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On Solutions of Representors in Reproducing Kernel Space $W_2^2(\mathbb{R})$*

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Abstract

In this article, we give a historical referencing overview and compressed illuminating procedure of deriving the repersentors $R_y(x)$ in Reproducing Kernel space $W_2^2(\mathbb{R})$, being needed to find the solutions of integral equations, which construct the wavelets in $L^2(\mathbb{R})$.

0. Historical Background and Introduction

Since in the mid-1980's, the wavelet era, the sensational first construction of a smooth orthonormal wavelet basis on $\mathbb{R}$ by Meyer [12] and on $\mathbb{R}^n$ by Lemarie and Meyer [10] was established, as is well known, orthonormal wavelet expansions are an attempt to improve on Fourier series and other classical expansions, wavelets, as a versatile tool with very rich mathematical content and great potential for applications, have recently provided a new and more efficient method for decomposing a function or signal, especially a convenient framework unifying various earlier ones.

For referencing brief overviews, Mallat and Meyer [11] placed the singular calculation within a general framework by formulating the notion of multiresolution analysis. Daubechies([6], [7]) used this approach to construct families of smoothness together with clarifying the relation between wavelets in the continuous context of $\mathbb{R}$ and wavelets in the discrete context of $\mathbf{Z}$, as needed in digital signal analysis.

And after, since the notion of integral wavelet transform $(W_\phi f)$ was first introduced

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by Grossmann and Morlet [8] by using the techniques based on the dyadic translations and integral dilations, it is generally accepted [1] that to research for more adaptable basic wavelets used to formulate the IWT and establishing the decomposing of elements in $L^2(\mathbb{R})$ in high resolution ratio remains with desired improvements.

Recently, by using the concept of the Reproducing Kernel space $W_2^2(\mathbb{R})$ in $L^2(\mathbb{R})$, Cui [4] has introduced a new approach to construct wavelets of two variables by means of eigenvalue problems of the homogeneous Fredholm linear integral equations of the second kind, for $\lambda=2$; $\phi(x)=\lambda\int_{\mathbf{R}} h(2x-y)\phi(y)dy$, the solution of which constructs wavelets in $L^2(\mathbb{R})$.

In this regard, we have observed that the solution of the equation is generated by the representor $R_y(x)$ in $W_2^2(\mathbb{R})$, and drawing new attention to construct wavelets $\psi(x)$ in $L^2(\mathbb{R})$ with better suited and more efficient approach in terms of representor $R_y(x)$ in $W_2^2(\mathbb{R})$ is devoted to carry out this procedure of the desired improvements.

For the Hilbert space theory with adjointness and the integral equations referred to [15] and wavelet analysis including Fourier analysis mainly to [1].

1. Reproducing Kernel Space $W_2^2(\mathbb{R})$

Throughout this paper, $L^2(\mathbb{R})$ will denote the Hilbert space of all Lebesgue square integrable functions on $\mathbb{R}$ with inner product $\langle f,g\rangle=\int_{\mathbf{R}} f(x)\overline{g(x)}\,dx$, and the norm $\|f\|=\left\{\int_{\mathbf{R}}|f(x)|^2dx\right\}^{1/2}$, $f,\ g\in L^2(\mathbb{R})$. The sign $\wedge$ and $\vee$ denote Fourier transform and the inversion, respectively. Operators mean bounded and linear.

Let us start with the definition of the subspace $W_2^1(\mathbb{R})$ of $L^2(\mathbb{R})$ needed in the main results as follows, by means of the properties of $W_2^1(\mathbb{R})$, we can easily establish the computations of eigenvalue problem:

Definition 1.1. $W_2^1(\mathbb{R})=\{u|\,u,u'\in L^2(\mathbb{R}),u(x)$: absolutey continuos for $\mathbb{R}\}$, where u' is differential of u. Obviously, the $W_2^1(\mathbb{R})$ becomes a Hilbert space[4] with inner product

$$\langle u, v\rangle_{L^2(\mathbb{R})} = \int_{\mathbb{R}} u(x)v(x)dx + \int_{\mathbb{R}} u'(x)v'(x)dx$$

Let us recall [4] that for given any $u \in W_2^1(\mathbb{R})$ and for all $y \in \mathbb{R}$, there exists two variable function $R_y(x)$ by Riesz-Representation theorem such that

$$u(y) = \langle u, R_y\rangle,$$

where $R_y(x) = \frac{1}{2} e^{-|x-y|}$, $x \in \mathbb{R}$. $R_y(x)$ is said to be reproducing kernel and $W_2^1(\mathbb{R})$ reproducing kernel space.

