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A New Upper Bound of Convolution-type for Median-Unbiased Estimators†

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ABSTRACT

We derive a new upper bound of convolution type for the median-unbiased estimators with respect to an arbitrary unimodal utility function. We also obtain the necessary and sufficient condition for the attainability of the information bound. Applications to general MLR (*Monotone Likelihood Ratio*) model and censored survival data are discussed as examples.

Keywords: Median-unbiased estimator; Information bound

1. INTRODUCTION

Let μ be a Lebesque measure on the Euclidean space R^n . Let $\mathcal{P} = \{P_\theta; \theta \in \Theta\}$ be the family of distributions on R^n with the density function $f(x;\theta)$ with respect to μ where $x = (x_1, \cdots, x_n) \in R^n$ and $\Theta \in R$ is an open interval in R.

Let $X = (X_1, \cdots, X_n)$ be a random vector having a joint density function $f(x;\theta)$ and let $g(\theta)$ be a real valued differentiable function of $\theta \in \Theta$.

Definition 1.1. *An estimator $\delta(X)$ of $g(\theta)$ is called median-unbiased if*

$$median_\theta(\delta(X)) = g(\theta) \quad for\ all \quad \theta \in \Theta. \tag{1.1}$$

For any estimator having continuous distribution , the condition (1.1) is equivalent to the following conditions :

$$P_\theta[\delta(X) \leq g(\theta)] = P[\delta(X) \geq g(\theta)] = 1/2 \quad for\ all \ \ \theta \in \Theta.$$

or

$$E_\theta[sgn(\delta(X) - g(\theta))] = 0 \quad for\ all \ \ \theta \in \Theta. \tag{1.2}$$

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where $sgn(x) = +1, 0, -1$ depending on the sign of the value x.

Let $\delta(X)$ be an arbitrary median-unbiased estimator of $g(\theta)$ and let $f_\delta(y;\theta)$ be the density function of the random variable $Y = \delta(X)$ with respect to the Lebesque measure in R.

In this framework several versions of the analogue of the Cramer-Rao lower bound for median-unbiased estimators were proposed in the literature. As a first step in this direction , Alamo (1964) introduced the quantity $f_\delta(g(\theta);\theta)$ as a new measure of *concentration* of the median-unbiased estimator $\delta(X)$ around the estimand $g(\theta)$ and obtained the following lower bound:

$$[2f_\delta(g(\theta);\theta)]^{-1} \geq |g'(\theta)|/I_2(\theta)^{1/2} \tag{1.3}$$

where $I_2(\theta)$ is the usual Fisher information which is defined by

$$I_2(\theta) = E_\theta[(\partial/\partial\theta)log f(X;\theta)]^2 \ .$$

Here we note that the quantity $[2f_\delta(g(\theta);\theta)]^{-1}$ can be interpreted as a natural measure of the *dispersion* of the estimator $\delta(X)$ around the estimand $g(\theta)$ corresponding to the *concentration* measure $f_\delta(g(\theta);\theta)$. See Sung et al. (1990) for more information on the properties of the quantity $[2f_\delta(g(\theta);\theta)]^{-1}$ as a new measure of *diffusivity* of the estimator.

Recently, increasingly sharper lower bounds for the left hand side of the above inequality (1.3) were proposed by several authors including Sung et al. (1990) and So (1994). For example Sung et al. (1990) obtained the improved result :

$$[2f_\delta(g(\theta);\theta)]^{-1} \geq |g'(\theta)|/I_1(\theta) \tag{1.4}$$

where $I_1(\theta)$ is a simple L_1-analogue of the Fisher information

$$I_1(\theta) = E_\theta|(\partial/\partial\theta)log f(X;\theta)|.$$

Finally , motivated by the generalized Neyman-Pearson Lemma, So (1994) obtained a sharper lower bound.

$$[2f_\delta(g(\theta);\theta)]^{-1} \geq |g'(\theta)|/I_1^*(\theta) \tag{1.5}$$

where $I_1^*(\theta)$ is a *centered* L_1-version of the Fisher information :

$$I_1^*(\theta) = E_\theta|(\partial/\partial\theta)log f(X;\theta) - k^*|.$$

and

$$k^* = median_\theta[(\partial/\partial\theta)log f(X;\theta)].$$

On the other hand, from a decision-theoretic view point, the quantity $f_\delta(y;\theta)$ has a serious drawback as a measure of concentration because it is based on a very special zero-one loss function and is overly sensitive to the value of the density $f_\delta(y;\theta)$ at a single point $g(\theta)$ and ignores all other values.

