⊠연구논문

Hypotheses Testing for the Shape Parameter of the Weibull Lifetime Data

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Abstract

In this paper, we address the Bayesian hypotheses testing for the shape parameter of weibull model. In Bayesian testing problem, conventional Bayes factors can not typically accommodate the use of noninformative priors which are improper and are defined only up to arbitrary constants. To overcome such problem, we use the recently proposed hypotheses testing criterion called the intrinsic Bayes factor. We derive the arithmetic and median intrinsic Bayes factors and use these results to analyze real data sets.

1. Introduction

For lifetime studies, Weibull model is perhaps the most widely used lifetime model. Its application in connection with lifetimes of many types of manufactured items has been widely advocated, and it has been used as a model with diverse types of items such as vacuum tubes, ball bearings and electrical insulation.

Statistical hypothesis testing for the parameters α and β for Weibull model has been considered in the literature from a frequentist viewpoint. Hypotheses about the shape parameter β often have interpretations that the hazard function is monotone increasing for $\beta>1$, decreasing for $\beta<1$, and constant for $\beta=1$. Thus the hypotheses $H_1: \beta=1$, $H_2: \beta>1$ and $H_3: \beta<1$ mean that the hazard function of Weibull model is constant, increasing and decreasing over time.

In this paper, we use a Bayesian approach to the test of shape parameter of

Weibull model using reference priors. In Bayesian testing problem, the Bayes factor under proper priors or informative priors have been very successful. However, limited information and time constraints often require the use of noninformative priors. Since noninformative priors such as Jeffrey's(1961) priors or reference priors(Berger and Bernardo(1989,1992)) are typically improper so that such priors are only defined up to arbitrary constants which affects the values of Bayes factors. Geisser and Eddy(1979), Spiegalhalter and Smith(1982), San Martini and Spezzaferri(1984) and O'Hagan(1995) et.al. have made efforts to compensate for that arbitrariness.

Berger and Pericchi(1996b) introduced a new model selection and hypotheses testing criterion, called the Intrinsic Bayes Factor(IBF) using a data-splitting idea, which would eliminate the arbitrariness of improper priors. This approach has shown to be quite useful (Berger and Pericchi(1996a), Varshavsky(1996) and Lingham and Sivaganesan (1997)).

This paper is arranged as follows. In Section 2, we introduce the intrinsic Bayes factor. In Section 3, we derive intrinsic Bayes factors to solve our problem. Finally, we give some numerical results with real data analysis to illustrate our results.

2. The Intrinsic Bayes Factor Methodology

In this Section, we firstly introduce the intrinsic Bayes factor in the general hypotheses testing. As a matter of convenience, we introduce the following notations.

$\boldsymbol{X}=(X_1,\cdots,X_n)$: observation with density $f(\boldsymbol{x}\mid\theta)$, where $\theta\in\Theta$ is a finite dimensional parameter and Θ is parameter space.

Θ_i : parameter space under ith hypothesis H_i , $i=1,2,\cdots,q$.

$\pi_i(\theta)$: the prior distribution under H_i , $i=1,2,\cdots,q$.

$m_i(\boldsymbol{x})$: the marginal density of $\boldsymbol{X}$ under H_i when use $\pi_i(\theta)$, $i=1,2,\cdots,q$.

p_i : the prior probability of H_i being true, $i=1,2,\cdots,q$.

$\pi_i^N(\theta)$: the improper prior distribution under H_i , $i=1,2,\cdots,q$.

$m_i^N(\boldsymbol{x})$: the marginal density of $\boldsymbol{X}$ under H_i when use $\pi_i^N(\theta)$, $i=1,2,\cdots,q$.

