gamma}, \tilde{\kappa}, \tilde{\varphi})$ and a reduced Kac algebra $\mathbb{K}_R$ of $\mathbb{K}$, we know that the dual Kac algebra $(\tilde{\mathbb{K}})\hat{}$ of $\tilde{\mathbb{K}}$ is isomorphic to the reduced Kac algebra $(\hat{\mathbb{K}})_R$ of $\hat{\mathbb{K}}$ for some projection R (see Proposition 3.7.9 in [ES2]). Conversely, the dual Kac algebra $(\mathbb{K}_R)\hat{}$ of $\mathbb{K}_R$ is isomorphic to the sub-Kac algebra of $\hat{\mathbb{K}}$ (see Proposition 3.7.10 in [ES2]).

The basic examples of Kac algebras that we shall need are ones associated to locally compact groups.

To any locally compact group G with a left Haar measure ds, we associate two concrete Kac algebras acting on the Hilbert space $L^2(G)$ of all square integrable functions on G with respect to ds. Let $L^\infty(G)$ be the algebra of all essentially bounded with respect to ds, which is an abelian von Neumann algebra acting, by multiplication, on $L^2(G)$. The left regular unitary representation λ_G of G on $L^2(G)$ is defined by $(\lambda_G(g)\xi)(h) = \xi(g^{-1}h), (\xi \in L^2(G), g, h \in G)$ and we have group von Neumann algebra $L(G) = \{\lambda_G(g) | g \in G\}''$. One is the abelian Kac algebra

$\mathbb{K}_a(G) = (L^\infty(G), \Gamma_a, \kappa_a, \varphi_a)$, where

$$\Gamma_a(f)(s,t) = f(st), \quad \kappa_a(f)(s) = f(s^{-1}), \quad \varphi_a(f) = \int_G f(s)ds.$$

The other is the symmetric Kac algebra $\mathbb{K}_s(G) = (L(G), \Gamma_s, \kappa_s, \varphi_s)$, where

$$\Gamma_s(\lambda_G(s)) = \lambda_G(s) \otimes \lambda_G(s), \quad \kappa_s(\lambda_G(s)) = \lambda_G(s^{-1}).$$

Note that $\mathbb{K}_a(G)$ and $\mathbb{K}_s(G)$ are dual to each other.

3. Correspondences between sub-Kac algebras and subgroups

We believe that the next proposition is widely known to specialists. The author, however, cannot find the exact statement. So we give it below, which will be a key to give uniqueness of Kac algebras up to isomorphism. The following proposition asserts that if there exists an $\mathbb{H}$-isomorphism between two Kac algebras $\mathbb{K}_1$ and $\mathbb{K}_2$ then $\mathbb{K}_1$ and $\mathbb{K}_2$ are isomorphic Kac algebras.

Proposition 3.1. *For two Kac algebras $\mathbb{K}_i = (M_i, \Gamma_i, \kappa_i, \varphi_i), (i = 1, 2)$, if there exists a unital normal isomorphism $u : M_1 \to M_2$ such that $\Gamma_2 u = (u \otimes u)\Gamma_1$ then $\mathbb{K}_1$ and $\mathbb{K}_2$ are isomorphic.*

Proof. By Theorem 5.5.6 in [ES2], u is an $\mathbb{H}$-isomorphism from $(M_1, \Gamma_1, \kappa_1)$ to $(M_2, \Gamma_2, \kappa_2)$. From Corollary 2.7.9 in [ES2], there exists $\alpha > 0$ such that $\varphi_2 \circ u = \alpha\varphi_1$, and so $\mathbb{K}_1$ and $\mathbb{K}_2$ are isomorphic. □

From now on, G denotes a locally compact group and $\mathbb{K}_a(G)$ (resp. $\mathbb{K}_s(G)$) the *abelian Kac algebra* (resp. the *symmetric Kac algebra*). It is natural to expect that for some subgroup H of G, we can associate sub-Kac algebras of $\mathbb{K}_a(G)$ and $\mathbb{K}_s(G)$.

Lemma 3.2. *We have the followings:*

(a) *For any open and closed subgroup H of G, $\mathbb{K}_s(H)$ is isomorphic to a sub-Kac algebra of $\mathbb{K}_s(G)$.*

(b) *For any compact and normal subgroup K of G, $\mathbb{K}_a(G/K)$ is isomorphic to a sub-Kac algebra of $\mathbb{K}_a(G)$.*

Proof. (a) For any open and closed subgroup H of G, consider the double commutant $\{\lambda_G(s) \in L(G) | s \in H\}''$ denoted by $\tilde{M}$, which is a von Neumann subalgebra of

$L(G)$. From the definitions of Γ_s and κ_s, it is a straightforward computation that $\tilde{\mathbb{K}} =(\tilde{M}, \tilde{\Gamma}_s, \tilde{\kappa}_s, \tilde{\varphi}_s)$ is a sub-Kac algebra of $\mathbb{K}_s(G)$.

