중복근을 갖는 구조물에 대한 개선된 부분공간 반복법

An Improved Subspace Iteration Method for Structures with Multiple Natural Frequencies

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요 지

본 논문에서는 중복근을 갖는 구조물에 대한 효율적이고 수치적으로 안정한 고유치해석 방법을 제안하였다. 제안방법은 널리 알려진 쉬프트를 갖는 부분공간 반복법을 개선한 방법이다. 쉬프트를 갖는 부분공간 방법의 주된 단점은 특이성 문제 때문에 어떤 고유치에 근접한 쉬프트를 사용할 수 없어서 수렴성이 저하될 가능성이 있다는 점이다. 본 논문에서는 부가조건식을 이용하여 위와 같은 특이성 문제를 수렴성의 저하없이 해결하였다. 이 방법은 쉬프트가 어떤 단일 고유치 또는 중복 고유치와 같은 경우일지라도 항상 비특이성인 성질을 갖고 있다. 이것은 제안방법의 중요한 특성중의 하나이다. 제안방법의 비특이성은 해석적으로 증명되었다. 제안방법의 수렴성은 쉬프트를 갖는 부분공간 반복법의 수렴성과 거의 같고, 두 방법의 연산횟수는 구하고자 하는 고유치의 개수가 많은 경우에 거의 같다. 제안방법의 효율성을 증명하기 위하여, 두개의 수치예제를 고려하였다.

핵심용어 : 부분공간 반복법, 역반복법, 쉬프트, 중복 고유진동수, 비특이성, 부가조건식

Abstract

An efficient and numerically stable eigensolution method for structures with multiple natural frequencies is presented. The proposed method is developed by improving the well-known subspace iteration method with shift. A major difficulty of the subspace iteration method with shift is that because of singularity problem, a shift close to an eigenvalue can not be used, resulting in slower convergence. In this paper, the above singularity problem has been solved by introducing side conditions without sacrifice of convergence. The proposed method is always nonsingular even if a shift is on a distinct eigenvalue or multiple ones. This is one of the significant characteristics of the proposed method. The nonsingularity is proved analytically. The convergence of the proposed method is at least equal to that of the subspace iteration method with shift, and the operation counts of above two methods are almost the same when a large number of eigenpairs are required. To show the effectiveness of the proposed method, two numerical examples are considered.

Keywords : *Subspace iteration method, Inverse iteration, Shift, Multiple natural frequencies, Nonsingularity, Side condition*

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1. Introduction

Eigensolution method is very important in dynamic analysis of structures when mode superposition method is used. Many eigensolution methods have been developed, and among these methods, the subspace iteration method has hitherto been known to be very efficient, so this method has been widely used.

The subspace iteration method was developed and named by Bathe[1),2)]. This method combines simultaneous inverse iteration method and Rayleigh-Ritz analysis. The following shortcomings have been identified after extensive use of the method[3)]. These include: (1) slow convergence and large computational and storage costs when a relatively large number of eigenirs required; (2) significantly computational effort required to form and solve the subspace eigenproblem when a large number of eigenpairs are required; and (3) missed eigenvectors caused by a poor choice of starting trial vectors.

To overcome the above shortcomings, many researchers have studied a variety of acceleration procedures of the subspace iteration method. The techniques employed include Chebyshev polynomials[4)], over-relaxation method [5),6)], shifting technique[7)], exploitation by partitioning a large structure into a number of substructures[8)], improving the selection initial vectors[9)], selective repeated inverse iteration and multiple inverse iteration[10),11)], and subspace iteration by omitting some of the Rayleigh-Ritz procedure from certain iteration steps[12),13)].

Among the above accelerated techniques, a shifting technique is well-known and effectively used in the commercial FEM programs such as ADINA[14)]. However, since the singularity may occur during the use of the shifting technique in the accelerated scheme such as the subspace iteration method with shift, the shift must be carefully chosen to avoid the singularity. It is a significant disadvantage of the subspace iteration method with shift.

Jung, Kim and Lee[15)] have developed a method that always guarantees the numerical stability and maintains the convergence rate of the subspace iteration method with shift even if a shift is an exact eigenvalue itself. However, the method can be only applied to the structures with distinct natural frequencies. If a structure with multiple natural frequencies is analyzed by the method, the singularity problem may still occur.

In this paper, when the eigenvalue analysis for a structure with multiple eigenvalues is performed, an eigensolution technique that always guarantees the numerical stability is developed by improving the method of Jung, Kim and Lee[15)]. That is, the proposed method is always numerically stable even if a shift is on a distinct eigenvalue or multiple ones.

