A note on M-groups

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Abstract

Every finite solvable group is only a subgroup of an M-group and all M-groups are solvable. Supersolvable group is an M-group and also subgroups of solvable or supersolvable groups are solvable or supersolvable. But a subgroup of an M-group need not be an M-group. It has been studied that whether a normal subgroup or Hall subgroup of an M-group is an M-group or not. In this note, we investigate some historical research background on the M-groups and also we give some conditions that a normal subgroup of an M-group is an M-group and show that a solvable group is an M-group.

0. Introduction

An irreducible complex character χ of a finite group G is monomial if it is induced from a linear(i.e. degree 1) character of some subgroup of G. A finite group G is M-group if all its irreducible characters are monomial. Let $\mathrm{Irr}(G)$ be the set of all irreducible complex character of a finite group G.

One of the remaining mysteries about M-group is whether of not normal subgroups of odd M-groups must, themselves, be M-groups. In [3], Dade constructed an example of an M-group of order $2^9 \cdot 7$ which has a non M-normal subgroup of index 2. A normal subgroup of an M-group must not be an M-group. I. Chubarov[1] proved that odd normal subgroups of M-groups are M-groups. Let G be an M-group and suppose $N \lhd G$. If N is an M-group then all of its primitive characters are linear. The converse of this statement is easily seen to be false.

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In [7], if G is an M-group and $N \triangleleft G$ with either $|N|$ or $|G:N|$ odd, then N is an M-group. In [6], Isaacs proved that if G is an M-group and suppose $S \triangleleft \triangleleft G$ is a subnormal subgroup of odd index then every primitive character of S is linear. Two are still the main problems on M-groups; are Hall subgroups of M-groups M-group? Under certain addness hypothesis are normal subgroups of M-groups M-group? In both cases there is evidence that this could be the case: the primitive characters of the subgroups in question are the linear characters.

Recently, some idea appears to take form. In [13], T. Okuyama proved that if G is an M-group and P is a Sylow p-subgroup of G, then $N_G(P)/P$ is an M-group. In [8]. M. Isaacs showed that if H is a Hall subgroup of an M-group then $N_G(H)/H'$ is also an M-group. In [12], G. Navarro proved that if H is a Hall subgroup of an M-group G and $\varphi \in \mathrm{Irr}(N_G(H))$ is primitive then φ is linear. In [10], M. Lewis proved that if H is a maximal subgroup of an M-group G such that $|G:H|$ is odd, and $\varphi \in \mathrm{Irr}(H)$ is primitive then $\varphi(1)^2$ divides $|G:H|$. In [10], he proved that if S is a subgroup of an M-group of G that is reachable by primes, H is a Hall subgroup of S and $\varphi \in \mathrm{Irr}(H)$ is primitive, then $\varphi(1)$ is a power of 2. Furthermore, if $|G:S|$ is odd, then $\varphi(1)=1$.

Recall that M-groups are necessarily solvable(Takeda, [16]).

A group G is said to be *supersolvable* if there is a normal subgroup series

$$G = G_1 \supseteq G_2 \supseteq \cdots \supseteq G_n = 1$$

with cyclic factor of prime order where each $G_i \triangleleft G$.

Since supersolvable groups are M-group[3], we have

$$\{\text{nilpotents}\} \subset \{\text{supersolvables}\} \subset \{M\text{-groups}\} \subset \{\text{solvables}\}.$$

The following remarks are clear([4], [14], [15]).

1. A subgroup of a supersolvable group is supersolvable.
2. Any factor group of a supersolvable group is also supersolvable.
3. A minimal normal subgroup of a supersolvable group is of prime order.
4. The index of maximal subgroup of a supersolvable group is a prime order.

Let $\mathrm{Irr}(G/\theta)$ be the set of all irreducible constituents of θ^G where θ^G is the induced characters of G for a character θ of normal subgroup N, and let for a character χ of G, χ_N be the restriction of χ to a normal subgroup N.

In this note, we show that under certain hypothesis the normal subgroup of M-group and the solvable group are M-groups.

