⊠연구논문

자기상관을 갖는 공정의 로버스트 누적합관리도*

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Robust CUSUM chart for Autocorrelated Process

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Abstract

Conventional SPC assumes that observations are independent. Often in industrial practice, however, observations are not independent. A common approach to building control charts for autocorrelated data is to apply conventional SPC to the residuals from a time series model of the process or is to apply conventional SPC to the weighted or unweighted subgroup means. In this paper, we propose a robust CUSUM control scheme for the detection of level change, without model identification or subgrouping of autocorrelated data. The proposed CUSUM chart and other conventional control charts are compared by a Monte Carlo simulation. It is shown that the proposed CUSUM chart is more effective than conventional CUSUM chart when the process is autocorrelated.

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1. Introduction

The cumulative sum(CUSUM) chart was first introduced by Page(1954). Other important early contributions are Barnard(1959), Ewan and Kemp(1960) and Johnson and Leone(1962). Ewan(1963) provided an excellent expository article on the CUSUM and a small text elucidating CUSUM procedures was authored by Van Dobben de Bruyn(1968).

Traditional statistical process control (SPC) assumes that consecutive observations from a process are independent. Often in industrial practice, however, observations are serially correlated. The effects of autocorrelation on various monitoring schemes were studied by Goldsmith and Whitfield(1961) who showed that negative autocorrelation can decrease false alarm rates for CUSUM charts. Conversely, positive autocorrelation increases false alarm rates. Additional studies were reported by Johnson and Bagshaw(1974), Bagshaw and Johnson(1975), and Vasilopoulos and Stamboulis(1978). Johnson and Bagshaw(1974) obtained the limit process for cumulative(or partial) sums of observations from ARMA processes and explored the effect of ARMA noise on the CUSUM statistics proposed by Page (1954). In a sequel, Bagshaw and Johnson(1975) examined the effect of ARMA noise on the run length distribution for CUSUM's. Alwan and Roberts(1988), Alwan(1992), Montgomery and Mastrangelo(1991), Montgomery(1992), Hariss and Ross(1991), Wardell *et al.*(1994) and Yashchin(1993) described the effects of autocorrelation on classical statistical process control. In the presence of positive autocorrelation, classical SPC, without compensation, generates too many false alarms.

Common approach to building control charts for autocorrelated data are to apply classical SPC to the residuals from a time series model of the process and/or is to apply classical SPC to the weighted or unweighted subgroup means. In practice, the autocorrelated data are modeled by an ARMA model, see Box *et al.*(1994), Montgomery(1992), and Wardell *et al.*(1994) among others. If the model is appropriate and well estimated, its residual approximate white noise, and classical methods can be applied.

One of the most interesting approaches to SPC for correlated processes using ARMA modeling was proposed by Alwan and Roberts(1988). They introduced two charts, which they referred to as the common cause control (CCC) chart and special cause control (SCC) chart. The CCC chart is a plot of forecasted values that are determined by fitting the correlated process with an ARMA model. This chart assumes that no special causes have occurred. The SCC only include the systematic variation in the data.

On the other hand, Harris and Ross(1991) investigated the effect of autocorrelation on the performance of a chart similar to the SCC chart that plots the CUSUM of the residuals. They determined a simulated average run length (ARL) for the CUSUM procedure when the process evolves according to an AR(1) model for various values of the AR parameter, concluding that residual analysis is insensitive to shifts in the mean when the process is positively autocorrelated. Similarly, Nikiforov(1979) approximated the ARL's for the CUSUM's of the residuals for general ARMA(p, q) process and verified the approximation via simulation. Yashchin's(1993) objective is to evaluate the performance of CUSUM charts applied to autocorrelated data. He considered charting the raw data directly when the autocorrelation is low. When the autocorrelation is high, he considered a transformation that essentially creates residuals of the type studied by Harris and Ross(1991), but he allowed for autocorrelation in the residuals due to model misspecification. He approximately accounted for moderate amounts of residual autocorrelation by increasing the CUSUM parameter H, the decision interval of the chart, to a level consistent with a desired ARL. He developed his adjustments using the method suggested by Johnson and Bagshaw(1974) of approximating the residuals by Brownian motion with drift.

Another approach can be found in Alwan and Radson(1992), and Runger and Willemain(1995, 1996), among others. Alwan and Radson(1992) proposed the monitoring of autocorrelation processes by plotting the averages of small subgroups separated by lengths of skipping observations. Runger and Willemain (1995, 1996) proposed subgroup means charts which plot averages of subgroups of the raw data with no skipping observations.

Unfortunately, one cannot completely escape the effects of autocorrelation by using charts based on residuals of time series model or subgroup means since the performance of those approaches are poor relative to charts based on independent data. Therefore, we propose a robust CUSUM control scheme without model identification or subgrouping of the autocorrelated data.