The subspace iteration method with shift is briefly reviewed in next chapter. Chapter three includes the theory, the proof of the numerical stability, the convergence analysis and the operation counts of the proposed method. The effectiveness of the proposed method is verified by the results of numerical examples in chapter four. In chapter five, we give concluding remarks.

2. Subspace Iteration Method with Shift

The eigenproblem of the structural dynamics may be written as follows[16)].

$$K X = M X \Lambda \tag{1}$$

where K and M are the stiffness and mass matrices of the structure of order n, respec-

tively, the columns of X the eigenvectors, and Λ a diagonal matrix with eigenvalues on its diagonal.

Applying a shift μ to eqn (1) gives

$$(K-\mu M)X = MX\Omega \tag{2}$$

where

$$\Omega = \Lambda - \mu I \tag{3}$$

a I is the unit matrix.

Suppose that the p smallest eigenvalues $\lambda_i (I = 1, 2, \ldots, p)$ and corresponding eigenvectors x_i are required. For faster convergence, q trial vectors are normally used with $q = \min\{2p, p+8\}$. Then, the algorithm of the subspace iteration method with the shift μ can be described as follows

Step 1. Find the eigenvector approximations $\overline{X}^{(k+1)}$ by the simultaneous inverse iteration method:

$$(K-\mu M)\,\overline{X}^{(k+1)} = M\,X^{(k)} \tag{4}$$

where $\overline{X}^{(k+1)}$ and $X^{(k)}$ are the $(n \times q)$ matrices.

Step 2. Compute the projections of the matrices $(K-\mu M)$ and M :

$$\overline{K}^{(k+1)} = \overline{X}^{(k+1)^T}(K-\mu M)\,\overline{X}^{(k+1)} \tag{5}$$

$$\overline{M}^{(k+1)} = \overline{X}^{(k+1)^T} M\,\overline{X}^{(k+1)} \tag{6}$$

where $\overline{K}^{(k+1)}$ and $\overline{M}^{(k+1)}$ are the $(q \times q)$ symmetric matrices.

Step 3. Solve the eigenproblem of reduced order q:

$$\overline{K}^{(k+1)}\,Q^{(k+1)} = \overline{M}^{(k+1)}\,Q^{(k+1)}\,\Omega^{(k+1)} \tag{7}$$

wher $Q^{(k+1)}$ and $\Omega^{(k+1)}$ are the $(q \times q)$ matrices.

Step 4. Find the improved eigenvectors $X^{(k+1)}$:

$$X^{(k+1)} = \overline{X}^{(k+1)}\,Q^{(k-1)} \tag{8}$$

and the improved eigenvalues $\Lambda^{(k+1)}$:

$$\Lambda^{(k+1)} = \Omega^{(k+1)} + \mu I\ . \tag{9}$$

$\Lambda^{(k+1)}$ convertges to Λ and $X^{(k+1)}$ converges to X as k approaches infinity. The convergence rate of the subspace iteration method with shift is

$$(\lambda_j - \mu)/(\lambda_{q+1} - \mu) \tag{10}$$

If a shift is an eigenvalue itself or very close to it, the iteration procedure becomes unstable because of the singularity problem occurred during $L\,D\,L^T$ factorization process of the coefficient matrix. To avoid this singularity problem, that is, to guarantee the stability of the subspace iteration method with shift, the following condition was adopted in the subspace iteration method[7]:

$$1.01\,\overline{\lambda}_{s-1} \le \mu \le 0.99\,\overline{\lambda}_s \tag{11}$$

where $\overline{\lambda}_{s-1}$ is the calculated approximation to $(s\text{-}1)$th eigenvalue and $\overline{\lambda}_s$ sth eigenvalue.

It means that a shift must be within a limited region resulting in slow convergence. Moreover, if the calculated approximation to an eigenvalue slightly differs from it, an eigenvalue may be inside of the limited region. Then, the singularity may occur although a shift is inside of the limited region. These are the significant disadvantages of the subspace iteration method with shift. The purpose of this paper is to remove the limitation in eqn (11) for choosing the value of a shift μ.

3. Proposed Method

3.1 Theory

Consider the simultaneous inverse iteration step in the subspace iteration method with shift:

$$(K-\mu M)\,\overline{X}^{(k+1)} = MX^{(k)} \,. \tag{12}$$

Since if a shift is very close to an eigenvalue in eqn (12) the singularity occurs during the decomposition process, The $(k+1)$th eigenvector approximations, $\overline{X}^{(k+1)}$, can not be acquired. This is a significant disadvantage of the subspace iteration method with shift.

Jung, Kim and Lee[15] proposed the eigensolution method which is always numerically stable in case of eigenvalue analysis for structures with distinct eigenvalue. However, if a shift is very close to multiple eigenvalues, the singularity still occurs during the decomposition process of the simultaneous inverse iteration step.