Definition 1.2. $W_2^2(\mathbb{R}) = \{u \mid u, u'$ and $u'' \in L^2(\mathbb{R}), u(x), u'(x)$: absolutely continuous for $\mathbb{R}\}$. Obviously, the $W_2^2(\mathbb{R})$ becomes a Hilbert space [4] with inner product

$$\langle u, v\rangle_{L^2(\mathbb{R})} = 4\int_{\mathbb{R}} u(x)v(x)dx + 5\int_{\mathbb{R}} u'(x)v'(x)dx + \int_{\mathbb{R}} u''(x)v''(x)dx. \quad (1)$$

Recall [4] that for given any $u \in W_2^2(\mathbb{R})$ and for all $y \in \mathbb{R}$, there exists two variable function $R_y(x)$ by Riesz-Representation theorem such that

$$u(y) = \langle u, R_y\rangle.$$

$R_y(x)$ is said to be reproducing kernel and $W_2^2(\mathbb{R})$ reproducing kernel space. It is well known [1] that good wavelets are usually constructed by means of multiresolution analysis.

Definitin 1.3. A multiresolution analysis is a sequence $(V_j)_{j\in Z}$ of norm-closed subspaces of $L^2(\mathbb{R})$ such that

i) $V_j \subset V_{j+1}$
ii) $u(x) \in V_j$ if and only if $u(2x) \in V_{j+1}$
iii) $u(x) \in V_0$ if and only if $u(x-k) \in V_0$
iv) $\overline{\bigcup_{j\in Z} V_j} = L^2(\mathbb{R})$
v) $\bigcap_{j\in Z} V_j = \{0\}$

vi) There exists a function $\phi \in V_0$, called a scaling function, such that the system $\{\phi(x-k)\}_{k\in Z}$ is a Riesz basis of V_0. That is, for all $u(x) \in V_0$, $u(x)$ have unique representation as follows: there exist C_k such that

$$u(x) = \sum_{k\in Z} C_k \phi(x-k).$$

Moreover, there exist constant A and B such that

$$A\|u\|^2_{L^2(\mathbb{R})} \leq \sum_{k\in\mathbb{Z}} |C_k|^2 \leq B\|u\|^2_{L^2(\mathbb{R})}.$$

As a result, a sequence $\{h_k\}$ exists such that the scaling function satisfies

$$\phi(x) = \sqrt{2} \sum_{k\in\mathbb{Z}} h_k \phi(2x-k). \quad (2)$$

This functional equations goes by several different names: the refinement equation, the dilation equation or the two-scale difference equation.

The following theorem clarifies the existence of the solution of the archetypical integral equation: $\phi(x) = \lambda \int_{\mathbb{R}} h(2x-y)\phi(y)dy$ for the case of $\lambda = 2$, by taking suitable function $h(x)$in $L^2(\mathbb{R})$:

Theorem 1.4. In case $\lambda = 2$, there exists a solution of $\phi(x) = \lambda \int_{\mathbb{R}} h(2x-y)\phi(y)dy$ for some $h(\cdot) \in L^2(\mathbb{R})$. (3)

Proof. Taking the Fourier transform of the given equation:

$$\begin{aligned}
\phi(x) &= \lambda \int_{\mathbb{R}} h(2x-y)\phi(y)dy \\
\hat{\phi}(\omega) &= \lambda \int_{\mathbb{R}} \{h[2(x-\tfrac{y}{2})]\}^{\wedge}\ \phi(y)dy \\
&= \frac{\lambda}{2}\hat{h}(\frac{\omega}{2}) \int_{\mathbb{R}} e^{-i\omega\frac{y}{2}} \phi(y)\,dy \\
&= \frac{\lambda}{2}\hat{h}(\frac{\omega}{2})\hat{\phi}(\frac{\omega}{2}),
\end{aligned}$$

where $\omega = (\omega_1, \omega_2)^T$ and

$$\hat{h}(\omega) = \frac{1}{2\pi} \int_{\mathbb{R}}\int_{\mathbb{R}} h(x_1, x_2) e^{-(\omega_1 x_1 + \omega_2 x_2)} dx_1 dx_2.$$

We take the following

$$\hat{h}(\omega) = \begin{cases} \frac{\lambda}{2} h(\omega)(= H(\omega), \lambda = 2), & \omega \in [-\pi, \pi] \\ 0, & \omega \notin [-\pi, \pi] \end{cases}$$

From the fact that the solution of the identity (2) exists, the identities (2) and (3) are equivalent, which completes the proof. □

2. Main Results

We now describe to construct the solution procedure of representor $R_y(x)$ in reproducin kernel space $W_2^2(\mathbb{R})$.