2. Properties of the Cumulative Sum for Dependent Processes

In this section we will show how to design the robust CUSUM chart for the detection of level change in the dependent processes. The control limits of the proposed control charts are obtained using the CUSUM of the observations via the concept of the mixingale and Brownian motion. We consider a process which follows a linear process.

$$X_t = \psi(B)\varepsilon_t = \sum_{k=0}^{\infty} \psi_k \varepsilon_{t-k}, \tag{1}$$

where $\psi(B) = 1 - \psi_1 B - \psi_2 B^2 - \psi_3 B^3 - \cdots$ and ε_t is independent and identically distributed with $E(\varepsilon_t) = 0$ and $E(\varepsilon_t^2) = \sigma_\varepsilon^2$. It can be readily seen from (1) that ψ_k is geometrically bounded, say, $|\psi_k| \le B\rho^k$ for some $B > 0$, $\rho \in (0, 1)$.

Let (Ω, F, P) denote a probability space and $\{X_t : t \ge 1\}$ be a sequence of random variables on (Ω, F, P). For weakly dependent data, the standard reference is McLeish(1975a). McLeish's results include that a maximal inequality for L_2-mixingales, a strong law of large numbers(SLLN) for L_2-mixingales, and α-mixing sequences. His mixingale requires L_2-bounded random variables. In this paper, however, maximal inequalities are provided for L_p-mixingales, $p > 1$. Mixingale covers cases which cannot be handled by the mixing process concept.

Let $\{F_n : 0 \le n \le \infty\}$ be any sequence of sub σ-algebras of F which are increasing in n. Andrews(1988) defined an L_p-mixingale as follows.

Definition 1. *(Mixingale) The sequence* (X_t, F_t) *is an* L_p*-mixingale if, for sequences of finite nonnegative constants* c_t *and* ξ_k *where* $\xi_k \to 0$ *as* $k \to \infty$, *we have for all* $t \ge 1$, $k \ge 0$,

(a) $\|E(X_t | F_{t-k})\|_p \le \xi_k c_t$ *and*

(b) $\|X_t - E(X_t | F_{t+k})\|_p \le \xi_{k+1} c_t$.

Definition 2. *(Size)* $\{\xi_k\}$ *is of size* $-p$ *if there exist a positive sequence* $\{L(k)\}$ *such that*

(a) $\sum_k (kL(k))^{-1} < \infty$,

(b) $L(k) - L(k-1) = O(k^{-1}L(k))$,

(c) $L(k)$ *is eventually nondecreasing and*

(d) $\xi_k = O((n^{1/2}L(k))^{-2p})$.

Theorem 1. *Let* $\{X_t\}$ *be a linear process in (1). Then* $\{(X_t, F_t)\}$ *is an* L_p*-mixingale of size* $-1/2$.

Remark. *If* $\sum_{n=1}^{\infty} \sum_{n \le i} |\psi_i| < \infty$, *then* $\sum |\xi_i| < \infty$. *For example,* $|\psi_i| \le Bi^{-\lambda}$, $\lambda > 2$, *then* $\sum |\xi_i|$ *is finite.*

Hence, we have the following theorem :

Theorem 2. *If* $\{(X_t, F_t)\}$ *is an* L_p*-mixingale with* ξ_m *of size* $-1/2$, *then*

$$\sigma_X^{-1} T^{-1/2} \sum_{t=1}^{[Tr]} X_t \rightarrow W(r), \tag{2}$$

where $\sigma_X^2 = E(X_t^2) + 2\sum_{k=1}^{\infty} E(X_0 X_k)$ *and* W *is the standard Brownian motion.*

For (2), we can easily show that $\sigma_X^2 = \sigma_\varepsilon \psi^2(1)$.

3. Proposed Robust Cusum Control Scheme

In this section, we will show how to obtain the empirical control limits of CUSUM chart for dependent processes when the underlying model is unknown.

As with other control chart procedures, the process parameters must be estimated from the process data so that the control limits can be determined. Since σ_X^2 is obtained from the historical data the control limits can be estimated more precisely with more data. To design a robust CUSUM chart for the purpose of detecting shifts in the process level, the following three steps are proposed.

To construct the control limits, we need an estimator of the variance of $S_k = \sum_{t=1}^{k} X_t$ and a K of the control limit constant. We may estimate σ_X^2 and lag-one autocorrelation, ρ_1 from the historical data when the process was in-control.

Step 1. *Estimate* ρ_1 *and* σ_X^2 *using spectral based estimator,*

$$\widehat{\sigma_X}^2 = 2\pi \hat{f}(0) = \hat{\gamma}_0 + 2\sum_{k=1}^{M} W_{n(k)}\, \hat{\gamma}_k, \tag{3}$$

where $\hat{f}(0)$ *is spectral density at frequency 0,* $W_n(k)$ *is Tukey-Hanning lag-window,* $\gamma_k = \mathrm{Cov}(X_0, X_k)$.

Selecting control limits has implications not only on in-control but also on out-of-control ARL.

In general, we favor the approach of developing control limits from an empirical reference distribution based on the process data acquired during in-control process. Hence, in this step, we consider whether historical data has no change point using the following procedure.

