

Fuzzy algebraic structures of L -fuzzy ideals of L -fuzzy ring*

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Abstract

In this paper, the concepts of semiprime L -fuzzy ideals and semiprimary L -fuzzy ideals of a L -fuzzy ring are introduced and some fundamental propositions proved. And we investigate the relation between fuzzy nil radical and semiprimary.

0. Historical background and introduction

The theory of fuzzy sets was introduced by Zadeh [16] in 1965. Since its inception, the theory of fuzzy subsets has developed in many directions and found applications in a wide variety of fields. The study of fuzzy subsets and its application to various mathematical contexts has given rise to what is now commonly called fuzzy mathematics. Fuzzy algebra is an important branch of fuzzy mathematics. In 1971, Rosenfeld [12] introduced the concepts of fuzzy subgroup of a group and formulated the concepts of a fuzzy subgroup and showed how some basic notions of group theory should be extended in an elementary manner to develop the theory of fuzzy groups.

Wang-Jin-Liu, in his pioneering paper [7] introduced the notions of fuzzy subrings and fuzzy ideals in a rings and proved some fundamental results pertaining to these notions.

Subsequently, Mukherjee and Sen [10], Swamy and Swamy [14], Zhang Yue [15], Dixit et al. and Rajesh [2] applied some basic concepts pertaining to ideals from classical ring theory and developed a theory of fuzzy ideals. In this paper, the definition of fuzzy semiprime ideal is given in the context of the theory of fuzzy ideals of fuzzy

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rings. The definitions of semiprimary fuzzy ideal and fuzzy radical of a fuzzy ideal are also given.

1. Preliminaries

The basic definitions of fuzzy ideal of a fuzzy ring and fuzzy quotient ring are the ones given [9]. For general results about rings and ideals see [4, 7, 9]. The reader interested in having an overview of fuzzy systems can consult [16]. We fix $L = (L, \leq, \vee, \wedge)$ as a totally distributive, which has least elements 0 and greatest elements 1. Throughout this paper, the ring that appear will be commutative. In this case, left, right and two-sided fuzzy ideals will be the same thing.

Definition 1.1. A L -fuzzy ring is a function $A: R \rightarrow L$, where $(R, +, \cdot)$ is a ring, that satisfies: (1) $A \neq 0$,

$$(2) A(x - y) \geq A(x) \wedge A(y) \text{ for every } x, y \text{ in } R,$$

$$(3) A(xy) \geq A(x) \wedge A(y) \text{ for every } x, y \text{ in } R,$$

$$(4) \text{ if } R \text{ is unitary, then } A(1) = A(0).$$

Definition 1.2. If $A: R \rightarrow L$ and $A^*: R^* \rightarrow L^*$ are L -fuzzy rings, a homomorphism between A and A^* is a ring homomorphism $f: R \rightarrow R^*$ that satisfies $A^*(f(x)) = A(x)$ for all x in R (i.e. $f^{-1}(A^*) = A$).

Definition 1.3. A L -fuzzy subring of a L -fuzzy ring A is a fuzzy ring $A^*: R \rightarrow L$ satisfying $A^*(x) \leq A(x)$ for all x in R .

Definition 1.4. Let $A: R \rightarrow L$ be a L -fuzzy ring; a L -fuzzy ideal of A is a map $I: R \rightarrow L$ such that the following properties hold:

$$(1) I \neq 0,$$

$$(2) I(x + y) \geq I(x) \wedge I(y) \text{ for every } x, y \text{ in } R,$$

$$(3) I(xy) \geq A(x) \wedge I(y) \text{ for every } x, y \text{ in } R,$$

$$(4) I(x) \leq A(x) \text{ for every } x \text{ in } R,$$

$$(5) I(-x) = I(x) \text{ for every } x \text{ in } R.$$

Note that (1) and (5) implies that $I(x-y) \geq I(x) \wedge I(y)$ for every x, y in R .