Let $L(H)$ be the group von Neumann algebra generated by λ_H of H and $\mathbb{K}_a(H)$ the abelian Kac algebra. Note that $L(H)$ can be identified with the von Neumann subalgebra $\tilde{M}$ of $L(G)$ by $u, u(\lambda_G(s)) = \lambda_H(s), (s \in H)$. From Proposition 3.1, u gives an isomorphism between $\mathbb{K}_s(H)$ and $\tilde{\mathbb{K}}$ and it follows the proof.

(b) For each $s \in G$ and $f \in L^\infty(G)$, we define $r(s)$ by $(r(s)f)(t) = f(s^{-1}t), (t \in G)$. For any compact and normal subgroup K of G, consider a subset $\tilde{M} = \{f \in L^\infty(G) | r(s)f = f, \forall s \in K\}$ of $L^\infty(G)$. Since K is compact $\tilde{M}$ is a closed self adjoint subalgebra of $L^1(G) = L(G)_*$ and so $\tilde{M}$ is a von Neumann subalgebra of $L^\infty(G)$ by λ and $(L(G))\hat{} = L^\infty(G)$ (see [TT]).

Takesaki and Tatsuuma [TT] have given a one-to-one correspondence between normal and closed subgroups of G and two sided invariant co-ideals (for the definition, see Definition 4.1 in [ILP]) of $L^\infty(G)$. This shows that $\tilde{M}$ is a two sided invariant co-ideal of $L^\infty(G)$ which implies $\Gamma_a(\tilde{M}) \subset \tilde{M} \otimes \tilde{M}$ and $\kappa_a(\tilde{M}) = \tilde{M}$. It follows that we have a sub-Kac algebra $\tilde{\mathbb{K}} =(\tilde{M}, \tilde{\Gamma}_a, \tilde{\kappa}_a, \tilde{\varphi}_a)$ of $\mathbb{K}_a(G)$.

On the other hand, the abelian von Neumann algebra $L^\infty(G/K)$ can be identified with the von Neumann subalgebra $\tilde{M}$ by $u : \tilde{M} \to L^\infty(G/K)$ defined by $u(f)(\bar{g}) = f(g), (\bar{g} \in G/K)$. Thanks to Proposition 3.1, u gives an isomorphism between $\mathbb{K}_a(G/K)$ and $\tilde{\mathbb{K}}$. Thus we have shown that for any compact and normal subgroup H of G, $\mathbb{K}_a(G/K)$ is isomorphic to a sub-Kac algebra of $\mathbb{K}_a(G)$. □

We need the following lemma in order to relate sub-Kac algebras and subgroups.

Lemma 3.3. *We have the followings:*

(a) *For any sub-Kac algebra $\tilde{\mathbb{K}}$ of $\mathbb{K}_s(G)$, there exists an open and closed subgroup H of G with $\tilde{\mathbb{K}} \cong \mathbb{K}_s(H)$.*

(b) *For any sub-Kac algebra $\tilde{\mathbb{K}}$ of $\mathbb{K}_a(G)$, there exists a compact and normal subgroup K of G with $\tilde{\mathbb{K}} \cong \mathbb{K}_a(G/K)$.*

Proof. (a) By Corollary 4.3.6 in [ES2] to $\tilde{\mathbb{K}}$, there exists H, open and closed subgroup of G and an $\mathbb{H}$-isomorphism from $\tilde{\mathbb{K}}$ onto $\mathbb{K}_s(H)$. By Proposition 3.1, $\tilde{\mathbb{K}}$ and $\mathbb{K}_s(H)$ are isomorphic.

(b) Thanks to Theorem 4.5.10 in [ES2], it is clear. □

Now we are ready to give our main result that makes clear the relation between subgroups and sub-Kac algebras. From Corollary 4.3.5 in [ES2] and Proposition 3.1, note that for two locally compact groups G_1 and G_2, the followings are equivalent:

(1) G_1 and G_2 are homeomorphic groups.
(2) $\mathbb{K}_a(G_1)$ and $\mathbb{K}_a(G_2)$ are isomorphic Kac algebras.
(3) $\mathbb{K}_s(G_1)$ and $\mathbb{K}_s(G_2)$ are isomorphic Kac algebras.

If we apply above equivalences to the following theorem and corollary, then one-to-one correspondences mean those between the set of all classes of homeomorphic subgroups and that of all classes of isomorphic Kac algebras.