Let $S_k = \sum_{t=1}^{k} X_t$ be the cumulative sum of a linear process $\{X_t\}$ with mean 0 and variance σ_X^2. By Theorem 2, the plot of $k^{-1/2}S_k$ against k will oscillate around 0. When there is a sudden change in mean, the plot of $(k,\ k^{-1/2}S_k)$ will exhibit a pattern going out of some specified control limits with high probability. As a result of change point detection, if historical data has no change point, we construct control limits using the historical data. Intuitively, this is necessary condition for empirical design of control charts, since estimation of in-control ARL would be impossible without the large number of observations available.

Step 2. *Determine the control limits using* $\widehat{\sigma_X}^2$ *obtained from Step 1, i.e.,*

$$UCL = +C \cdot K\widehat{\sigma_X},$$

$$LCL = -C \cdot K\widehat{\sigma_X},$$

where UCL is the upper control limit, LCL is the lower control limit, K *is the control limit constant, and* C *is a correction constant for the autocorrelation.*

We will show how to obtain K and C in Section 4.1.

Step 3. *Calculate the cumulative sum of deviations from target after the k-th observation,*

$$k^{-1/2}S_k = k^{-1/2}\sum_{t=1}^{k}(X_t - \mu_0),$$

where μ_0 *is a target value for the process, and plot* $(k,\ k^{-1/2}S_k)$.

Perform a sensitivity analysis by comparing out-of-control ARL's for the optimal K to other choices of K producing the same in-control ARL. From this choice pick K with the most desirable performance overall in terms of out-of-control ARL's. The parameter K is required to implement the CUSUM. This parameter is most conveniently determined by calculating the ARL for the test procedure.

ARL can be calculated via Markov chain considerations, as a solution of a set of integral equations, or via a Monte Carlo simulation, Van Dobben de Bruyn (1968), once the distributional properties of the observations are specified. Tables and nomograms are given in Lucas(1976) and Goel and Wu(1971) for the CUSUM procedure when the inherent variability is independently and identically normally distributed.

A control chart based on a robust CUSUM is fairly easy to operate and has several implementation advantages over conventional procedures; (1) The important implication of robust CUSUM chart is that one does not need to estimate an ARMA model of the data. As distinct from residuals plots, robust CUSUM charts retain the basic simplicity of cumulative summing of observations to form point on the control chart. (2) The procedure of partial summing of successive observations to generate a plotted point is simple and consistent with conventional approaches to control charting both conceptually and mechanically. Because robust CUSUM charts signal assignable causes when the plotted point is far from the centerline, these charts provide a familiar signal when the estimate of the current processes level deviates significantly from the historical process mean level or target. (3) Features provided in most commercial SPC software can be used to plot these control charts. We regard this as an important practical advantage.

Consequently, control charts based on robust CUSUM can be constructed and interpreted according to traditional guidelines for uncorrelated data, namely, robust CUSUM charts are easy to implement for dependent process, and they can be interpreted as traditional control charts.

4. Simulation Study

In this section we obtain the ARL's of the proposed robust CUSUM procedures and compare the performance of four control charts, CUSUM type, conventional CUSUM, EWMA and robust CUSUM when the process can be described by either the AR(1) or MA(1). The usual performance criterion for any control chart is the ARL. In order to compare the performances of the charts more meaningfully for different autoregressive and moving average parameter values, the σ_X multipliers of the control charts (K) are manipulated so that the ARL's, when there is no shift in the mean, is the same for all four charts. Then the chart with lowest ARL when a shifts in the mean occurred is considered superior. This is analogous to matching the Type I errors so that the Type II errors can be compared in a more meaningful way. The ARL when the process mean deviates from the target value, i.e., out-of-control, should be as short as possible, subject to a specified ARL when the process is in-control. However, the autocorrelation in the process data degrades the ARL performance, as shown for conventional CUSUM charts by Johnson and Bagshaw(1974) and Bagshaw and Johnson(1975).

For dependent processes, computation of the ARL's of the control charts is analytically intractable and they are thus determined via simulation. For the simulation, it is assumed that only one observation is available at each time period and that all parameters are known exactly. Time series of sample size 2100 are generated and the first 100 observations are discarded to reduce the effect of initialization. A step change in the mean is introduced at time 0. The white noise are drawn from normal distribution with zero mean and unit variance. In order to obtain a comprehensive view of the effect of autocorrelation on the three control charts, an experiment was designed over the AR(1) and MA(1) model. The run length is measured until the first out-of-control condition is signaled for chart. The process is repeated 5000 times in order to obtain the ARL.