Then the posterior probability that H_i is true is given as

$$P(H_i \mid \boldsymbol{x}) = \left(\sum_{j=1}^{q} \frac{p_j}{p_i} B_{ji}\right)^{-1}, \tag{1}$$

where B_{ji}, the Bayes factor of H_j to H_i, is defined by

$$B_{ji} = \frac{m_j(\boldsymbol{x})}{m_i(\boldsymbol{x})} = \frac{\int_{\Theta_j} f(\boldsymbol{x} \mid \theta)\pi_j(\theta)d\theta}{\int_{\Theta_i} f(\boldsymbol{x} \mid \theta)\pi_i(\theta)d\theta}. \tag{2}$$

The posterior probabilities in (1) are then used to select the most plausible hypothesis.
If one were to use some noninformative priors, then (2) becomes

$$B_{ji}^N = \frac{m_j^N(\boldsymbol{x})}{m_i^N(\boldsymbol{x})} = \frac{\int_{\Theta_j} f(\boldsymbol{x} \mid \theta)\pi_j^N(\theta)d\theta}{\int_{\Theta_i} f(\boldsymbol{x} \mid \theta)\pi_i^N(\theta)d\theta}. \tag{3}$$

Being usually improper prior, π_i^N are defined only up to arbitrary constants.

Hence, the corresponding Bayes factor, B_{ji}^N, is indeterminate. One solution to this indeterminancy problem, due to Berger and Pericchi(1996), begins with the assumption that we can split the data vector $\boldsymbol{x}$ into $\boldsymbol{x}(l)$, the so called *training sample*, and the remainder of the data $\boldsymbol{x}(-l)$, such that

$$0 < m_i^N(\boldsymbol{x}(l)) < \infty, \quad i = 1, \cdots, q. \tag{4}$$

In view (4), the posteriors $\pi_i^N(\theta \mid \boldsymbol{x}(l))$ are well defined. Now, consider the Bayes factor, $B_{ji}(l)$, for the rest of the data $\boldsymbol{x}(-l)$, using $\pi_i^N(\theta \mid \boldsymbol{x}(l))$ as the priors:

$$B_{ji}(l) = \frac{\int_{\Theta_j} f(\boldsymbol{x}(-l) \mid \theta, \boldsymbol{x}(l))\pi_j^N(\theta \mid \boldsymbol{x}(l))d\theta}{\int_{\Theta_i} f(\boldsymbol{x}(-l) \mid \theta, \boldsymbol{x}(l))\pi_i^N(\theta \mid \boldsymbol{x}(l))d\theta} = B_{ji}^N \times B_{ij}^N(\boldsymbol{x}(l)) \tag{5}$$

where B^N_{ji} is given by (3) and

$$B^N_{ij}(\boldsymbol{x}(l)) = \frac{m^N_i(\boldsymbol{x}(l))}{m^N_j(\boldsymbol{x}(l))}. \tag{6}$$

In (5), any arbitrary ratio, c_j/c_i say, that multiples B^N_{ji} would be cancelled by the ratio c_i/c_j forming the multiplicand in $B^N_{ij}(\boldsymbol{x}(l))$. Also, while the expression (6) renders $B_{ji}(l)$ in terms of the simpler marginal densties of $\boldsymbol{x}(l)$.

As training samples play a fundamental role in our testing H_i , $i=1,\cdots,q$, we will need.

Definition 1.(Berger and Pericchi(1996b)) A training sample $\boldsymbol{x}(l)$, will be called *proper* if (4) holds and *minimal* if it is proper and none of its subsets is proper.

Berger and Pericchi(1996b) advocated various summaries based on $B_{ji}(l)$'s in (5) from many training samples to test H_i, $i=1,\cdots,q$. Generically termed the Intrinsic Bayes Factor(IBF), summaries is given by

Definition 2.(Berger and Pericchi(1996b)) The Arithmetic Intrinsic Bayes factor of H_j to H_i is

$$B^{AI}_{ji} = B^N_{ji} \cdot \frac{1}{L} \sum_{l=1}^{L} B^N_{ij}(\boldsymbol{x}(l)). \tag{7}$$

where L is the number of all possible minimal training samples.

One can calculate the posterior probability of H_i using (1), where B_{ji} is replaced by B^{AI}_{ji} from (7). Other versions of the IBF are available(Berger and Pericchi(1996b)).