In this paper, to solve the above singularity problem that may occur in case of structures with multiple eigenvalues, the following procedures are proposed. First, let us consider a shift close to multiple eigenvalues. To simplify the notation in this discussion, assume that the multiplicity of the lowest eigenvalue is s, that is, $\lambda_1 = \lambda_2 = \cdots = \lambda_s$. Then, the inverse iteration step on the multiple eigenvalues can be expressed as follows:

$$(K-\mu M)\,\overline{X}_s^{(k+1)} = M\,X_s^{(k)} D_s^{(k+1)} \tag{13}$$

where the $(n\times s)$ matrices $X_s^{(k)} = [x_1^{(k)}, x_2^{(k)}, \ldots, x_s^{(k)}]$, $\overline{X}_s^{(k+1)} = [\overline{x}_1^{(k+1)}, \overline{x}_2^{(k+1)}, \ldots, \overline{x}_s^{(k+1)}]$, the $(s\times s)$ matrix $D_s^{(k+1)} = diag(d_{11}^{(k+1)}, d_{22}^{(k+1)}, \ldots, d_{ss}^{(k+1)})$ and the scalar $d_{ii}^{(k+1)}$ controls the length of the vector $\overline{x}_i^{(k+1)}$.

Because there are only $(n\times s)$ equations with $((n+1)\times s)$ unknowns, $(n\times s)$ components of $\overline{X}_s^{(k+1)}$ and s components of $d_{ii}^{(k+1)}$, in eqn(13), s side conditions must be introduced for the solution of eqn (13). These conditions are that the current vector set $(X_s^{(k)})$ is orthogonal to the incremental vector set $(\Delta X_s^{(k)})$ wth respect to M ; that is,

$$X_s^{(k)T} M \Delta X_s^{(k)} = 0 \,. \tag{14}$$

Adding the mass orthonormality relation, $X_s^{(k)T} M X_s^{(k)} = I_s$, to the side conditions, eqn (14), yields

$$X_s^{(k)T} M \overline{X}_s^{(k+1)} = I_s \tag{15}$$

where

$$\overline{X}_s^{(k+1)} = X_s^{(k)} + \Delta X_s^{(k)} \tag{16}$$

The inverse iteration step on the other eigenvalues make use of the existing equation, eqn (12); that is,

$$(K-\mu M)\,\overline{X}_{q-s}^{(k+1)} = M X_{q-s}^{(k)} \tag{17}$$

where

$$X_{q-s}^{(k)} = [x_{s+1}^{(k)}, x_{s+2}^{(k)}, \ldots, x_q^{(k)}] \tag{18}$$

Writing eqns (13), (15) and (17) in matrix form gives

$$\begin{bmatrix} K-\mu M & MX_s^{(k)} \\ X_s^{(k)T} M & 0 \end{bmatrix} \begin{bmatrix} \overline{X}^{(k+1)} \\ \overline{D}^{(k+1)} \end{bmatrix} = \begin{bmatrix} MX^{(k)} \\ E \end{bmatrix} \tag{19}$$

where the unknown (s×q) matrix $\overline{D}^{(k+1)} = [D_s^{(k+1)}, 0, \ldots, 0]$ and the (s×q) matrix $E = [I_s, 0, \ldots, 0]$.

Note that $\overline{X}^{(k+1)}$ from eqn (19) is used in

eqns (5) and (6) instead of $\overline{X}^{(k+1)}$ in eqn (4). Eqn (19) is the main linear algebraic equation used in the proposed method.

The coefficient matrix of eqn (19) is of order $(n+s)$, symmetric, and nonsingular. The nonsingularity is one of the significant advantages of the proposed method and will be shown in the next section.

3.2 Proof of the Nonsingularity of the Coefficient Matrix[17)-19)]

The most remarkable characteristic of the proposed method is that nonsingularity is always guaranteed. Let the coefficient matrix of eqn (19) be denoted by C, that is

$$C=\begin{bmatrix} K-\mu M & MX_s^{(k)} \\ X_s^{(k)^T}M & 0 \end{bmatrix}. \tag{20}$$

If C is nonsingular when the shift μ becomes multiple eigenvalues, that is, $\mu\cong\lambda_1=\cdots=\lambda_s$, then it will be also nonsingular for a non-close shift. The resulting C^* will be

$$C^*=\begin{bmatrix} K-\lambda_s M & MX_s \\ X_s^T M & 0 \end{bmatrix}. \tag{21}$$