4.1 Correction Constant and Design of Robust CUSUM Chart

It was shown by Johnson and Bagshaw(1974) that the ARL is much affected by the autocorrelation structure of the underlying process as follows,

$$ARL(\phi_1) = \frac{(1-\phi_1)}{(1+\phi_1)} \cdot ARL(0) \quad \text{for AR(1) model,}$$

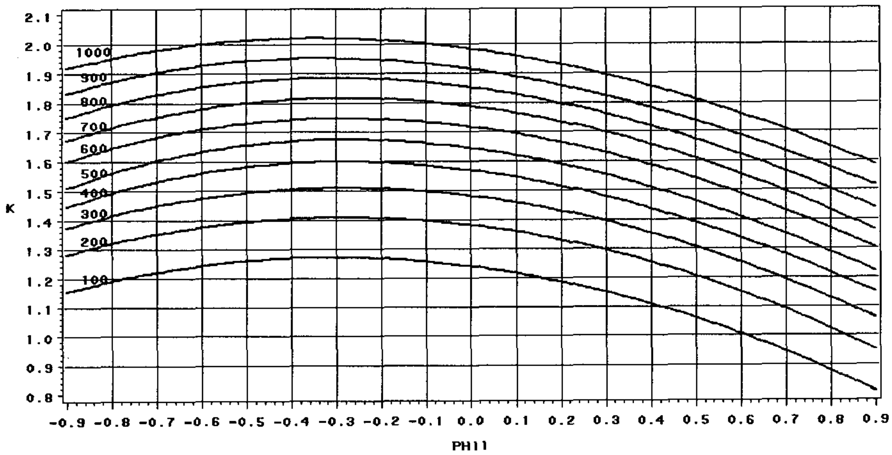
$$ARL(\theta_1) = \frac{(1+\theta_1^2)}{(1-\theta_1)^2} \cdot ARL(0) \quad \text{for MA(1) model,}$$

where ARL(0) is the theoretical ARL for the independent process.

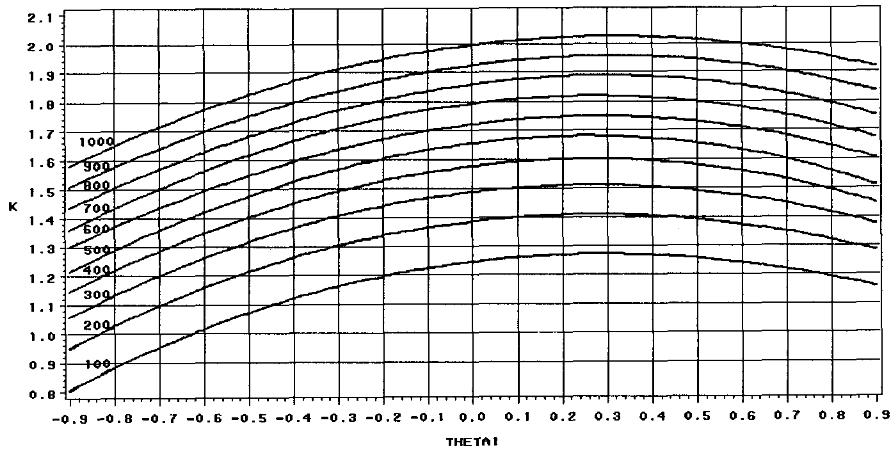
As mentioned above, the ARL of autocorrelated process depends on the pattern of autocorrelation function(ACF). For that reason, we adjust the control limits by correction constant C, as follows,

$$C = 1 - \operatorname{sign}(\rho_1 \rho_2) \cdot (\rho_1 - \rho_2)^2 / 5$$

where ρ_k is a lag-k ACF. Other choice of correction constants can be employed.



< Figure 1 > Nomogram for the Choice of Optimal K for Robust CUSUM Charts of AR(1) Processes



< Figure 2 > Nomogram for the Choice of Optimal K for Robust CUSUM Charts of MA(1) Processes

Then, the design strategy is to choose the control limit constant K which for a given in-control ARL, minimizes the out-of-control ARL for a shift in the process mean. Generally speaking, both Type I and Type II errors are characterized by these ARL's. The corresponding analogue of the Type I error is the average number of observations before an out-of-control signal is given when the process is actually in control. The ARL corresponding to the Type II error is the average number of observations that must be taken to detect a true process shift once one has occurred. In general, optimal choice of K can be chosen according to the following steps; (1) Choose the smallest acceptable ARL for the case there is no shift in the process level. This corresponds to fixing the false alarm rate (Type I error) and (2) Find the control limit constant K which satisfies the in-control ARL constraint from (1). Choosing an acceptable ARL will often be based informally on economic considerations such as the cost associated with a false alarm.

New nomograms using the in-control ARL's for dependent processes are provided in Figures 1 and 2, which cover the independent cases provided by Ewan and Kemp(1960) and Goel and Wu(1971). In Figures 1 and 2, optimal K's are given for robust CUSUM charts with in-control ARL's ranging from 100 to 1000.

The contour nomogram is a convenient device for the design of robust CUSUM charts, since the method of construction is relatively simple and straightforward. The ARL surfaces can be easily visualized to get an intuitive insight when designing a robust CUSUM chart.

Suppose a robust CUSUM chart is to be designed for controlling a process mean such that the chart will yield an ARL of approximately 400 when the process is in-control. Choice of K will depend on the magnitude of autocorrelation, which should be detected quickly. From Figures 1 and 2, K is approximately 1.2 to 1.5 ranging when a process was a positive autocorrelation ($0.3 \le \phi_1 \le 0.8$ or $-0.8 \le \theta_1 \le -0.3$), approximately 1.4 to 1.6 ranging when a process was a negative autocorrelation ($-0.8 \le \phi_1 \le -0.3$ or $0.3 \le \theta_1 \le 0.8$), and approximately 1.57 when the process is independent ($\phi_1 = \theta_1 = 0$).