In order to overcome this difficulty, we will consider a general class of unimodal utility functions $h(\delta(X)-g(\theta))$ in this paper. Then we exploit the representation of the arbitrary unimodal density function as a scale mixture of uniform densities and derive a new information bound of *convolution-type* for the expected utility of the median-unbiased estimators. Essentially we will prove the following inequality:

$$E_\theta[h(\delta(X)-\theta)] \leq (1/2)I(\theta,G) \quad for\ all\ \ \theta\in\Theta, \tag{1.6}$$

where $h(\cdot)$ is an arbitrary unimodal utility function with an unique mode at zero and $I(\theta,G)$ is a new kind of Fisher information which will be defined explicitly in Section 2. Note that the left hand side of (1.6) can be represented formally as the *convolution* $\int h(y-g(\theta))f_\delta(y;\theta)dy$ of the two densities $h(\cdot)$ and $f_\delta(y;\theta)$.

For the special zero-one utility function $h(\cdot)$, our result (1.6) will be shown to include the previous inequality (1.5) in the limit. We will also identify the necessary and sufficient conditions for the attainability of the convolution bound (1.6) which will be very useful in constructing optimal mean-unbiased estimators in many practical problems.

Finally in Section 4. we show that these optimality conditions are satisfied in several examples including general family of density functions having MLR(*Monotone Likelihood Ratio*) property and some models with censored survival data.

2. UPPER BOUNDS

In this section we construct a class of general information inequalities of convolution-type for the median-unbiased estimators with respect to arbitrary unimodal utility functions.

Let Δ be a real number such that both θ and $\theta+\Delta$ belong to Θ. By the definition of median-unbiased estimator, we have

$$E_\theta[sgn(\delta(X)-g(\theta))]=0. \tag{2.1}$$

$$E_{\theta+\Delta}[sgn(\delta(X)-g(\theta+\Delta))]=0\ . \tag{2.2}$$

Subtracting (2.1) from (2.2) and rearranging suitable expressions, we obtain

$$\int [f(x;\theta+\Delta) - f(x;\theta)] sgn[\delta(X) - g(\theta+\Delta)] d\mu$$

$$+ E_\theta\{sgn[\delta(X) - g(\theta+\Delta)] - sgn[\delta(X) - g(\theta)]\} = 0. \qquad (2.3)$$

Multiplying (2.2) by $k\Delta$ and subtracting it from (2.3), we get the identity :

$$\int [f(x;\theta+\Delta) - f(x;\theta) - k\cdot\Delta f(x;\theta+\Delta)] sgn[\delta(X) - g(\theta+\Delta)] d\mu$$

$$+ E_\theta\{sgn[\delta(X) - g(\theta+\Delta)] - sgn[\delta(X) - g(\theta)]\} = 0. \qquad (2.4)$$

We now summarizes above results in the following lemma.

Lemma 2.1. *Let $g(\theta)$ be a real valued function on Θ. Let $\delta(X)$ be a median-unbiased estimator of $g(\theta)$ having continuous distribution function $F_\delta(\cdot;\theta)$. Then for an arbitrary constant k, we have the identity :*

$$[F_\delta(g(\theta+\Delta);\theta) - F_\delta(g(\theta);\theta)]$$

$$= (1/2)\int |f(x;\theta+\Delta) - f(x;\theta) - k\Delta f(x;\theta+\Delta)| s_1 \cdot s_2 d\mu \qquad (2.5)$$

where $s_1 = sgn[f(x;\theta+\Delta) - f(x;\theta) - k\Delta f(x;\theta+\Delta)]$ and
$s_2 = sgn[\delta(X) - g(\theta+\Delta)]$.

Proof: This identity follows immediately from (2.4) . □

Now we are ready to obtain a general upper bound for the expected utility function of the median-unbiased estimator $\delta(X)$ with respect to an arbitrary unimodal utility function.