1. Normal subgroups

Proposition 1. Let $N \lhd G$ and assume that G/N is solvable. If for $\chi \in \mathrm{Irr}(G)$, θ is an irreducible constituent of χ_N then $\chi(1)/\theta(1)$ divides $|G:N|$.

proof. We induct on $|G:N|$. If $|G:N|=1$ then $\chi=\theta$ and so $\chi(1)/\theta(1)=1$. Thus it is clear.

We assume that $N<G$. Let M is an maximal normal subgroup of G containing N. Since G/N is solvable, $|G:M|=p$ is prime.

Let $\varphi \in \mathrm{Irr}(M)$ be a constituent of χ_M such that θ is a constituent of φ_N.

By inductive hypothesis, $\varphi(1)/\theta(1)$ divides $|M:N|$.

Now we need $\chi(1)/\varphi(1)$ divides $|G:M|$. Hence since $|G:M|=p$. Thus we have either $\chi_M=\varphi$ is irreducible or $\chi_M=\sum_{i=1}^{p}\varphi_i$ [5].

If $\chi_M=\varphi$, then $\chi(1)=\varphi(1)$ and $\chi(1)/\varphi(1)=p$ divides $|G:M|$, otherwise $\chi(1)=p\varphi(1)$ and so $\chi(1)/\varphi(1)=p$ divides $|G:M|$. Hence the proof is complete.

Corollary 2. Let $N \lhd G$ and assume that G/N is solvable. Let $\chi \in \mathrm{Irr}(G)$ if $(\chi(1), |G:M|)=1$ then χ_N is irreducible.

proof. Let θ be an irreducible constituent of χ_N. Then by Proposition 1, $\chi(1)/\varphi(1)$ divides $|G:N|$.

Thus we have $\chi(1)/\theta(1)=1$ since $(\chi(1), |G:M|)=1$. So $\chi(1)=\varphi(1)$.

Thus $\theta_N=\theta$ is irreducible.

Theorem 3. Let G be an M-Group and suppose that $N \lhd G$ with $(|N|, |G:M|)=1$. Then N is an M-Group.

proof. Let $\theta \in \mathrm{Irr}(N)$ and let χ be an irreducible constituent of θ^G. Since G is an M-Group, χ is monomial. So $\chi=\lambda^G$ where $\lambda \in \mathrm{Irr}(N)$ is linear for some $H \subseteq G$.

Let $\varphi=\lambda^{NH}$. Then we have $\varphi^G=(\lambda^{NH})^G=\lambda^G=\chi \in \mathrm{Irr}(G)$.

Thus $\varphi \in \mathrm{Irr}(NH)$. Hence we obtain

$\varphi(1)=\lambda^{NH}(1)=|NH:H|\,\lambda(1)=|NH:H|=|N:N\cap H|$.

This divides $|N|$. Since $|N|$ is coprime to $|G:N|$, $(\varphi(1), |G:N|)=1$. But since $|NH:N|$ divides $|G:N|$, we have $(\varphi(1), |NH:N|)=1$. Note that M-Group is solvable(Takeda, [5]). Hence G is solvable. So NH/N is solvable. Thus by corollary 2, φ_N is irreducible.

But $\varphi_N=(\lambda^{NH})_N=(\lambda_{N\cap H})^N$. So φ_N is monomial. Since $\varphi^G=\chi$, by Frobenius

Reciprocity φ is a constituent of χ_{NH}. Thus φ_N is an irreducible constituent of $(\chi_{NH})_N = \chi_N$. Since φ is irreducible constituent of χ_N, by Clifford's theorem $\theta = (\varphi_N)^g$ for some $g \in G$. Hence θ is a monmial. The proof is now complete.

2. Characters of solvables

Theorem 4. Let $N \lhd G$ and suppose that G/N is supersolvable. Let $\chi \in \mathrm{Irr}(G)$. Then

(1) If χ_N is reducible then there exists a subgroup H with $N \subseteq H \subseteq G$ such that $|G:H|$ is prime and χ is induced from irreducible character of H.

(2) There exists a subgroup U with $N \subseteq U \subseteq G$ and a character $\varphi \in \mathrm{Irr}(U)$ such that $\psi^G = \chi$ and ψ_N is irreducible.