Nonsingularity of the proposed method is, therefore, proved by introducing the new eigenvalue problem of the resulting matrix such as

$$C^*Y=M^*YD \tag{22}$$

where D and Y are the eigenvalue and the associate eigenvector matrices of the new eigenvalue problem, respectively, and

$$M^*=\begin{bmatrix} M & 0 \\ 0 & I_s \end{bmatrix} \tag{23}$$

$$Y=[y_1\ y_2\ \cdots\ y_{n+s}] \text{ and } D=diag(\gamma_1,\gamma_2,\cdots,\gamma_{n+s}) \tag{24,25}$$

The eigenpairs of the eigenvalue problem eqn (22), y_j and γ_j for $j=1,2,\cdots,n+s$, are as follows:

- Eigenvector $y_j:\begin{Bmatrix} x_i \\ e_i \end{Bmatrix}, \begin{Bmatrix} x_i \\ -e_i \end{Bmatrix}, \begin{Bmatrix} x_k \\ 0 \end{Bmatrix}$,

$$i=1,2,\cdots,s:\ k=s+1,s+2,\cdots,n \tag{26}$$

- Eigenvalue, $\gamma_j:\begin{cases} 1,\cdots,1\ (s) \\ -1,\cdots,-1\ (s) \\ \lambda_k-\lambda_s\ (n-s) \end{cases}$

$$k=s+1,s+2,\cdots,n \tag{27}$$

where λ_i and x_i are the eigenvalues and eigenvectors of the system $KX=MX\Lambda$, and e_i is $(s\times 1)$ vector that all elements are zero except for ith element with unity.

Considering the determinant of eqn (22), the relationship can be obtained as follows:

$$det[C^*]=det[M^*]det[D]$$
$$=(-1)^s det[M]\prod_{k=s+1}^{n}(\lambda_k-\lambda_s) \tag{28}$$

The determinant of C^* is not zero because of $det[M]\neq 0$ by definition. The nonsingularity of the coefficient matrix in eqn (19) is shown. That is, the numerical stability of the proposed method is proved analytically. The proposed method, therefore, makes up for the disadvantage that the subspace iteration method with shift has the limitations: no limited regions are needed in the proposed method.

3.3 Operation Count and Summary of Algorithm

Let one operation equal to one multiplication which is nearly always followed by an addition. Assume that the half-bandwidths of K and M are m_k and m_M, respectively. The

steps for the subspace iteration method with shifting with the operations are summarized in Table 1, and for the proposed method in Table 2.

The number of operations of the subspace iteration method with shifting is $T_s qn(2m_k+4m_M+2q+4)+n(m_k^2+3m_k+2m_M+2)$, and that for the proposed method $T_p\{qn(2m_k+4m_M+2q+4+s)+sn(m_k+(s+1)/2)\}+n(m_k^2+3m_k+2m_M+2)$. s is the multiplicity of the multiple eigenvalues which is on or very close to a shift. The proposed method needs more operations per each iteration step, $qn+n(m_k+1)$, than the subspace iteration method with shift. Assume that the ratio is composed of the operation count per iteration of the proposed method(N_p), that of the subspace iteration method with shift(N_s), and the difference of the operation count per iteration for above two methods(N_p-N_s) as follows:

$$ratio=\frac{N_p-N_s}{N_p}=\frac{qn(1+s)+sn\{m_k+(s+1)/2\}}{qn\{(2m_k+4m_M+2q+4+s)+sn(m_k+(s+1)/2)\}} \quad (29)$$

Then, if the half-bandwidth of the stiffness matrix(m_k) is equal to that of the mass matrix (m_M), the above ratio can be approximated as follows:

$$ratio\approx\frac{s}{6q} \quad (30)$$

This ratio means that the larger the number of the required eigenpairs, the smaller is the difference of the operation count between the proposed method and the subspace iteration method with shift. That is, the number of operations for the aforementioned two methods, the subspace iteration method with shift and the proposed method, is almost the same when the number of eigenpairs to be required is large.