Definition 1.5. A L -fuzzy ideal I of a L -fuzzy ring A is said to be prime if $A \neq 1$ and it satisfies $I(xy) \wedge A(x) \wedge A(y) \leq I(x) \vee I(y)$ for every x, y in R .

Definition 1.6. Let A be a L -fuzzy subset and $t \in L$. The t -level cut A_t will be $\{x \in R \mid A(x) \geq t\}$. Analogously the t -level strong cut $A_{\bar{t}}$ will be $\{x \in R \mid A(x) > t\}$.

2. Main results

Definition 2.1. A L -fuzzy ideal I of a L -fuzzy ring A is said to be semiprime if $I \neq 1$ and it satisfies $I(x^2) \wedge A(x) \leq I(x)$ for every x in R .

We will analyze the relation of this definition with some of the ones already established in the literature for the case in which $A \equiv 1$, taking into account that in this case definition 2.1 because

$$I(x^2) \leq I(x) \tag{1}$$

for every x in R . In [6], a L -fuzzy ideal I of a L -fuzzy ring A said to be semiprime if for every x in R ,

$$I(x^2) = I(x) \tag{2}$$

If the lattice L is totally ordered, the it is evident that (2) implies (1).

From the above definition 1.5, we can get the following proposition.

Proposition 2.2. If I is a semiprime L -fuzzy ideal of A , then I is a prime L -fuzzy ideal of L -fuzzy ring A .

Proposition 2.3. Let L be a totally ordered set. If I is a semiprime L -fuzzy ideal of A , then I_t is a semiprime ideal of A_t for every t in L such that $t \leq I(0)$.

Proof. Let x be in A_t with $x^2 \in I_t$.

Then $t \leq A(x)$ and $t \leq I(x^2)$.

Since I is a semiprime L -fuzzy ideal of A ,

$$t \leq I(x^2) \wedge A(x) \leq I(x).$$

Therefore $x \in I_t$. □

Unless stated otherwise R and R^* are rings, $f: R \rightarrow R^*$ ring homomorphism and $A: R \rightarrow L$ and $A^*: R^* \rightarrow L$ are L -fuzzy rings.

We define the image of A under f to be fuzzy set $f(A): R^* \rightarrow L$ such that

$$f(A)(x^*) = \begin{cases} V\{A(x) \mid x \in f^{-1}(x^*)\} & \text{if } f^{-1}(x^*) \neq \emptyset, \\ 0 & \text{otherwise} \end{cases}$$

The inverse image of A^* is a fuzzy subset $f^{-1}(A^*): R \rightarrow L$, defined by

$$f^{-1}(A^*)(x) = A^*(f(x)).$$

Proposition 2.4. If A and A^* are L -fuzzy rings, $f: R \rightarrow R^*$ is a homomorphism between them and I^* is a semiprime L -fuzzy ideal of A^* , then $f^{-1}(I^*)$ is a semiprime L -fuzzy ideal of A .

Proof. Since I^* is a semiprime L -fuzzy ideal of A^* ,

$$I^*(x^{*2}) \wedge A(x) \leq I^*(A^*).$$

Let $x \in R$. Then We have

$$\begin{aligned} f^{-1}(I^*)(x^2) \wedge A(x) &= I^*(f^2(x)) \wedge A(x) \\ &= I^*(f(x^2)) \wedge A(x) \\ &\leq I^*(f(x)) \\ &= f^{-1}(I^*)(x) \end{aligned} \quad \square$$

Lemma 2.5. [2] (1) $f(I)(0^*) = I(0)$.

(2) If I is constant on $\text{Ker } f$, then $f(I)f(x) = I(x)$ for all $x \in R$.

Theorem 2.6. Let $f: A \rightarrow A^*$ be an epimorphism and I be a semiprime L -fuzzy ideal of a L -fuzzy ring A , which is constant on $\text{Ker } f$. Then $f(I)$ is a semiprime L -fuzzy ideal of A^* .