Theorem 3.4. *The following two assertions hold:*

(a) *There exists a one-to-one correspondence between open and closed subgroups H of G and sub-Kac algebras $\tilde{\mathbb{K}}$ of $\mathbb{K}_s(G)$ with $\tilde{\mathbb{K}} \cong \mathbb{K}_s(H)$.*

(b) *There exists a one-to-one correspondence between compact and normal subgroups K of G and sub-Kac algebras $\tilde{\mathbb{K}}$ of $\mathbb{K}_a(G)$ with $\tilde{\mathbb{K}} \cong \mathbb{K}_a(G/K)$.*

Proof. Proofs are completed by Lemma 3.2 and Lemma 3.3. □

We shall apply the preceding theorem to obtain one-to-one correspondences between reduced Kac algebras and subgroups. Similar correspondences are valid for reduced Kac algebras.

Corollary 3.5. *We have the followings:*

(a) *There exists a one-to-one correspondence between open and closed subgroups H of G and reduced Kac algebras $\mathbb{K}_a(G)_R$ of $\mathbb{K}_a(G)$ with $\mathbb{K}_a(G)_R \cong \mathbb{K}_a(H)$.*

(b) *There exists a one-to-one correspondence between compact and normal subgroups K of G and reduced Kac algebras $\mathbb{K}_s(G)_R$ of $\mathbb{K}_s(G)$ with*

$$\mathbb{K}_s(G)_R \cong \mathbb{K}_s(G/K).$$

Proof. (a) We have noted that the dual Kac algebra $(\mathbb{K}_a(G)_R)\hat{}$ of $\mathbb{K}_a(G)_R$ is isomorphic to the sub-Kac algebra of the dual Kac algebra $(\mathbb{K}_a(G))\hat{} = \mathbb{K}_s(G)$ and the dual Kac algebra $(\tilde{\mathbb{K}})\hat{}$ of a sub-Kac algebra $\tilde{\mathbb{K}}$ of $(\mathbb{K}_s(G))\hat{}$ is isomorphic to the reduced Kac algebra of the dual Kac algebra $(\mathbb{K}_s(G))\hat{} = \mathbb{K}_a(G)$. Thus by Theorem 3.4, there exists a one-to-one correspondence between open and closed subgroups H of G and sub-Kac algebras of $\mathbb{K}_s(G)$ with $\mathbb{K}_s(H) \cong (\mathbb{K}_a(G)_R)\hat{}$ and so $\mathbb{K}_a(H) \cong \mathbb{K}_a(G)_R$ gives the proof.

(b) Similarly we note that the dual Kac algebra $(\mathbb{K}_s(G)_R)\hat{}$ of $\mathbb{K}_s(G)_R$ is isomorphic to the sub-Kac algebra of the dual Kac algebra $(\mathbb{K}_s(G))\hat{} = \mathbb{K}_a(G)$. By applying

Theorem 3.4 to $\mathbb{K}_a(G)$, there exists a one-to-one correspondence between compact and normal subgroups K of G and sub-Kac algebras of $\mathbb{K}_a(G)$ with $\mathbb{K}_a(G/K) \cong (\mathbb{K}_s(G)_R)\hat{}$ and so $\mathbb{K}_s(G/K) \cong \mathbb{K}_s(G)_R$, which completes the proof. □

We will end this paper with a remark which shows the explicit description of two isomorphic Kac algebras.

Remark 3.6. For any open and closed subgroup H of G, let R be a characteristic function χ_H on H then trivially R is a projection in the center of $L^\infty(G)$. Since, for $s,t \in H, \Gamma_a(\chi_H)(s,t) = \chi_H(s,t), (\chi_H \otimes \chi_H)(s,t) = \chi_H(s)\chi_H(t)$ and $\kappa_a(\chi_H)(s) = \chi_H(s^{-1}) = \chi_H(s)$, we get that $\Gamma_a(R) \geq R \otimes R$ and $\kappa_a(R) = R$.

A surjection $r : L^\infty(G) \to L^\infty(G)\chi_H$ defind by $r(f) = f\chi_H$ gives Γ_R (resp. κ_R and φ_R) such that $\Gamma_R(r(f)) = (r \otimes r)\Gamma_a(f)$ (resp. $\kappa_R(r(f)) = r(\kappa_a(f))$ and $\varphi_R(r(f)) = \int_H r(f)ds$).

Thus we have the reduced Kac algebra $\mathbb{K}_a(G)_R$ of $\mathbb{K}_a(G)$.

The fact of $L^\infty(G)\chi_H \cong L^\infty(H)$ follows $\mathbb{K}_a(G)_R \cong \mathbb{K}_a(H)$.

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