Table 1 Operation count for subspace iteration method with shift

Operation	Calculation	Number of Operations
Multiplication	$K-\mu M$	$n(m_M+1)$
Factorization	$LDT^T=K-\mu M$	$nm_M(m_M+3)/2$
Iteration		
Multiplication	$MX^{(k)}$	$qn(2m_M+1)$
Solve for $\overline{X}^{(k+1)}$	$(K-\mu M)\,\overline{X}^{(k+1)}=MX(k)$	$qn(2m_K+1)$
Multiplication	$\overline{K}^{(k+1)}=\overline{X}^{(k+1)^T}MX^{(k)}$	$qn(q+1)/2$
Multiplication	$M\overline{X}^{(k+1)}$	$qn(2m_M+1)$
Multiplication	$\overline{M}^{(k+1)}=\overline{X}^{(k+1)^T}M\overline{X}^{(k+1)}$	$qn(q+1)/2$
Solve for $Z^{(k+1)}$ & $\Omega^{(k+1)}$	$\overline{K}^{(k+1)}Z^{(k+1)}=\overline{M}^{(k+1)}Z^{(k+1)}\Omega^{(k+1)}$	$O(q^3)$ □□□□□□
Multiplication	$X^{(k+1)}=\overline{X}^{(k+1)}Z^{(k+1)}$	nq^2
Sturm Sequence Check	$K-\lambda_p M$	$n(m_M+1)$
Multiplication		
Factorization	$LDT^T=K-\lambda_p M$	$nm_M(m_M+3)/2$
Total	$T_s qn(2m_K+4m_M+2q+4)+n(m_K^2+3m_K+2m_M+2)$	

Table 2 Operation count for the proposed method

Operation	Calculation	Number of Operations
Iteration		
$k=0$		
Multiplication	$K-\mu M$	$n(m_M+1)$
Multiplication	$MX^{(0)}$	$qn(2m_M+1)$
Change the last s columns of $K-\mu M$ into $MX_s^{(0)}$		neglected
Factorization	$LDT^T=F^{(1)}$	$\{nm_K(m_K-3+2_s)+(s+1)\}2$
$k=1,2,\cdots$		
Multiplication	$MX^{(k)}$	$qn(2m_M+1)$
Change the last s columns of $K-\mu M$ into $MX_s^{(k)}$		neglected
Factorization	$LDT^T=F^{(k+1)}$	$sn\{m_K+(s-1)/2\}$
Solve for $\bar{X}^{(k+1)}$	$F^{(k+1)}\,\bar{X}^{(k+1)}=R$	$qn(2m_K+s-1)$
Multiplication	$\overline{K}^{(k+1)}=\overline{X}^{(k+1)^T}MX^{(k)}$	$qn(q+1)/2$
Multiplication	$M\overline{X}^{(k+1)}$	$qn(2m_M+1)$
Multiplication	$\overline{M}^{(k+1)}=\overline{X}^{(k+1)^T}M\overline{X}^{(k+1)}$	$qn(q+1)/2$
Solve for $Z^{(k+1)}$ & $\Omega^{(k+1)}$	$\overline{K}^{(k+1)}Z^{(k+1)}=\overline{M}^{(k+1)}Z^{(k+1)}\Omega^{(k+1)}$	$O(q^3)$ □□□□□□
Multiplication	$X^{(k+1)}=\overline{X}^{(k+1)}Z^{(k+1)}$	nq^2
Sturm Sequence Check		
Multiplication	$K=\lambda_p M$	$n(m_M+1)$
Factorization	$LDT^T=K-\lambda_p M$	$nm_K(m_K+3)/2$
Total	$T_p\{qn(2m_K+4m_M+2q+4+s)+sn(m_K+(s+1)/2)\}$ $+n(m_K^2+3m_K+2m_M+2)$	

where $F^{(k+1)}=\begin{bmatrix} K-\mu M & MX_s^{(k)} \\ X_s^{(k)^T}M & 0 \end{bmatrix}$, $\bar{X}^{(k+1)}=\begin{bmatrix} \overline{X}^{(k+1)} \\ \overline{D}_s^{(k+1)} \end{bmatrix}$, $R=\begin{bmatrix} MX^{(k)} \\ E_s \end{bmatrix}$ (see eqn(19))

4. Numerical Examples

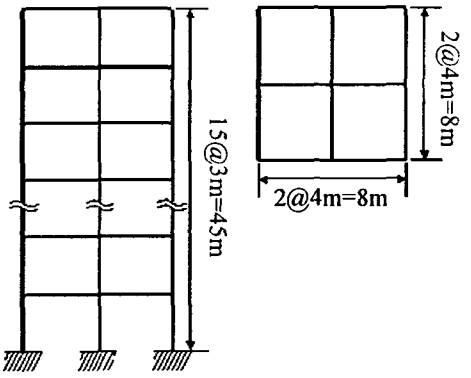
The three-dimensional framed structure and the simply supported square plate are analyzed to verify the effectiveness of the proposed method. With the predetermined error norm of 10^{-6}, the structures are analyzed by two methods: the subspace iteration method with shift which is not used the limited region(see eqn (11)) and the proposed method, where the error norm is computed by the following equation:

$$error\ norm=\frac{||(K-\lambda_i^{(k)}M)x_i^{(k)}||_2}{||Kx_i^{(k)}||_2} \tag{31}$$

Even if a shift is on or very close to multiple eigenvalues, it is shown that the proposed method can obtain the solutions without any singularity. When a shift is not close to

multiple eigenvalues, each convergence rate for calculating the first ten eigenpairs is compared. All runs are executed in the IRIS4D-20-S17 with 10 Mips and 0.9 Mflops.