(3) If G is metabelian($G''=1$) then G is an M-group.

proof. (1) Let $L \subseteq G$ be maximal with $N \subseteq L \lhd G$ and χ_L is reducible. Then G/N is supersolvable. If we take $K \lhd G$ such that K/L is chief factor(K/L is minimal normal subgroup of G/L), then by the supersolvability of G/L, K/L is cyclic with order prime p and $\chi_K \in \mathrm{Irr}(G)$.

Since $(\chi_K)_L = \chi_L$ is reducible, we have

$\chi_L = \varphi_1 + \varphi_2 + \cdots + \varphi_p$

where $\varphi_i \in \mathrm{Irr}(L)$ are distinct [5].

On the other hand, $\chi_L = e\sum_{i=1}^{t} \theta_i$ where $\{\theta_1, \cdots, \theta_t\}$ is the conjugacy classes of $\theta = \theta_1$ via the action of G on $\mathrm{Irr}(G)$ and $t = |G:I_G(\theta)|$, where $I_G(\theta)$ is inertia group [5].

Hence we have $e=1$ and $t=p$. If $H=I_G(\theta)$, then $N \subseteq H \subseteq G$ and $|G:H| = t = p$ prime. Since $|\chi_L, \theta| = 1 \neq 0$, $\chi \in \mathrm{Irr}(G \mid \theta)$ and thus by Clifford's correspondence, χ is induced from some irreducible character of H.

(2) Let $U \subseteq G$ be minimal such that $N \subseteq H \subseteq G$ and χ is induced from some irreducible character of U. Let $\psi \in \mathrm{Irr}(U)$ such that $\psi^G = \chi$. Assume that ψ_N is reducible. Then by (1), there is a subgroup $V \subseteq U$ with $N \subseteq V \subseteq U$, $|U:V|$ is prime and $\psi = \theta^U$ for some $\theta \in \mathrm{Irr}(V)$. Thus we have $V \subseteq U$ and $\chi = \psi^G = (\theta^V)^G = \theta^G$ which contradicts to the minimality of U. Hence ψ_N is irreducible.

(3) Let $\chi \in \mathrm{Irr}(G)$, we have $G' \lhd G$ and G/G' is abelian. Thus by (2), there exists $U \subseteq G$ with $G' \subseteq U \subseteq G$ and $\psi \in \mathrm{Irr}(U)$ such that $\chi = \psi^G$ and $\psi_{G'} \in \mathrm{Irr}(G')$.

But by hypotheses $G''=1$, that is, G' is abelian.

Hence all irreducible characters are linear. In particular $\psi_{G'} = \lambda$ is linear. It follows that $\psi(1)=\psi_{G'}(1)=\lambda(1)=1$. Hence ψ itself was linear.

Note that $G'\subseteq U\subseteq G$ implies $U/G'\subseteq G/G'$ be abelian, so $U/G'\lhd G/G'$ and so $U\lhd G$ conclude that all $\chi\in\mathrm{Irr}(G)$ is induced from an irreducible character ψ of a normal subgroup $U\lhd G$. Thus G is M-group.

Lemma 5. Let $\chi\in\mathrm{Irr}(G)$ be primitive and $N\lhd G$. Then χ_N is homogeneous.

proof. Let θ be an irreducible constituent of χ_N and $T=I_G(\theta)$. Then there is $\psi\in\mathrm{Irr}(T\mid\theta)$ such that $\psi^G=\chi$. Primitivity of χ yields that $T=G$. Hence θ is invariant in G, so $\{\theta\}$ is a G-orbit in $\mathrm{Irr}(N)$ and thus θ is the only irreducible constituent of χ_N. Therefore χ_N is homogeneous.

Corollary 6. Let $\chi\in\mathrm{Irr}(G)$ be primitive and $A\lhd G$ is abelian. Then $A\subseteq Z(\chi)$.

proof. By Lemma 5, we have $\chi_A=e\lambda$, where $\lambda\in\mathrm{Irr}(A)$ is linear. Thus we obtain $e=\chi(1)$ and if $a\in A$ then

$|\chi(a)|=|\chi(1)\lambda(a)|=\chi(1)|\lambda(a)|=\chi(1)$,

hence $A\subseteq Z(\chi)$.