Proof. Let $x^* \in A^*$ and $f(x) = x^*$ for some $x \in A$.

By above Lemma, we have the following condition.

$$\begin{aligned}
 f(I)(x^*) \wedge A(x) &= f(I) f^2(x) \wedge A(x) \\
 &= f(I) f(x^2) \wedge A(x) \\
 &= I(x^2) \wedge A(x) \\
 &\leq I(x) \\
 &= f(I) f(x)
 \end{aligned}$$

Hence $f(I)$ is a semiprime L -fuzzy ideal of A^* . □

Definition 2.7. A L -fuzzy ideal I is said to be semiprimary if

$$I(xy) \wedge A(x) \wedge A(y) \leq \left(\bigvee_{n \in N} I(x^n) \right) \vee \left(\bigvee_{m \in N} I(y^m) \right)$$

for all x, y in R and for all n, m in N .

When $A \equiv 1$, definition 2.7 reduces to $I(xy) \leq \left(\bigvee_{n \in N} I(x^n) \right) \vee \left(\bigvee_{m \in N} I(y^m) \right)$.

In [2], another definition of semiprimary fuzzy ideal is given as follow: for any fuzzy ideals I_1 and I_2 of A , $I_1 I_2 \subseteq I$ and $I_1(x) \triangleright I(x^m)$ for some $x \in A$ and for all $m \in N$, together imply that given $y \in A$, there exists $n \in N$ such that $I_1(y) \leq I(y^n)$.

Proposition 2.8. If I is a semiprimary L -fuzzy ideal and L is finite and totally ordered, then I_t is a semiprimary L -fuzzy ideal of A_t for every $t \in L$.

Proof. Let $x, y \in A_t$ and suppose that $xy \in I_t$.

Then $t \leq A(x)$, $t \leq A(y)$ and $t \leq I(xy)$.

Since I is a semiprimary L -fuzzy ideal,

$$t \leq \left(\bigvee_{n \in N} I(x^n) \right) \vee \left(\bigvee_{m \in N} I(y^m) \right)$$

for all x, y in R and for all n in N .

Hence $x^n \in I_t$ or $y^m \in I_t$ for some $n, m \in N$. □

Inductively, we can state the following proposition.

Proposition 2.9. If I is a semiprime L -fuzzy ideal, then I is semiprimary.

Definition 2.10. [8] Let I be a L -fuzzy ideal of a L -fuzzy ring A . The fuzzy nil

radical of I is the fuzzy subset of R defined by

$$\sqrt{I}(x) = \left(\bigvee_{n \in N} I(x^n) \right) \wedge A(x)$$

When $A \equiv 1$, this definition takes the form

$$\sqrt{I}(x) = \bigvee_{n \in N} I(x^n).$$

Lemma 2.11. Let I be a L -fuzzy ideal of a L -fuzzy ring A . Then

$$\bigvee_{n \in N} I(x+y)^n \geq \bigvee_{n \in N} (I(x^n) \wedge I(y^n)).$$

Proof. Let $x, y \in R$.

$$\bigvee_{n \in N} I(x+y)^n = \left(\bigvee_{n \in N} I(x+y)^{2n} \right) \vee \left(\bigvee_{n \in N} I(x+y)^{2n+1} \right)$$

But

$$\begin{aligned} \bigvee_{n \in N} I(x+y)^{2n} &= I\left(\sum_r \binom{2n}{r} x^{2n-r} y^r\right) \\ &\geq I(x^{2n}) \wedge I(x^{2n-1}y) \wedge \cdots \wedge I(x^n y^n) \wedge I(x^{n-1}y^{n+1}) \wedge \cdots \wedge I(y^{2n}) \\ &\geq I(x^{2n}) \wedge I(x^{2n-1}) \wedge \cdots \wedge I(x^n) I(y^{n+1}) \wedge \cdots \wedge I(y^{2n}) \\ &\geq I(x^n) \wedge I(y^{n+1}) \quad (\text{since } I(z^m) \geq I(z^n), \text{ for all } z \in R \text{ and } m \geq n) \\ &\geq I(x^n) \wedge I(y^n). \end{aligned}$$

Similarly, $I(x+y)^{2n+1} \geq I(x^n) \wedge I(y^n)$.