Theorem 2.1. Let $R_y(x)$ be any reproducing kernel in $W_2^2(\mathbb{R})$. Then, we have

$$R_y(x)=\frac{1}{6}e^{-|x-y|}-\frac{1}{12}e^{-2|x-y|}.$$

Proof. From the above discussion, we have

$$\begin{aligned}
u(y) &= \langle u, R_y\rangle \\
&= 4\int_{\mathbf{R}} u(x)R_y(x)dx+5\int_{\mathbf{R}} u'(x)R_y'(x)dx+\int_{\mathbf{R}} u''(x)R_y''(x)dx \\
&= 4\int_{\mathbf{R}} u(x)R_y(x)dx+[5u(x)R_y'(x)]_{\mathbf{R}}-5\int_{\mathbf{R}} u(x)R_y''(x)dx \\
&\quad +[u'(x)R_y''(x)]_{\mathbf{R}}-\int_{\mathbf{R}} u'(x)R_y'''(x)dx \\
&= 4\int_{\mathbf{R}} u(x)R_y(x)\,dx+[5u(x)R_y'(x)]_{\mathbf{R}}-5\int_{\mathbf{R}} u(x)R_y''(x)dx \\
&\quad +[u'(x)R_y''(x)]_{\mathbf{R}}-[u(x)R_y'''(x)]_{\mathbf{R}}+\int_{\mathbf{R}} u(x)R_y^{(4)}(x)dx \\
&= \int_{\mathbf{R}} u(x)[4R_y(x)-5R_y''(x)+R_y^{(4)}(x)]dx+[5u(x)R_y'(x)]_{\mathbf{R}} \\
&\quad +[u'(x)R_y''(x)]_{\mathbf{R}}-[u(x)R_y'''(x)]_{\mathbf{R}},
\end{aligned}$$

where by using $f(y)=\int_{\mathbf{R}} f(x)\delta(x-y)dx$, we have the followings

$$4R_y(x)-5R_y''(x)+R_y^{(4)}(x)=\delta(x-y) \tag{4}$$

$$[u(x)R_y'(x)]_{\mathbf{R}}=0 \tag{5}$$

$$[u'(x)R_y''(x)]_{\mathbf{R}}=0 \tag{6}$$

$$[u(x)R_y'''(x)]_{\mathbf{R}}=0 \tag{7}$$

Let $x\neq y$, then the solutions of auxiliary equation of (4): $\lambda^4-5\lambda^2+4=0$, obtained by $\lambda_1=1, \lambda_2=-1, \lambda_3=2, \lambda_4=-2$.

From these solutions we have the following relations:

$$R_y(x)=\begin{cases} a_1e^x+a_2e^{-x}+a_3e^{2x}+a_4e^{-2x} & (x>y) \\ b_1e^x+b_2e^{-x}+b_3e^{2x}+b_4e^{-2x} & (x<y)\end{cases}$$

Also, from the equation (4) and for $\varepsilon > 0$, we have

$$\int_{y-\varepsilon}^{y+\varepsilon} 4R_y(x) - 5R_y''(x) + R_y^{(4)}(x)dx = \int_{y-\varepsilon}^{y+\varepsilon} \delta(x-y)dy = 1 \tag{8}$$

and from (8) as $\varepsilon \to 0$, the following yields

$$R_y'''(y+0) - R_y'''(y-0) = 1$$

$$\text{i.e.,}\ (-a_2 - 8a_4) - (b_2 + 8b_4) = 1$$

By the equation (5), (6) and (7)

$$\begin{aligned} a_2 + a_4 &= b_1 + b_3 \\ -a_2 - 2a_4 &= b_1 + 2b_3 \\ a_2 + 4a_4 &= b_1 + 4b_3 \end{aligned}$$

are easily obtained.

Consequently, we have $a_2 = \frac{1}{6}, a_4 = -\frac{1}{12}, b_1 = \frac{1}{6}, b_3 = -\frac{1}{12}$.

Hence,

$$R_y(x) = \begin{cases} \frac{1}{6} e^{-x+y} - \frac{1}{12} e^{-2(x-2y)} & (x > y) \\ \frac{1}{6} e^{x-y} - \frac{1}{12} e^{2(x-y)} & (x < y) \end{cases}$$

is obtained.

Therefore

$$R_y(x) = \frac{1}{6} e^{-|x-y|} - \frac{1}{12} e^{-2|x-y|}, \tag{9}$$

which completes the proof. □

We now conclude with the following not surprising characteristics of reproducing kernel spaces.

Theorem 2.2. Let $L: W_2^2(\mathbb{R}) \to W_2^1(\mathbb{R})$ be a bounded linear operator defined by

$$Lu(x) = \lambda \int_{\mathbf{R}} k(x, y)u(y)dy + \alpha u'(x) + \beta u(x).$$

Then we have $L^* v(x) = \langle LR_y, v \rangle$.