Suppose $g(\theta) = \theta$ without loss of generality and let $\Delta > 0$. Then we can obtain the following inequality directly from Lemma 2.1.

$$P_\theta[\theta \le \delta(X) \le \theta+\Delta] \le (1/2)\int |f(x;\theta+\Delta) - f(x;\theta) - k\Delta f(x;\theta+\Delta)| d\mu \qquad (2.6)$$

for an arbitrary constant k. Dividing (2.6) by Δ and taking infimum with respect to k on the right hand side of (2.6). we obtain the bound :

$$E_\theta I[0 \le \delta(X) - \theta \le \Delta]/\Delta \le (1/2) I(\theta,\Delta)/\Delta \qquad (2.7)$$

where $I(\theta, \Delta)$ is an analogue of Fisher information defined by :

$$I(\theta, \Delta) = inf_{k \in R} \int |f(x; \theta + \Delta) - f(x; \theta) - k\Delta f(x; \theta + \Delta)| d\mu. \tag{2.8}$$

Now if we integrate both sides of (2.7) with respect to Δ for an arbitrary distribution function $G(\Delta)$, $\Delta > 0$, we get the following inequality :

$$E_\Delta E_\theta [h_\Delta(\delta(X) - \theta)] \ \le \ (1/2) E_\Delta [I(\theta, \Delta)/\Delta] \tag{2.9}$$

where $h_\Delta(u) = I(0 \le u \le \Delta)/\Delta$. By the Fubini's Theorem, we can interchange the order of integration in (2.9) and obtain the following inequality immediately:

$$E_\theta [h_G(\delta(X) - \theta)] \ \le \ (1/2) I(\theta, G) \tag{2.10}$$

where

$$h_G(u) = \int_0^\infty h_\Delta(u) dG(\Delta) \tag{2.11}$$

and

$$I(\theta, G) = \int_0^\infty I(\theta, \Delta)/\Delta \ dG(\Delta).$$

Here we note that the function $h_G(u)$ defined in (2.11) is a monotonically decreasing density function of $u > 0$ and it has an unique maximum at $u = 0$. Hence we can consider $h_G(u)$ as a kind of utility function which measures the degree of concentration of the estimator $\delta(X)$ around the true value θ.

For general $\Delta \in R$, we introduce the following notations. Let $h_G(u)$ be an arbitrary unimodal density function in R with an unique mode at zero such that for some distribution function $G(\cdot)$ in R with no atom at zero ;

$$h_G(u) = \begin{cases} \int_0^\infty h_\Delta(u) dG(\Delta) & \text{if } u \ge 0, \\ \int_{-\infty}^0 h_\Delta(u) dG(\Delta) & \text{if } u < 0. \end{cases} \tag{2.12}$$

where $h_\Delta(u) = I(0 \le u \le \Delta)/\Delta$ for $\Delta > 0$ and $h_\Delta(u) = -I(\Delta \le u \ < 0)/\Delta$ for $\Delta < 0$.

We also introduce the following definition of the information number :

$$I(\theta, G) = E_\Delta \ [\ I(\theta, \Delta)/ \ |\Delta| \ ] = \int_{-\infty}^\infty I(\theta, \Delta)/|\Delta| \ dG(\Delta), \tag{2.13}$$

where

$$I(\theta,\Delta) = inf_{k\in R}\int |f(x;\theta+\Delta) - f(x;\theta) - k\Delta f(x;\theta+\Delta)|d\mu.$$

Now, if we apply the same arguments to the case $\Delta < 0$ as in (2.10), we can obtain an universal upper bound for the concentration measure of the median-unbiased estimator for an arbitrary unimodal utility function. This result is summarized in the following theorem.

Theorem 2.1. *Let $\delta(X)$ be a median-unbiased estimator of θ having a continuous distribution function. Then we have :*

$$E_\theta[h_G(\delta(X)-\theta)] \le (1/2)I(\theta,G). \tag{2.14}$$

Remark 2.1. It is well known in the theory of unimodal function that every unimodal density $h(\cdot)$ with an unique mode at zero can be represented by (2.13) for some distribution $G(\cdot)$. See Feller(1968) for more details.