Corollary 7. Let $\chi\in\mathrm{Irr}(G)$ be primitive and $N=\mathrm{Ker}\,\chi$. Then every abelian normal subgroup of G/N is central and cyclic.

Proof. If $N=1$($\Leftrightarrow$ $\mathrm{Ker}\,\chi=1$ $\Leftrightarrow$ χ is faithful), then by Corollary 6, $A\subseteq Z(\chi)=Z(G)$. But $Z(\chi)$ is cyclic. Thus A is central and cyclic.

In general, let $A/N\lhd G/N$ and let A/N be abelian, then $A\lhd G$ and by Lemma 5, $\chi_A=e\theta$ for $\theta\in\mathrm{Irr}(A)$. Hence we have $\chi(1)=e\theta(1)$. If $n\in N$, then we get $\chi(1)=\chi(n)=e\theta(n)$ and thus we obtain $\theta(n)=\theta(1)$. Hence $N\subseteq\mathrm{Ker}\,\theta$. But θ comes from some irreducible character of A/N. Since A/N is abelian, θ is linear. Thus we have $\chi(a)=\chi(1)$ for $a\in A$, so $A\subseteq Z(\chi)$. But $Z(\chi)/N=Z(\chi)/\mathrm{Ker}\,\chi$ is central and cyclic in G/N [5]. Hence A/N is central and cyclic in G/N.

Theorem 8. Let G be a solvable. Suppose $N\lhd G$ such that G/N is supersolvable and every Sylow subgroup of N for all prime is abelian. Then

(1) There exists an abelian normal subgroup A of G such that $A=C_G(A)$.

(2) G is an M-group.

proof. (1) Let $A\lhd G$ be abelian and maximal with the property. Write $C=C_G(A)$. Then $A\subseteq C$. Assume that $A<C$. Then $C/A\lhd G/A$. Let M/A be minimal normal in G/A with $M/A\subseteq C/A$. Then $A\subseteq M\subseteq C$ and M/A is p-group, since G is solvable.

Case I. $M \subseteq NA$

$M = (M \cap N)A$, and also $M \cap N/A \cap N$ is p-group. Thus for some $S \in \mathrm{Sylow}(M \cap N)$, $M \cap N = S(A \cap N)$. Let $M = S(A \cap N)A = SA$. By hypothesis, S is abelian. Since $S \subseteq M \subseteq C = C_G(A)$ and A and S are abelian, $M = AS$ is abelian and also $M \lhd G$, $M > A$. Hence it contradicts to the maximality of A.

Case II. $M \not\subseteq NA$

$NA \cap M \lhd G$ and also $A \subseteq NA \cap M \subsetneq M$. By minimality of M/A, we have $NA \cap M = A$ and $NAM = NM$. Claim that NM/NA is minimal normal subgroup of G/NA. But G/NA is a homomorphic image of G/N. So it is supersolvable. It follows that $NM/NA \cong M/A$ has prime order and is hence cyclic. Thus $M = A\langle m \rangle$ for $m \in M$. Note that $\langle m \rangle \subseteq M \subseteq C_G(A)$ and $\langle m \rangle$, A are abelian. Hence M is abelian which contradicts to the maximality of A. Therefore, $A = C = C_G(A)$ and the proof is complete.

(2) Let $\chi \in \mathrm{Irr}(G)$ for a group G. Then there is $N \subseteq G$ such that for some $\psi \in \mathrm{Irr}(N)$, $\psi^G = \chi$ and ψ is primitive. But N is a subgroup of G with the hypothesis. We put $K = \mathrm{Ker}\,\psi$. Then N/K satisfies the hypothesis. Hence N/K has the property that all of its abelian normal subgroup are central and cyclic. By (1), N/K is abelian. Since $K = \mathrm{Ker}\,\psi$, ψ comes from an irreducible character of the abelian group N/K and thus $\psi(1) = 1$.

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