4.1 Three-dimensional Framed Structure



(a) Elevation (b) Plan

$A=0.2787m^2$, $I=8.631\times10^{-3}m^4$

$E=2.068\times10^{10}\ Pa$, $\rho=5.154\times10^{2}\ kg/m^3$

Fig.1 Three-dimensional framed structure

The first example is a three-dimensional framed structure. The geometric configuration and the material properties are shown in Fig.1 The structure discretized using 315 beam elements resulting in system of dynamic equations with a total of 810 degrees of freedom. The consistent mass matrix is used for *M*.

The lowest ten eigenvalues of the model are shown in Table 3. The eigenvalues of the model are distinct root or multiple ones.

Some results are shown in Table 4 and in Fig.2 to Fig.7 The solution time for two methods are summarized in Table 4. When a shift is on 1.01 λ_4, the subspace iteration method with shift and the proposed method obtain the required ten eigenpairs. However, when the shift is on 1.00001 λ_4 or on λ_4, the subspace iteration method with shift does not calculate the solutions while the proposed method finds the solutions. It shows that the iteration procedure for the proposed method can converge without any singularity even if the shift is the same exactly to multiple eigenvalues, as analytically proved in the article 3.2.

For each solution method, the convergence of each eigenpair is depicted in Fig.2 to Fig.7. Fig.2 to Fig.3 show that when the shift is on 1.01 λ_4 the convergence of the proposed method is nearly equal to that of the subspace iteration method with shift. Fig.4 to Fig.5 show that when the shift is on 1.00001 λ_4 the proposed method converges

Table 3 The lowest ten eigenvalues of the three-dim. framed structure

Mode number	Eigenvalues
1	0.1556E+03
2	0.1556E+03
3	0.3112E+03
4	0.1623E+04
5	0.1623E+04
6	0.2840E+04
7	0.5736E+04
8	0.5736E+04
9	0.8942E+04
10	0.1202E+05

Table 4 Solution time for the lowest ten eigenvalues of the three-dim. framed structure

Analysis methods	Shift=1.01 λ_4	Shift=1.00001 λ_4	Shift= λ_4
Subspace iteration method with shift	409.86(1.00)	No solution	No solution
Proposed method	421.58(1.03)	421.69	421.19

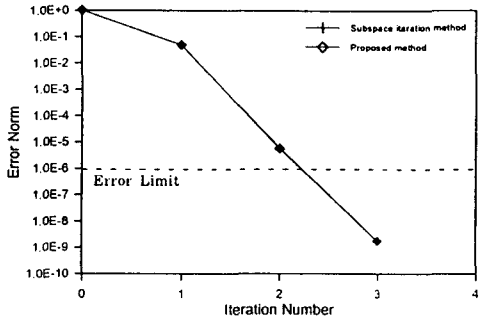


Fig. 2 Error norm versus iteration number of the 4th eigenpair in case of shift = $1.01\lambda_4$

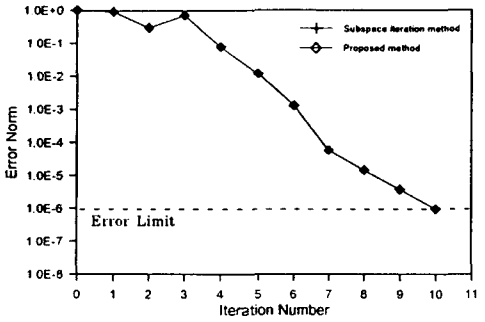


Fig. 3 Error norm versus iteration number of the 10th eigenpair in case of shift = $1.01\lambda_4$

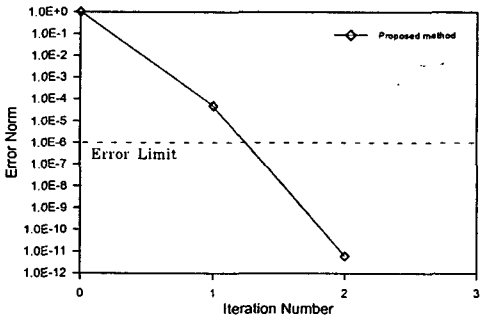


Fig. 4 Error norm versus iteration number of the 4th eigenpair in case of shift = $1.00001\lambda_4$