3. Intrinsic Bayes Factor for Weibull Lifetime Data

The Weibull model with parameters α and β is given by

$$f(x \mid \alpha, \beta) = \frac{\beta}{\alpha} \left(\frac{x}{\alpha} \right)^{\beta-1} \exp\left(- \left(\frac{x}{\alpha} \right)^{\beta} \right), \tag{8}$$

where $x \geq 0$, and $\alpha > 0$ and $\beta > 0$ are parameters refered to as the scale and shape parameters of the distribution, respectively. We are interested in test of the shape parameter of Weibull data. Consider samples of sizes n from weibull populations with parameters α and β. Then the observed sample consists of the failure times $x_1, \cdots, x_n$. We want to test the hypotheses of $H_1 : \beta = 1$, $H_2 : \beta > 1$ and $H_3 : 0 < \beta < 1$. Since this problem is nonnested hypothesis, it is often no clear which hypothesis is more comlex. We therefore introduce the *encompassing hypothesis* $H_0 : \beta > 0$. This idea is to formulate a hypothesis which is more complex than the hypotheses being tested(See Beger and Pericchi(1996b)).

Suppose that H_i's, $i = 1, \cdots, q$ are nonnested hypotheses. Suppose also that H_i is nested in a certain hypothesis, H_0 say, so that H_0 is encompassing hypothesis. Assume further that $\pi_0(\theta)$ is the prior for θ under H_0. Then we have

Definition 3.(Beger and Pericchi(1996b)). The Encompassing Arithmetic Intrinsic Bayes factor of H_j to H_i is

$$B_{ji}^{0AI} = \frac{B_{0i}^{AI}}{B_{0j}^{AI}} = B_{ji}^{N} \cdot \frac{\overline{B_{i0}^{N}}}{\overline{B_{j0}^{N}}}, \tag{9}$$

where $\overline{B_{ij}^{N}} = \frac{1}{L} \sum_{l=1}^{L} B_{ij}^{N}(\boldsymbol{x}(l))$.

3.1 Minimal Training Sample

The goal here is to determine the set of all possible MTS's for the data $\boldsymbol{x} = (x_1, \cdots, x_n)$. To this end, we use Definition 1 and the reference priors $\pi_i^N(\theta), i = 0,1,2,3$, say, corresponding respectively to $H_0 : \beta > 0$, $H_1 : \beta = 1$, $H_2 : \beta > 1$ and $H_3 : 0 < \beta < 1$. The reference priors for $H_i, i = 0, 1, 2, 3$ are respectively given by

$$\pi_0^N(\alpha, \beta) = \frac{1}{\alpha\beta} I(\alpha > 0, \ \beta > 0), \tag{10}$$

$$\pi_1^N(\beta) = \frac{1}{\alpha} I(\alpha > 0), \tag{11}$$

$$\pi_2^N(\alpha, \beta) = \frac{1}{\alpha\beta} I(\alpha > 0,\ \beta > 1), \tag{12}$$

$$\pi_3^N(\alpha, \beta) = \frac{1}{\alpha\beta} I(\alpha > 0,\ 0 < \beta < 1), \tag{13}$$

where $I(A)$ means the indicator function of A for any set A.

We now derive the marginals with respect to the reference priors given by (10) to (13). For this, we first observe that the joint pdf of $(X_1, \cdots, X_n)$ is given by

$$f(\boldsymbol{x} \mid \alpha, \beta) = \left(\frac{\beta}{\alpha}\right)^n \left[\prod_{i=1}^n \left(\frac{x_i}{\alpha}\right)\right]^{\beta-1} \exp\left(-\sum_{i=1}^n \left(\frac{x_i}{\alpha}\right)^\beta\right). \tag{14}$$

Moreover, the joint pdf of $(X_k, X_l), 1 \le k < l \le n$, is given by

$$f(x_k, x_l \mid \alpha, \beta) = \left(\frac{\beta}{\alpha}\right)^2 \left(\frac{x_k x_l}{\alpha^2}\right)^{\beta-1} \exp\left[-\left(\left(\frac{x_k}{\alpha}\right)^\beta + \left(\frac{x_l}{\alpha}\right)^\beta\right)\right]. \tag{15}$$

Now, we introduce some notation for the marginals that we will use. For $i = 0, 1, 2, 3$, let $m_i^N(x_k, x_l)$ and $m_i^N(\boldsymbol{x})$ be the marginal densities of (X_k, X_l) and $\boldsymbol{X}$ under the hypothesis H_i, respectively. In the following lemma, we give the marginal densities for any two observations.