Therefore $\bigvee_{n \in N} I(x+y)^n \geq \bigvee_{n \in N} (I(x^n) \wedge I(y^n))$. □

Theorem 2.12. Let I be a L -fuzzy ideal of a L -fuzzy ring A . Then \sqrt{I} is L -fuzzy ideal of A .

Proof. Let $x, y \in R$ and $n \in N$. From Lemma 2.11,

$$\begin{aligned} \sqrt{I}(x+y) &= \left(\bigvee_{n \in N} I((x+y)^n) \right) \wedge A(x+y) \\ &\geq \left(\bigvee_{n \in N} (I(x^n) \wedge I(y^n)) \right) \wedge A(x) \wedge A(y) \\ &\geq \left(\bigvee_{n \in N} I(x^n) \right) \wedge \left(\bigvee_{n \in N} I(y^n) \right) \wedge A(x) \wedge A(y) \\ &\geq \left(\bigvee_{n \in N} I(x^n) \right) \wedge A(x) \wedge \left(\bigvee_{n \in N} I(y^n) \right) \wedge A(y) \\ &\geq \sqrt{I}(x) \wedge \sqrt{I}(y). \end{aligned}$$

Since $(xy)^n = x^n y^n$, we have

$$\begin{aligned}
 \sqrt{I}(xy) &= \left(\bigvee_{n \in N} I((xy)^n) \right) \wedge A(xy) \\
 &\geq \left(\bigvee_{n \in N} I(x^n y^n) \right) \wedge A(xy) \\
 &\geq \left(\bigvee_{n \in N} (A(x^n) \wedge I(y^n)) \right) \wedge A(x) \wedge A(y) \\
 &\geq \left(\bigvee_{n \in N} A(x^n) \right) \wedge \left(\bigvee_{n \in N} I(y^n) \right) \wedge A(x) \wedge A(y) \\
 &\geq \left(\bigvee_{n \in N} A(x^n) \right) \wedge A(x) \wedge \left(\bigvee_{n \in N} I(y^n) \right) \wedge A(y) \\
 &\geq A(x) \wedge \left(\bigvee_{n \in N} I(y^n) \right) \wedge A(y) \\
 &\geq A(x) \wedge \sqrt{I}(y).
 \end{aligned}$$

Since $I(x) = I(-x)$, it follows that $\sqrt{I}(x) = \sqrt{I}(-x)$.

$$\begin{aligned}
 \sqrt{I}(x) &= \left(\bigvee_{n \in N} I(x^n) \right) \wedge A(x) \\
 &\leq A(x) \text{ for all } x \text{ in } R.
 \end{aligned}$$

□

Proposition 2.13. If I is semiprimary, then \sqrt{I} is a L -fuzzy prime ideal.

Proof. Let $x, y \in R$. We have that

$$\begin{aligned}
 \sqrt{I}(xy) \wedge A(x) \wedge A(y) &= \left(\bigvee_{n \in N} (I(xy)^n) \right) \wedge A(xy) \wedge A(x) \wedge A(y) \\
 &= \bigvee_{n \in N} (I(x^n) I(y^{nm})) \wedge A(xy) \wedge A(x) \wedge A(y), \\
 &\quad \text{when } m \text{ depends on } n. \\
 &\leq \left(\bigvee_{n \in N} I(x^n) \right) \vee \left(\bigvee_{n \in N} I(y^m) \right) \wedge A(x) \wedge A(y) \\
 &\leq \sqrt{I}(x) \vee \sqrt{I}(y).
 \end{aligned}$$

Hence \sqrt{I} is a L -fuzzy prime ideal.

□

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