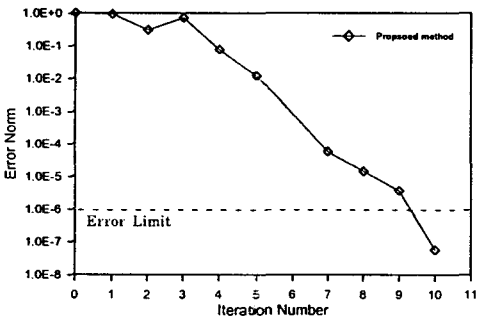


Fig. 5 Error norm versus iteration number of the 10th eigenpair in case of shift = $1.00001\lambda_4$

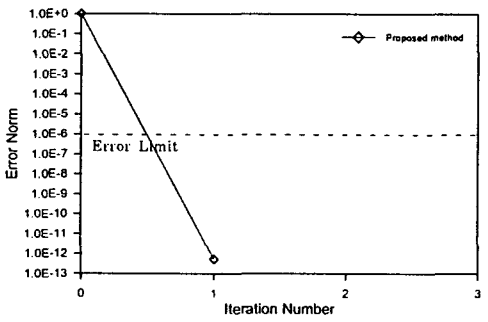


Fig. 6 Error norm versus iteration number of the 4th eigenpair in case of shift = λ_4

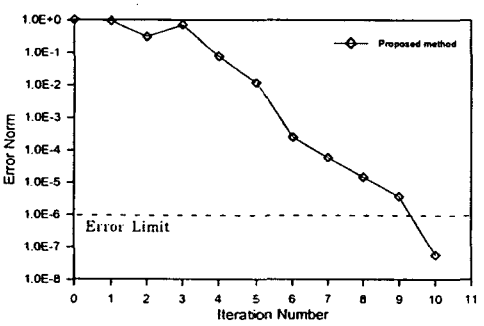


Fig. 7 Error norm versus iteration number of the 10th eigenpair in case of shift = λ_4

well without any singularity while the subspace iteration method with shift can not converge due to the singularity. Fig.6 to Fig.7 show that when the shift is the same exactly to the second eigenvalue the proposed method only converges well without any singularity. This results are the same as a shift is on 1.00001 λ_4 As the above results, the proposed method can choose a more exact shift than the subspace iteration method with shift, thus the proposed method may be the more computationally efficient.

4.2 Simply Supported Square Plate

The second example is the simply supported square plate. Fig.8 shows the geometric configuration and material properties. The structure discretized using 36 shell elements (nine node/element) resulting in system of dynamic equations with a total of 701 degrees of freedom. The consistent mass matrix is used for M.

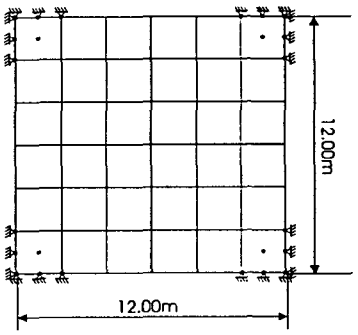
The lowest ten eigenvalues of the model are shown in Table 5. The eigenvalues of the model are distinct root or multiple ones.

Some results are shown in Table 6 and in Fig.9 to Fig.14. The solution time for two methods are summarized in Table 6. When a shift is on 1.01 λ_2, the subspace iteration method with shift and the proposed method obtain the required ten eigenpairs. However, when the shift is on 1.0001 λ_2 or on λ_2, the subspace iteration method with shift does not calculate the solutions while the proposed method finds the solutions. It shows that the iteration procedure for the proposed method can converge without any singularity even if the shift is the same exactly to multiple eigenvalues.

For each solution method, the convergence of each eigenpair is depicted in Fig.9 to Fig.14. Fig.9 to Fig.10 show that when the

Table 5 The lowest ten eigenvalues of the simply supported square plate

Mode number	Eigenvalues
1	0.4435E+01
2	0.2914E+02
3	0.2914E+02
4	0.7367E+02
5	0.1305E+03
6	0.1305E+03
7	0.2087E+03
8	0.2087E+03
9	0.4010E+03
10	0.4418E+03



$E=2.0\times10^{11}Pa$, $\rho=7.85\times10^{3}\,kg/m^{3}$

poisson ratio = 0.3, Shell thickness=0.01m

Fig. 8 Simply supported square plate

Table 6 Solution time for the lowest ten eigenvalues of the three-dim. framed structure

Analysis methods	Shift=1.01 λ_2	Shift=1.00001 λ_2	Shift= λ_2
Subspace iteration method with shift	720.95(1.00)	No solution	No solution
Proposed method	751.05(1.04)	750.64	751.25