Lemma 1. We have the marginal density $m_i^N(x_k, x_l)$ under $H_i, i = 0, 1, 2, 3$ as follows.

$$m_0^N(x_k, x_l) = \frac{1}{2 x_k x_l \log(x_l / x_k)}, \tag{16}$$

$$m_1^N(x_k, x_l) = \frac{1}{(x_k + x_l)^2}, \tag{17}$$

$$m_2^N(x_k, x_l) = \frac{1}{x_l (x_k + x_l) \log(x_l / x_k)}, \tag{18}$$

$$m_3^N(x_k, x_l) = \frac{1}{x_k x_l \log(x_l/x_k)}\left(\frac{1}{2} - \frac{x_k}{(x_k+x_l)}\right),$$

(19)

where $1 \le k < l \le n$.

Proof. We firstly derive the marginal density $m_0(x_k, x_l)$ under H_0.

$$\begin{aligned} m_0(x_k, x_l) &= \int_0^\infty \int_0^\infty f(x_k, x_l \mid \alpha, \beta)\pi_0(\alpha, \beta) d\alpha d\beta \\ &= \int_0^\infty \int_0^\infty \left(\frac{\beta}{\alpha}\right)^2 \left(\frac{x_k x_l}{\alpha^2}\right)^{\beta-1} \exp\left[-\left(\left(\frac{x_k}{\alpha}\right)^\beta\right) + \left(\frac{x_l}{\alpha}\right)^\beta\right)\right] \frac{1}{\alpha\beta} d\alpha d\beta \\ &= \int_0^\infty \frac{(x_k x_l)^{\beta-1}}{(x_k^\beta + x_l^\beta)^2} d\beta \\ &= \frac{1}{2 x_k x_l \log(x_l/x_k)} . \end{aligned}$$

Secondly compute the marginal density $m_1(x_k, x_l)$ under H_1.

$$\begin{aligned} m_1(x_k, x_l) &= \int_0^\infty f(x_k, x_l \mid \alpha)\pi_1(\alpha) d\alpha \\ &= \int_0^\infty \left(\frac{1}{\alpha}\right)^2 \exp\left(-\left(\frac{x_k}{\alpha} + \frac{x_l}{\alpha}\right)\right)\frac{1}{\alpha} d\alpha \\ &= \frac{1}{(x_k + x_l)^2} . \end{aligned}$$

Finally $m_2(x_k, x_l)$ and $m_3(x_k, x_l)$ are easily computed by changing the range of integration on β, to $(1, \infty)$ and $(0, 1)$ in $m_0(x_k, x_l)$, respectively.

It is clear from the above that the marginal density of (X_k, X_l) is finite for all $1 \le k < l \le n$ under each hypothesis, and hence we conclude that any training sample of size two is an MTS.

3.2 Arithmetic and Median Bayes Factors

The marginal densities corresponding to the full data $\boldsymbol{X}$ can also be expressed in the following lemma.

Lemma 2. For the full data, we have the marginal density $m_i^N(\boldsymbol{x})$ under $H_i, i=0,1,2,3$ as follows.

$$m_0^N(\boldsymbol{x}) = \Gamma(n)A_0[\prod_{i=1}^n x_i]^{-1}, \tag{20}$$

$$m_1^N(\boldsymbol{x}) = \Gamma(n)[\sum_{i=1}^n x_i]^{-n}, \tag{21}$$

$$m_2^N(\boldsymbol{x}) = \Gamma(n)A_2[\prod_{i=1}^n x_i]^{-1}, \tag{22}$$

$$m_3^N(\boldsymbol{x}) = \Gamma(n)A_3[\prod_{i=1}^n x_i]^{-1}, \tag{23}$$

where

$$A_0 = \int_0^\infty \beta^{n-2}[\prod_{i=1}^n x_i]^\beta[\sum_{i=1}^n x_i^\beta]^{-n}d\beta,$$

$$A_2 = \int_1^\infty \beta^{n-2}[\prod_{i=1}^n x_i]^\beta[\sum_{i=1}^n x_i^\beta]^{-n}d\beta,$$

$$A_3 = \int_0^1 \beta^{n-2}[\prod_{i=1}^n x_i]^\beta[\sum_{i=1}^n x_i^\beta]^{-n}d\beta.$$