4.2 ARL of the Control Charts When the Process is In-Control

Various criteria for designing a quality control chart for monitoring a process mean have been suggested. We perform a simulation study to compare the performance of the proposed control scheme with the CUSUM type, the conventional CUSUM chart proposed by Bagshaw and Johnson(1974), and the

EWMA chart. Since no results, except Bagshaw and Johnson(1974), are available on the exact distribution of run length for the CUSUM charts in the presence of autocorrelation, we investigated the ARL of another schemes based on simulated trials.

The design of control chart procedure is usually based on the ARL of the scheme, although other factors should be taken into consideration. Page(1961) recommended the design of CUSUM chart to have specified ARL values at the target mean $\mu=0$. Robinson and Ho(1978) made the same recommendation for constructing an EWMA chart. Woodall(1985) proposed a strategy for designing a CUSUM chart that applies equally well to the design of EWMA charts. His approach was to first specify a region of acceptable values for the process mean. A control chart was then designed to have specified ARL values at two particular shifts in the underlying process mean. Tables of ARL's have been given in the various literature, Montgomery(1991) for CUSUM charts, Crowder(1987a, 1987b) for EWMA charts, and Lucas and Saccucci(1990) for both EWMA charts and composite Shewhart-EWMA charts. For the majority of the comparisons, the ARL associated with no shift in the mean is set approximately equal to 400 which is the ARL of the standard Shewhart chart when there is no shift in the mean and when observations are independent. To be complete, we also obtained results for the case when the ARL with no shift and no autocorrelation in the mean is approximately equal to 400. Many economic design results suggest that a smaller value for K is more appropriate, Montgomery(1980).

In this simulation, parameters of each control charts are $K=1.57$ for the CUSUM type and the robust CUSUM chart, $\lambda=0.12$ and $K=2.75$ for the EWMA chart, $h=19.3528$ and $k=0.0408$ for the conventional CUSUM chart. Under the specified parameters of each chart, ARL is approximately equal to 400 when the processes are independent and in-control state.

The ARL will depend on the actual model the process follows and on the values of control limit constant K. In the followings, we will investigate the effect of autocorrelation structure upon the ARL. We concentrate on the models

$$\mathrm{AR}(1):\ X_t = \phi_1 X_{t-1} + \varepsilon_t,$$

$$\mathrm{MA}(1):\ X_t = \varepsilon_t - \theta_1 \varepsilon_{t-1}.$$

The ARL's of the resulting scheme for various values of ϕ_1's and θ_1's are given in Tables 1 and 2. Table 1 is obtained from the simulation of the AR(1) model with $-1 < \phi_1 < 1$. From Table 1, we see that the four charts have the same ARL's when $\rho = 0$, i.e., independent case. The ARL's of conventional CUSUM chart are exact values given by Johnson and Bagshaw(1974).

< Table 1 > AR(1) Process : $X_t = \phi_1 X_{t-1} + \varepsilon_t$

ϕ_1	CUSUM Type	NonCorrection CUSUM*	Robust CUSUM	EWMA	Conventional CUSUM**
0.75	15.04(0.38)	805.91(12.16)	792.32(12.11)	31.21(0.38)	57.14
0.60	22.65(1.04)	668.70(11.76)	640.75(11.59)	41.94(0.54)	100.00
0.45	48.90(2.55)	608.07(11.54)	584.76(11.39)	62.77(0.81)	151.72
0.30	104.87(4.76)	520.42(10.97)	502.85(10.83)	98.69(1.34)	215.38
0.15	211.16(7.27)	469.11(10.70)	461.41(10.64)	183.38(2.57)	295.65
0.00	398.59(10.08)	405.60(10.17)	405.60(10.17)	405.32(5.41)	400.00
-0.15	655.02(12.27)	341.61(9.50)	349.70(9.62)	936.07(9.48)	541.18
-0.30	1024.11(13.47)	266.97(8.61)	320.57(9.38)	1600 ↑	742.86
-0.45	1521.29(14.82)	218.40(7.87)	343.35(9.78)	1600 ↑	1054.55
-0.60	1600 ↑	148.34(6.71)	373.13(10.31)	1600 ↑	1600.00
-0.75	1600 ↑	79.67(4.88)	479.61(11.61)	1600 ↑	2800.00

( ): standard error of ARL
* : The correlation constant, C does not exact
**: exact value given by Johnson and Bagshaw (1974)