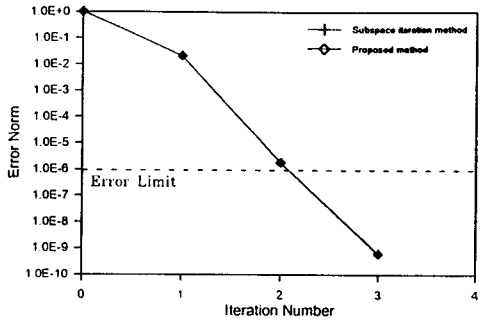


Fig. 9 Error norm versus iteration number of the 2nd eigenpair in case of shift = $1.01\lambda_2$

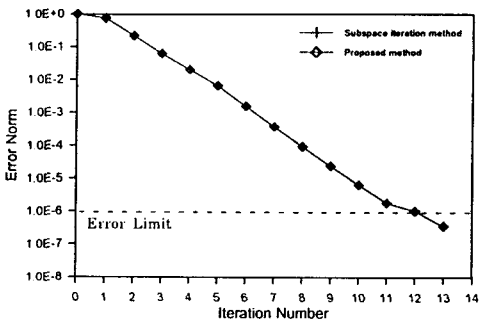


Fig. 10 Error norm versus iteration number of the 10th eigenpair in case of shift = $1.01\lambda_2$

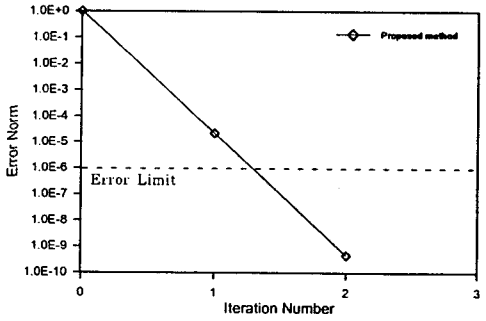


Fig. 11 Error norm versus iteration number of the 2nd eigenpair in case of shift = $1.00001\lambda_2$

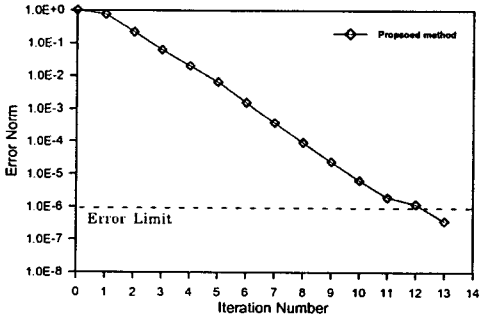


Fig. 12 Error norm versus iteration number of the 10th eigenpair in case of shift = $1.00001\lambda_2$

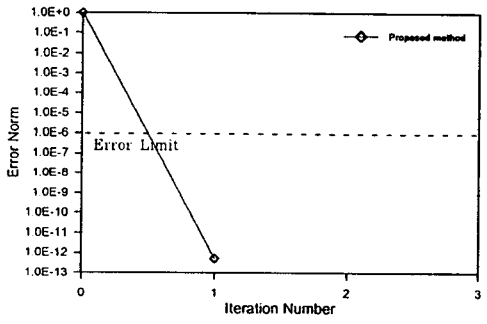


Fig. 13 Error norm versus iteration number of the 2nd eigenpair in case of shift = λ_2

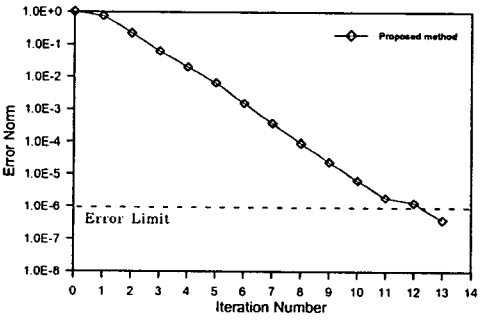


Fig. 14 Error norm versus iteration number of the 10th eigenpair in case of shift = λ_2

shift is on 1.01 λ_2 the convergence of the proposed method is nearly equal to that of the subspace iteration method with shift. Fig.11 to Fig.12 show that when the shift is on 1.00001 λ_2 the proposed method converges well without any singularity while the subspace iteration method with shift can not converge due to the singularity. Fig.13 to Fig.14 show that show that when the shift is the same exactly to the second eigenvalue the proposed method only converges well without any singularity.

5. Conclusions

A numerically stable technique using side conditions for improving the subspace iteration method with shift has been presented. The characteristics of the proposed method identified by the analytical and the numerical results from numerical examples are summarized as follows:

1) The nonsingularity of the proposed method is always guaranteed, which is proved analytically; even if the shift is on or very close to multiple eigenvalues, the proposed method can obtain the solutions without any singularity.

2) The convergence rate of the proposed method is at least equal to that of the subspace iteration method with shift, and the operation counts of the proposed method and the subspace iteration method with shift are almost the same when the number of eigenpairs to be required is large.

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