Remark 2.2. We note that the information quantity $I(\theta,G)$ is always bounded above by the following number ;

$$I(\theta,G) \le \int\int |f(x;\theta+\Delta) - f(x;\theta)|/|\Delta|\ d\mu\ dG(\Delta) \tag{2.15}$$

which is again bounded by a number given by ;

$$I(\theta,G) \le \int\int\int_0^1 |\partial f(x;\theta+u\Delta)/\partial\theta|\ du\ d\mu\ dG(\Delta). \tag{2.16}$$

Remark 2.3. If we choose $G(u) = I[u \ge \Delta]$, $\Delta > 0$, then (2.14) reduces to (2.7). Moreover if we let $\Delta \to 0$ in (2.7), the left hand side of the inequality approaches the density height $f_\delta(g(\theta);\theta)$ at $g(\theta) = \theta$ and the right hand side $I(\theta,\Delta)/\Delta$ reduces to $I_1^*(\theta)$ in (1.5). Thus our result generalizes previous bound (1.5) for the density height.

Remark 2.4. In the usual large sample asymptotic framework, our result (2.14) reduces to the well known bound of Phanzagl (1970) for the asymptotic risks of the median-unbiased estimators.

Now as a non-trivial application of the above results, we list some special case of Theorem 2.1 in the following corollary.

Corollary 2.1. *Under the same assumption as in Theorem 2.1, we have the inequality :*

$$E_\theta[|\delta(X) - \theta|^{-p} - 1]^+ \leq (1/2)\int_{-1}^{+1}[I(\theta,\Delta)/\Delta]p|\Delta|^{-p}d\Delta, \tag{2.17}$$

for any $0 < p < 1$ *where* $x^+ = max(x, 0)$.

Proof: Let $dG(x) = p/|x|^p dx$, $-1 < x < 1$ and let $h_G(x) = [1/|x|^p - 1]^+$. Then it follows immediately from (2.14). □

Remark 2.5. Note that the utility function $h_G(x) = [1/|x|^p - 1]^+$ is an unbounded function which becomes infinity as x tends to 0. But the right hand side of (2.17) is usually finite as is noted in the above remark.

3. OPTIMALITY CONDITION

In this section we identify the necessary and sufficient conditions for the attainability of the information bounds derived in Section 2 which will be useful in obtaining optimal median-unbiased estimators in many problems.

We can easily obtain the following results on the attainability of the information bounds from Lemma 2.1.

Theorem 3.1. *We have the equality in* (2.14) *if and only if for all* Δ *in the support of the distribution* $G(u)$, *we have the identity*

$$sgn[[f(x;\theta+\Delta) - f(x;\theta)]/\Delta - kf(x;\theta+\Delta)] = sgn[\delta(X) - \theta - \Delta]\cdot sgn(\Delta) \ a.e.\ \mu \tag{3.1}$$

for some k.

Proof: (3.1) follows immediately from Lemma 2.1 if we note that $|s_1 s_2| \leq 1$ and $|s_1 s_2| = 1$ holds if and only if $s_1 = s_2$ *a.e.* μ. □

Remark 3.6. By the median unbiasedness of $\delta(X)$, we note that the optimality condition (3.1) implies

$$k = median_{\theta+\Delta}\{[f(x;\theta+\Delta) - f(x;\theta)]/\Delta f(x;\theta+\Delta)\}. \tag{3.2}$$

Now we introduce the definition of optimal median-unbiased estimator $\delta(X)$ in terms of the general information bound (2.14).

Definition 3.1. *A median-unbiased estimator $\delta(X)$ of θ is called G-optimal (or h_G-optimal) if we have ;*

$$E_\theta[h_G(\delta(X) - \theta)] = (1/2)I(\theta, G) \ for \ all \ \theta \in \Theta. \tag{3.3}$$

4. EXAMPLES

In this section we give several examples where the optimal median-unbiased estimators can be found by the optimality conditions derived in Section 3.

Example 4.1. (MLR family) Let $\{f(x;\theta), \theta \in \Theta\}$ be a MLR family with respect to $T(X)$ with a continuous distribution function.

Now we show that $T(X)$ is an optimal median-unbiased estimator of $g(\theta)$ where $g(\theta) = median_\theta[T(X)]$. Then by the well known monotonicity property of $g(\theta)$ in the MLR family, we can conclude immediately that $\delta(X) = g^{-1}(T(X))$ is an optimal median-unbiased estimator of θ.

Note that the MLR property implies that for all $\Delta > 0$,

$f(x;\theta+\Delta)/f(x;\theta)$ is an increasing function $h(T(X);\theta,\theta+\Delta)$ of $T(X)$.