< Table 2 > MA(1) Process : $X_t = \varepsilon_t - \theta_1 \varepsilon_{t-1}$

θ_1	CUSUM Type	NonCorrection CUSUM*	Robust CUSUM	EWMA	Conventional CUSUM**
0.75	1600 ↑	1.44(0.02)	1.40(0.01)	1600 ↑	10000.00
0.60	1600 ↑	16.01(2.07)	11.68(1.69)	1600 ↑	3400.00
0.45	1690.68(14.92)	95.85(5.35)	80.10(4.84)	1600 ↑	1590.08
0.30	1145.43(13.45)	230.22(8.16)	206.14(7.71)	1600 ↑	889.80
0.15	701.74(12.56)	330.68(9.44)	322.12(9.33)	1048.44(9.95)	566.09
0.00	400.63(10.05)	399.67(10.04)	399.67(10.04)	399.71(5.37)	400.00
-0.15	240.77(7.85)	454.95(10.58)	448.00(10.52)	202.56(2.77)	309.26
-0.30	161.54(6.22)	485.34(10.74)	456.20(10.48)	133.03(1.78)	257.99
-0.45	111.92(4.92)	522.20(11.07)	457.21(10.48)	104.56(1.38)	228.78
-0.60	89.04(4.25)	521.05(11.06)	433.05(10.25)	89.56(1.19)	212.50
-0.75	80.54(4.02)	512.42(10.99)	416.53(10.08)	83.32(1.11)	204.08

( ): standard error of ARL
* : The correlation constant, C does not exact
**: exact value given by Johnson and Bagshaw (1974)

Since we estimate the process variance using (3) instead of using the usual estimator $\tilde{\sigma}_{X^2}=(T-1)^{-1}\sum_{t=1}^{T}(X_t-\overline{X})^2$, the simulated ARL's of the CUSUM type chart show the same pattern of the theoretical results, i.e., the ARL is exponentially decreasing as the value of ϕ_1 gets larger. This is because in the CUSUM type chart the variance σ_X^2 is estimated under the independence assumption, when $\phi_1 < 0$, it is exponentially increasing because of high fluctuations. For ρ_1 in opposite direction to that suggested by Johnson and Bagshaw (1974) is necessary if spurious out-of-control decision should be avoided. The ARL of the EWMA chart shows the similar pattern to the conventional one. For the NonCorrection CUSUM chart, ARL shows the different pattern to the CUSUM type and conventional CUSUM chart because there use the difference estimation method of σ_X^2. In contrast, the ARL of the robust CUSUM chart is relatively flat regardless of the magnitude of the values of ϕ_1. Hence proposed control scheme is more robust than the other control scheme in AR(1) model.

Table 2 is obtained from the simulation of the MA(1) model with $-1 < \theta_1 < 1$. From Table 2, the simulated ARL's of the CUSUM type chart show the same pattern of the theoretical results of Johnson and Bagshaw(1974), i.e., the ARL is exponentially increasing as the value of θ_1 gets larger. The ARL of the EWMA chart shows the similar pattern to the conventional one. For the NonCorrection CUSUM chart, ARL shows the different pattern to the CUSUM type and conventional CUSUM chart because there use the difference estimation method of σ_X^2. In contrast, the ARL of the robust CUSUM chart is relatively flat regardless of the magnitude of the values of θ_1. Hence proposed control scheme also is more robust than the other control scheme in MA(1) model.

As can be seen from Tables 1 and 2, the suggested method approximates the true ARL's fairly well. Also, the ARL of the proposed control scheme is flatter than the other control scheme in Tables 1 and 2. For this reason, we called robust CUSUM chart for the proposed control chart.

4.3 Power Comparison

Relative performances of the four control charts, CUSUM type, robust CUSUM, EWMA, and conventional CUSUM chart, are compared by a Monte Carlo simulation. The values of step change ranged from 0 to 5 standard deviations of the process are introduced at time 0 and the number of samples until the point fall outside control limits are counted. For each chart, the above procedure is repeated 5000 times in order to obtain the ARL's. Simulation results are summarized in Table 3. Table 3 (a) for the independent process ($\phi_1=0.0$), Table 3 (b) for positively autocorrelated process ($\phi_1=0.6$), and Table 3 (c) negatively autocorrelated process ($\phi_1=-0.6$) when the underlying process is AR(1). We choose the control limit constant $K=1.5686$ for independent case, $K=1.3359$ for positively autocorrelated case, and $K=1.5630$ for negatively autocorrelated case for the robust CUSUM using Figure 1 and the same constant K is used for the CUSUM type chart. For the EWMA chart, parameters of control limits used $\lambda=0.12$ and $K=2.75$ and for the conventional CUSUM chart, parameters of control limits used reference value, $k=0.0408$ and decision value, $h=19.3528$. These parameters are selected such that the ARL associated with no shift in the mean is approximately equal to 400. Table 3 compares the ARL of each of the four charts versus the size of the shift in the mean for three cases.

< Table 3 > Comparison of Chart Performance : AR(1) Process

(a) Random Process : $\phi_1=0.0$

Δ	CUSUM Type $K=1.5686$	Robust CUSUM $K=1.5686$	EWMA* $\lambda=0.12$ $K=2.75$	Conventional CUSUM** $h=19.3528$ $k=0.0408$
0.0	431.45(10.43)	434.32(10.46)	400.00	400.00
0.5	9.19(0.14)	9.20(0.14)	30.37	42.35
1.0	3.43(0.04)	3.43(0.04)	9.77	20.93
1.5	2.03(0.02)	2.02(0.02)	5.68	13.95
2.0	1.46(0.01)	1.46(0.01)	4.05	10.50
2.5	1.20(0.01)	1.20(0.01)	3.18	8.45
3.0	1.08(0.00)	1.08(0.00)	2.65	7.10
3.5	1.03(0.00)	1.03(0.00)	2.29	6.14
4.0	1.01(0.00)	1.01(0.00)	2.07	5.42
4.5	1.00(0.00)	1.00(0.00)	1.94	4.87
5.0	1.00(0.00)	1.00(0.00)	1.81	4.41