Proof. We firstly derive the marginal density $m_0(\boldsymbol{x})$.

$$\begin{aligned} m_0(\boldsymbol{x}) &= \int_0^\infty\int_0^\infty f(\boldsymbol{x} \mid \alpha,\beta)\pi_0(\alpha,\beta)d\alpha d\beta \\ &= \int_0^\infty\int_0^\infty (\frac{\beta}{\alpha})^n[\prod_{i=1}^n(\frac{x_i}{\alpha})]^{\beta-1}\exp[-\sum_{i=1}^n(\frac{x_i}{\alpha})^\beta]\frac{1}{\alpha\beta}d\alpha d\beta \\ &= \Gamma(n)[\prod_{i=1}^n x_i]^{-1}\int_0^\infty \beta^{n-2}[\prod_{i=1}^n x_i]^\beta[\sum_{i=1}^n x_i^\beta]^{-n}d\beta \\ &= \Gamma(n)A_0[\prod_{i=1}^n x_i]^{-1}, \end{aligned}$$

where $A_0 = \int_0^\infty \beta^{n-2}[\prod_{i=1}^n x_i]^\beta[\sum_{i=1}^n x_i^\beta]^{-n}d\beta.$

Secondly compute the marginal density $m_1(\boldsymbol{x})$.

$$
\begin{aligned}
m_1(\boldsymbol{x}) &= \int_0^\infty f(\boldsymbol{x} \mid \alpha)\pi_1(\alpha)\, d\alpha \\
&= \int_0^\infty \left(\frac{1}{\alpha}\right)^n \exp\left[-\left(\sum_{i=1}^n \frac{x_i}{\alpha}\right)\right]\frac{1}{\alpha}\, d\alpha \\
&= \Gamma(n)\left[\sum_{i=1}^n x_i\right]^{-n}.
\end{aligned}
$$

Finally $m_2(\boldsymbol{x})$ and $m_3(\boldsymbol{x})$ are easily computed by changing the range of integration on β in $m_0(\boldsymbol{x})$.

Now, we give the expressions for the Bayes factors. In lines with the notation in Section 2, we let $B_{10}^N(\boldsymbol{x}(l))$, $B_{20}^N(\boldsymbol{x}(l))$, $B_{30}^N(\boldsymbol{x}(l))$ represent the Bayes factors computed using the MTS, $\boldsymbol{x}(l) = (x_k, x_l)$ and let $B_{01}^N, B_{02}^N, B_{03}^N$ represent the Bayes factor computed using the full data. Thus we get the following theorem from Lemmas 1 and 2.

Theorem 1. (i) The Bayes factor computed using the full data is given by

$$
B_{01}^N = \frac{A_0\left[\prod_{i=1}^n x_i\right]^{-1}}{\left[\sum_{i=1}^n x_i\right]^{-n}}, \tag{24}
$$

$$
B_{02}^N = \frac{A_0}{A_2}, \tag{25}
$$

$$
B_{03}^N = \frac{A_0}{A_3}. \tag{26}
$$

(ii) The Bayes factor computed using the $\boldsymbol{x}(l) = (x_k, x_l)$ is given by

$$
B_{10}^N(\boldsymbol{x}(l)) = \frac{2x_k x_l \log(x_l/x_k)}{(x_k + x_l)^2}, \tag{27}
$$

$$
B_{20}^N(\boldsymbol{x}(l)) = \frac{2x_k}{x_k + x_l}, \tag{28}
$$

$$
B_{30}^N(\boldsymbol{x}(l)) = \left(1 - \frac{2x_k}{x_k + x_l}\right). \tag{29}
$$