This in turn implies that for all $\Delta > 0$,

$$\begin{aligned} sgn[\ f(x;\theta+\Delta)/f(x;\theta) - k\ ] &= sgn[T(X) - g(\theta+\Delta)] \\ &= sgn[\delta(X) - \theta - \Delta] \ a.e. \ \mu \end{aligned} \tag{4.1}$$

where we choose $k = h(g(\theta+\Delta);\theta,\theta+\Delta)$. This together with a similar identity for $\Delta < 0$ finishes the proof of the optimality of $T(X)$ and $\delta(X)$ with respect to an arbitrary unimodal utility function $h_G(\cdot)$. As a specific example of MLR family, we consider the following example.

Example 4.2. Let $X = (X_1, \cdots, X_n)$ be a random sample from the exponential distribution $f(x;\theta) = \theta^{-1}exp(-x/\theta)$, $x > 0$, $\theta > 0$. Then the joint distribution of $X = (X_1, \cdots, X_n)$ is an exponential family and has an MLR property in $T(X) = \sum_{i=1}^n X_i$. Thus $T(X)$ is an optimal median-unbiased estimator of $C_n\theta$ where C_n denotes the median of the Gamma distribution with shape parameter n and scale parameter 1.

Therefore $\delta(X) = \sum_{i=1}^n X_i/C_n$ is an optimal median-unbiased estimator of θ.

Example 4.3. (Censored Data) Let $X = (X_1, \cdots, X_n)$ be a random sample from the distribution of the form $f(x;\theta) = f(x-\theta)$, $x, \theta \in R$. Let the density $f(x-\theta)$ of X_i have a strongly unimodal property . Then we have $l(x) = log f(x)$ is a concave function of x. Let $G_\Delta(u) = I_{[\Delta,\infty)}(u)$, $h_\Delta(u) = I_{[0,\Delta]}(u)/\Delta$. Suppose we only observe first m-order statistics $(X_{(1)}, \cdots, X_{(m)})$, $m < n$ which represent censored data. Then the joint p.d.f. of the censored data $(X_{(1)}, \cdots, X_{(m)})$ is given by ;

$$f(x,\theta) = \prod_{i=1}^{m} f(x_{(i)} - \theta) \cdot (1 - F(x_{(m)} - \theta))^{n-m}. \tag{4.2}$$

Now it follows from (4.2) that for each $\Delta > 0$,

$$f(x;\theta+\Delta)/f(x;\theta)$$

$$= \prod_{i=1}^{m} [\ f(x_{(i)} - \theta - \Delta)/f(x_{(i)} - \theta)\ ]\ [\ \overline{F}(x_{(m)} - \theta - \Delta)/\overline{F}(x_{(m)} - \theta)\ ]^{n-m}$$

$$= exp\ [\ \sum_{i=1}^{m} [\ l(x_{(i)} - \theta - \Delta) - l(x_{(i)} - \theta)\ ]$$

$$+\ (n-m)\ [\ log\overline{F}(x_{(m)} - \theta - \Delta) - log\overline{F}(x_{(m)} - \theta)\ ] \tag{4.3}$$

where $\overline{F} = 1 - F$.

We note that (4.3) is a monotonically decreasing function of θ by the concavity of $l(\cdot)$. See Ross(1983) for more details.

Now let $\delta(X)$ be the solution of the equation

$$f(x;\theta+\Delta)/f(x;\theta) - k = 0 \tag{4.4}$$

where $k^{-1} = median_{\theta+\Delta}[f(x;\theta)/f(x;\theta+\Delta)]$. Here we note that k in (4.4) is independent of θ by the invariance property of the distribution of $(x_i - \theta)$, $i = 1, \cdots, n$. Then, by the monotonicity of the likelihood ratio, we have, for any $\Delta > 0$,

$$sgn[f(x;\theta+\Delta)/f(x;\theta) - k] = sgn[\delta(x) - \theta] = sgn[\delta(x) + \Delta - \theta - \Delta]\ a.e.\ \mu.$$

This shows that $\delta_\Delta(X) = \delta(X) + \Delta$ is a G_Δ-optimal (h_Δ-optimal) median-unbiased estimator of θ for an arbitrary $\Delta > 0$.

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