( ): standard error of ARL
* : exact value given by Crowder (1987a, b)
**: exact value given by Lucas and Crosier (1982)

< Table 3 > (continued)

(b) Positively Autocorrelated Process : $\phi_1 = 0.6$

Δ	CUSUM Type $K=1.3359$	Robust CUSUM $K=1.3359$	EWMA $\lambda=0.12$ $K=2.75$	Conventional CUSUM $h=19.3528$ $k=0.0408$
0.0	13.23(0.59)	367.93(9.11)	41.59(0.54)	86.00(1.01)
0.5	7.33(0.15)	28.94(0.48)	23.59(0.29)	35.58(0.32)
1.0	3.85(0.07)	9.78(0.13)	10.98(0.11)	17.61(0.11)
1.5	2.20(0.03)	4.80(0.06)	6.43(0.06)	11.50(0.06)
2.0	1.49(0.02)	3.04(0.03)	4.56(0.03)	8.75(0.04)
2.5	1.20(0.01)	2.05(0.02)	3.46(0.02)	7.03(0.03)
3.0	1.06(0.00)	1.52(0.01)	2.81(0.01)	5.88(0.02)
3.5	1.01(0.00)	1.24(0.01)	2.39(0.01)	5.06(0.01)
4.0	1.00(0.00)	1.10(0.00)	2.14(0.01)	4.47(0.01)
4.5	1.00(0.00)	1.03(0.00)	1.97(0.01)	4.02(0.01)
5.0	1.00(0.00)	1.01(0.00)	1.82(0.01)	3.67(0.01)

( ): standard error of ARL

(c) Negatively Autocorrelated Process : $\phi_1 = -0.6$

Δ	CUSUM Type $K=1.5630$	Robust CUSUM $K=1.5630$	EWMA $\lambda=0.12$ $K=2.75$	Conventional CUSUM $h=19.3528$ $k=0.0408$
0.0	1648.97(10.63)	393.82(10.51)	1600 ↑	856.00(7.94)
0.5	30.35(0.35)	7.38(0.14)	1209.30(10.10)	31.42(0.08)
1.0	8.34(0.09)	3.12(0.04)	51.09(0.51)	15.98(0.03)
1.5	4.20(0.04)	2.05(0.02)	16.54(0.09)	10.84(0.02)
2.0	2.77(0.02)	1.64(0.01)	10.31(0.04)	8.27(0.01)
2.5	2.07(0.02)	1.41(0.01)	7.71(0.02)	6.70(0.01)
3.0	1.71(0.01)	1.28(0.01)	6.36(0.02)	5.67(0.01)
3.5	1.50(0.01)	1.19(0.01)	5.49(0.01)	4.95(0.01)
4.0	1.35(0.01)	1.14(0.00)	4.87(0.01)	4.33(0.01)
4.5	1.25(0.01)	1.08(0.00)	4.43(0.01)	3.98(0.00)
5.0	1.17(0.01)	1.05(0.00)	4.11(0.01)	3.66(0.01)

( ): standard error of ARL

For the independent process, Table 3 (a), it is observed that the CUSUM type and the robust CUSUM charts performs equally well and performs better than the conventional CUSUM and EWMA charts. The ARL's of the conventional CUSUM and EWMA charts are exact values given by Lucas and Crosier (1982) and Crowder (1987a, 1987b), respectively. For the conventional CUSUM and EWMA charts, the ARL is calculated using the integral equation method with 24 Gaussian points.

From Table 3 (b) for the positively autocorrelated case, we can clearly see that the robust CUSUM chart outperforms the other charts. The small ARL's of the CUSUM type, conventional CUSUM and EWMA charts for $\Delta=0.0$ indicate that they give the false alarm too often when there is no change in the process mean.

While the ARL of the robust CUSUM chart is not much different from the independent case. Furthermore the larger ARL's of the EWMA chart for $\Delta>0.0$ than the robust CUSUM chart indicated that it cannot detect the shift in the process as quickly as the robust CUSUM chart.

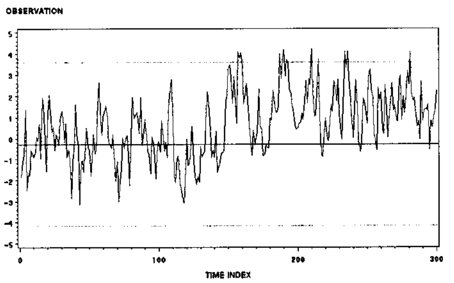
For the negatively autocorrelated case, Table 3 (c), the value of ARL's for $\Delta=0.0$ as too large compared to the independent case. Hence the ARL's for $\Delta>0.0$ are much different from the positively autocorrelated case.