Proof. As before,

$$\begin{aligned} u(y) &= \langle u, R_y \rangle \\ &= 4\int_{\mathbf{R}} u(x)R_y(x)dx + 5\int_{\mathbf{R}} u'(x)R_y'(x)dx + \int_{\mathbf{R}} u''(x)R_y''(x)dx \end{aligned}$$

$$
\begin{aligned}
&Lu(y)\\
&= \lambda \int_{\mathbb{R}} k(y,s)[4\int_{\mathbb{R}} u(x)R_s(x)dx + 5\int_{\mathbb{R}} u'(x)R_s'(x)dx + \int_{\mathbb{R}} u''(x)R_y''(x)dx]dx\\
&\quad + \alpha \frac{d}{dy}[4\int_{\mathbb{R}} u(x)R_y(x)dx + 5\int_{\mathbb{R}} u'(x)R_y'(x)dx + \int_{\mathbb{R}} u''(x)R_y''(x)dx]\\
&\quad + \beta[4\int_{\mathbb{R}} u(x)R_y(x)dx + 5\int_{\mathbb{R}} u'(x)R_y'(x)dx + \int_{\mathbb{R}} u''(x)R_y''(x)dx]\\
&= 4\int_{\mathbb{R}} u(x)[\lambda\int_{\mathbb{R}} k(y,s)R_s(x)ds + \alpha R_y'(x) + \beta R_y(x)]dx\\
&\quad +5\int_{\mathbb{R}} u'(x)\frac{d}{dx}[\lambda\int_{\mathbb{R}} k(y,s)R_s(y)\,ds + \alpha\frac{d}{dy}R_y(x) + \beta R_y(x)]dx\\
&\quad + \int_{\mathbb{R}} u''(x)\frac{d^2}{dx^2}[\lambda\int_{\mathbb{R}} k(y,s)R_s(x)\,ds + \alpha\frac{d}{dy}R_y(x) + \beta R_y(x)]dx\\
&= 4\int_{\mathbb{R}} u(x)LR_y(x)\,dx + 5\int_{\mathbb{R}} u'(x)\frac{d}{dx}[LR_y(x)]dx + \int_{\mathbb{R}} u''(x)\frac{d^2}{dx^2}[LR_y(x)]dx\\
&= \langle u, LR_y\rangle
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
\langle Lu, v\rangle &= \int_{\mathbb{R}} v(x)\langle u, LR_y\rangle\,dx + \int_{\mathbb{R}} v'(x)\frac{d}{dy}\langle u, LR_y\rangle\,dy\\
&= \int_{\mathbb{R}} v(x)[4\int_{\mathbb{R}}\langle u, LR_y\rangle\,dx + 5\int_{\mathbb{R}} u'(x)\frac{d}{dx}[LR_y]dx\\
&\quad + \int_{\mathbb{R}} u'(y)\frac{d^2}{dx^2}[LR_y(x)]dx]dx + \int_{\mathbb{R}} v'(y)\frac{d}{dy}[4\int_{\mathbb{R}} u(x)LR_y(x)dx\\
&\quad +5\int_{\mathbb{R}} u'(x)\frac{d}{dx}[LR_y(x)]dx + \int_{\mathbb{R}} u''(x)\frac{d^2}{dx^2}[LR_y(x)dy]dx]\,dy\\
&= 4\int_{\mathbb{R}} u(x)[\int_{\mathbb{R}} v(x)LR_y(x)dy + \int_{\mathbb{R}} v'(x)\frac{d}{dy}LR_y(x)]dx\\
&\quad +5\int_{\mathbb{R}u'(x)}\frac{d}{dx}[\int_{\mathbb{R}} v(x)LR_y(x)dy + \int_{\mathbb{R}} v'(x)\frac{d}{dx}[LR_y(x)]dy]dx\\
&\quad + \int_{\mathbb{R}} u''(x)\frac{d^2}{dx^2}[\int_{\mathbb{R}} v(x)LR_y(x)dx + \int_{\mathbb{R}} v'(x)\frac{d}{dy}u'(x)\frac{d}{dy}R_y(x)dy]dx\\
&= 4\int_{\mathbb{R}} u(x)\langle LR_y, v\rangle\,dx + 5\int_{\mathbb{R}} u'(x)\frac{d}{dx}\langle LR_y, v\rangle\,dx\\
&\quad + \int_{\mathbb{R}} u''(x)\frac{d^2}{dx^2}\langle LR_y, v\rangle\,dx
\end{aligned}
$$

Therefore $L^*v(x) = \langle LR_y, v\rangle$, which is desired. □

Remark: By employing the representor $R_y(x)$ in $W_2^2(\mathbb{R})$ just described above, we construct the solution of our integral equation, the solution of which, may construct the wavelet function in $L^2(\mathbb{R})$.

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