From the Theorem 1, the encompassing arithmatic intrinsic Bayes factor B_{ji}^{AI} is given by

$$B_{12}^{0AI} = B_{12}^{N} \cdot \frac{\sum_l B_{20}^{N}(\boldsymbol{x}(l))}{\sum_l B_{10}^{N}(\boldsymbol{x}(l))}, \tag{30}$$

$$B_{13}^{0AI} = B_{13}^{N} \cdot \frac{\sum_l B_{30}^{N}(\boldsymbol{x}(l))}{\sum_l B_{10}^{N}(\boldsymbol{x}(l))}, \tag{31}$$

$$B_{23}^{0AI} = B_{23}^{N} \cdot \frac{\sum_l B_{30}^{N}(\boldsymbol{x}(l))}{\sum_l B_{20}^{N}(\boldsymbol{x}(l))}. \tag{32}$$

Next we use the another intrinsic Bayes factor, is called median intrinsic Bayes factor(Berger and Pericchi(1998)). They showed that the median intrinsic Bayes factor seems to be a simple and very generally applicable intrinsic Bayes factor, which works well for nested or nonnested models, and even for small or moderate sample sizes.

Definition 4.(Berger and Pericchi(1998)) The Median Intrinsic Bayes factor of H_j to H_i is

$$B_{ji}^{MI} = B_{ji}^{N} \cdot ME[B_{ij}^{N}(\boldsymbol{x}(l))], \tag{33}$$

where ME indicates the median, here to be taken over all the training sample Bayes factors.

From the Definition 4, Lemma 1, Lemma 2 and Theorem 1, we can dervie the median Bayes factors as follow:

$$B_{12}^{MI} = B_{12}^{N} \cdot ME[\frac{B_{20}^{N}(\boldsymbol{x}(l))}{B_{10}^{N}(\boldsymbol{x}(l))}], \tag{34}$$

$$B_{13}^{MI} = B_{13}^{N} \cdot ME[\frac{B_{30}^{N}(\boldsymbol{x}(l))}{B_{10}^{N}(\boldsymbol{x}(l))}], \tag{35}$$

$$B_{23}^{MI} = B_{23}^{N} \cdot ME[\frac{B_{30}^{N}(\boldsymbol{x}(l))}{B_{20}^{N}(\boldsymbol{x}(l))}]. \tag{36}$$

4. Illustrative Examples

In this section, some examples are presented to illustrate for our findings regarding the test $H_1 : \beta = 1,\ H_2 : \beta > 1,\ H_3 : 0 < \beta < 1$.

Example 1 : Consider the results of tests on the endurance of deep-groove ball bearings. The data, given by Lieblein and Zelen(1956), consists of a complete sample of size $n = 23$. The results of the test, in millions of revolutions before failure, are as follow.

Ball Bearing	17.88, 28.92, 33.00, 41.52, 42.12, 45.60, 48.48, 51.84, 51.96, 54.12, 55.56, 67.80, 68.64, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04, 173.40

For the above data, the maximum likelihood estimate of β is $\hat{\beta} = 2.10$. In <Table 1>, we provide the Bayes factors and posterior probabilities for the test $H_1 : \beta = 1,\ H_2 : \beta > 1,\ H_3 : 0 < \beta < 1$ for the failures times for ball bearing data. From this table, there is strong evidence for H_2 in terms of the posterior probability.

< Table 1 > Bayes factors and $P(H_1 \mid \boldsymbol{x}),\ P(H_2 \mid \boldsymbol{x}),\ P(H_3 \mid \boldsymbol{x})$ for testing $H_1 : \beta = 1,\ H_2 : \beta > 1,\ H_3 : 0 < \beta < 1$ under the Ball Bearing Data

B_{12}^{0AI}	B_{13}^{0AI}	B_{23}^{0AI}	$P^{0AI}(H_1 \mid \boldsymbol{x})$	$P^{0AI}(H_2 \mid \boldsymbol{x})$	$P^{0AI}(H_3 \mid \boldsymbol{x})$
0.0064	28.1531	4392.9768	0.0064	0.9934	0.0002
B_{12}^{MI}	B_{13}^{MI}	B_{23}^{MI}	$P^{MI}(H_1 \mid \boldsymbol{x})$	$P^{MI}(H_2 \mid \boldsymbol{x})$	$P^{MI}(H_3 \mid \boldsymbol{x})$
0.0064	25.3539	3966.9162	0.0064	0.9934	0.0002

Example 2 : The following data are time to breakdown of a type of electrical insulating fluid subject to a constant voltage stress(Nelson(1970)).