Therefore, for the positively and negatively autocorrelated processes, from Table 3 (b) and (c) the CUSUM type, conventional CUSUM and EWMA chart are often useless in the autocorrelated environments. The robust CUSUM chart is a very simple and useful tool when the process is autocorrelated.

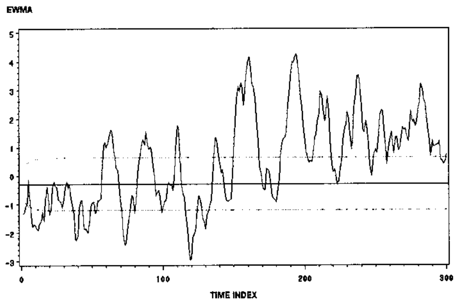
4.4 Example

We simulate the AR(1) process with $\phi_1=0.6$, 300 observation, single readings taken every time point. The simulated process undergoes a step change with $\Delta=1\sigma_X$ at the middle of the process of observation 150. Figures 3 to 6 are Shewhart, EWMA, conventional CUSUM, and robust CUSUM charts, respectively.

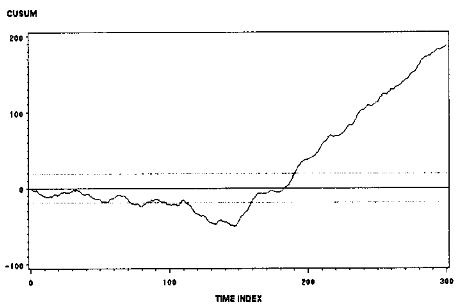
Figure 3 is the Shewhart chart for the simulated process. From visual examination alone, we see that the series is obviously out-of-control in the middle of the series, with strong evidence of positively autocorrelated behavior. The sample mean is -0.2886 and the sample standard deviation is 1.2988, so conventional 3σ control limits would extend from -4.1850 to 3.6078, limits that are shown in Figure 3. If a half of the data are viewed without regard to time sequence, they conform very closely to the normal random variables. It is seen not only that the data are positively autocorrelated, but that it is not even obvious that the data should be regarded as coming from stationary process. We see in Figure 3 that these limits are wider than real limits. It is seen that no individual points fall outside the 3σ control limits, indicating that the process is in-control.



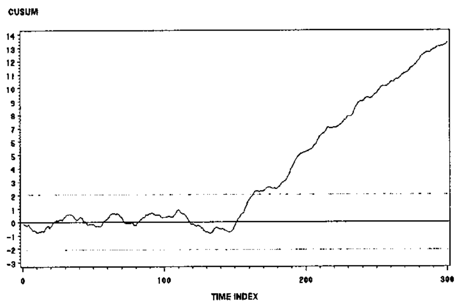
< Figure 3 > Shewhart Chart for Simulated Process



< Figure 4 > EWMA Chart for Simulated Process



< Figure 5 > Conventional CUSUM Chart for Simulated Process



< Figure 6 > Robust CUSUM Chart for Simulated Process

Figure 4 is the EWMA chart in which control parameters are $\lambda=0.12$ and $K=2.75$. And also Figure 5 is the conventional CUSUM chart which control limits are ± 19.3528. Both charts give the false alarm too often when the process is in-control, i.e., too many points fall outside the control limits. While the robust CUSUM chart with control limits ± 2.0633 indicates the process is out-of-control after observation 162. Consequently, it indicates that all but the robust CUSUM chart have too many out-of-control points or no out-of-control points. Hence, traditional means of monitoring quality for the detection of level change of dependent processes may have yielded misleading results.

5. Concluding Remarks

The result of this paper makes it clear that conventional control charts such as the CUSUM chart are not completely robust to deviations from the assumption of process randomness, that is, when observations are autocorrelated. The robust CUSUM chart is being designed and implemented without relying of the assumption of independence for the data. We showed that the level of in-control ARL of the robust CUSUM is better than other control charts.

Strong law of large numbers, derived from our main result, Theorems 1 and 2, are given in Section 2. All results were obtained under the assumption that the control variable follows a normal distribution. In terms of algebra, our results are unaffected by a change of this distribution.

The robust CUSUM chart is very effective in detecting shifts in the mean of autocorrelated processes and performs especially well when the process is positively autocerrelated. The robust CUSUM chart is thus a good alternative to the more realizable method proposed by Alwan and Roberts (1988). Their procedure may warrant the extra expense and efforts of modeling the correlated systems. The robust CUSUM charts is an alternative tool for the process control due to its practical convenience and robustness toward the autocorrelations. The effect of serial correlations, represented by $\rho_1, \rho_2, \cdots$, on the ARL is discussed in this paper on the basis of simulated ARL. Positive serial correlation drastically increase the ARL as compared to the case when process is independent.

If the process can be described by a linear process, then the robust CUSUM chart should be useful in determining which chart would be most beneficial. If there is not a significant distinction between chart performance for the given degree of autocorrelation, the EWMA chart, and possibly the conventional CUSUM chart, may be more appealing because of its simplicity. On the other hand, in the case where the robust CUSUM chart performs very well, implementation of this chart should be considered strongly.

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