30KV	7.74, 17.05, 20.46, 21.02, 22.66, 43.40, 47.30, 139.07, 144.12, 175.88, 194.90
32KV	0.27, 0.40, 0.69, 0.79, 2.75, 3.91, 9.88, 13.95, 15.93, 27.80, 53.24, 82.85, 89.29, 100.58, 215.10

The maximum likelihood estimates of β are $\hat{\beta}=1.058$ for 30KV and $\hat{\beta}=0.561$ for 32KV. In <Table 2> and <Table 3>, we provide the Bayes factors and posterior probabilities for the test $H_1 : \beta=1$, $H_2 : \beta<1$, $H_3 : 0<\beta<1$ for the failures times for ball bearing data. From these tables, there are strong evidence for H_1 in terms of the posterior probability for first data, 30KV and strong evidence for H_3 in terms of the posterior probability for second data, 32KV.

< Table 2 > Bayes factors and $P(H_1 \mid \boldsymbol{x})$, $P(H_2 \mid \boldsymbol{x})$, $P(H_3 \mid \boldsymbol{x})$ for testing $H_1 : \beta=1$, $H_2 : \beta>1$, $H_3 : 0<\beta<1$ under the Voltage Breakdown Data - 30KV

B_{12}^{0AI}	B_{13}^{0AI}	B_{23}^{0AI}	$P^{0AI}(H_1 \mid \boldsymbol{x})$	$P^{0AI}(H_2 \mid \boldsymbol{x})$	$P^{0AI}(H_3 \mid \boldsymbol{x})$
5.4927	4.6842	0.8528	0.7165	0.1305	0.1530
B_{12}^{MI}	B_{13}^{MI}	B_{23}^{MI}	$P^{MI}(H_1 \mid \boldsymbol{x})$	$P^{MI}(H_2 \mid \boldsymbol{x})$	$P^{MI}(H_3 \mid \boldsymbol{x})$
3.9371	3.6024	0.9150	0.6529	0.1658	0.1813

< Table 3 > Bayes factors and $P(H_1 \mid \boldsymbol{x})$, $P(H_2 \mid \boldsymbol{x})$, $P(H_3 \mid \boldsymbol{x})$ for testing $H_1 : \beta=1$, $H_2 : \beta>1$, $H_3 : 0<\beta<1$ under the Voltage Breakdown Data - 32KV

B_{12}^{0AI}	B_{13}^{0AI}	B_{23}^{0AI}	$P^{0AI}(H_1 \mid \boldsymbol{x})$	$P^{0AI}(H_2 \mid \boldsymbol{x})$	$P^{0AI}(H_3 \mid \boldsymbol{x})$
25.0606	0.0184	0.0007	0.0180	0.0007	0.9812
B_{12}^{MI}	B_{13}^{MI}	B_{23}^{MI}	$P^{MI}(H_1 \mid \boldsymbol{x})$	$P^{MI}(H_2 \mid \boldsymbol{x})$	$P^{MI}(H_3 \mid \boldsymbol{x})$
10.7357	0.0144	0.0013	0.0142	0.0013	0.9845

5. Concluding Remark

We have suggested a Bayesian hypotheses testing criterion for the shape parameter of weibull model via the intrinsic Bayes factor. We have derived the arithmetic and median intrinsic Bayes factors and used these results to analyze real data sets.

As we see from the numerical results, the arithmetic and median intrinsic Bayes factors are computed based on entire observations so that they give accurate interpretations and fairly steady answers.

In general, there has been a considerable amount of literature on the controversy between a P-value and a Bayes factor. It has been noticed that a P-value does not agree with the posterior probability that the null hypothesis is correct. Delampady and Berger(1990) have shown that the lower bounds of posterior probabilities in favor of null hypotheses are much larger than the corresponding P-values.

IBF methodology can be easily applied to nonnested as well as to irregular problems. They can also be applied in general when the samples come from any distribution.

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