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Some Limit Theorems for Fractional Lévy Brownian Motions on Rectangles in the Plane†

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Abstract

In this paper we establish some limit theorems for a two-parameter fractional Lévy Brownian motion on rectangles in the Euclidean plane via estimating upper bounds of large deviation probabilities on suprema of the two-parameter fractional Lévy Brownian motion.

Keywords: Fractional Lévy Brownian motion; Gaussian process; Borel-Cantelli lemma

1. INTRODUCTION AND RESULTS

We are interested in studying limiting behaviors for the increments of a two-parameter fractional Lévy Brownian motion on rectangles in the Euclidean plane, whose increments are composed of such a type that two sides of the rectangle increase to infinity as time passes by infinitely.

The starting point of this work was some limit theorems on the increments of a two-parameter Wiener process in the books of Csörgő and Révész (1981, pp. 57–87) or Lin and Lu (1992, pp. 40–51). Our results for the two-parameter fractional Lévy Brownian motion in this paper are different from those results in the previous books, because the structures of two processes differ from each other. Also the methods of their proofs are completely different from each other.

Let $\{W(x,y), 0 \le x, y < \infty\}$ be a two-parameter Wiener process on the probability space $(\Omega, \mathcal{S}, \mathcal{P})$, i.e., for the rectangle $R := [x_1, x_2] \times [y_1, y_2] \subset \mathbb{R}_+^2 := [0,\infty) \times [0,\infty)$, the W-measure

$$W(R) = W(x_2, y_2) - W(x_1, y_2) - W(x_2, y_1) + W(x_1, y_1)$$

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satisfies the following conditions:

(1) $W(R)$ has a centered Gaussian distribution with variance $\lambda(R) := (x_2 - x_1)(y_2 - y_1)$,

(2) $W(0,y) = W(x,0) = 0$ for all $0 \leq x, y < \infty$,

(3) $\{W(x,y)\}$ has independent increment process, that is, $W(R_1)$, $W(R_2)$, $\cdots$, $W(R_n)$ $(n = 2, 3, \cdots)$ are independent random variables, if $R_1, R_2, \cdots, R_n$ are disjoint rectangles,

(4) the sample path $W(\omega; x, y)$ is continuous in $(x,y) \in \mathbb{R}_+^2$ with probability one, where $\omega \in \Omega$ is fixed.

For our purpose we shall first quote one of the results in Csörgő and Révész (1981): Let a_T be a nondecreasing function of $T(0 < T < \infty)$ for which

(a) $0 < a_T \leq T$,

(b) T/a_T is nondecreasing.

Let $R_T = R(a_T)$ be the set of rectangles

$$R = [x_1, x_2] \times [y_1, y_2] \quad (0 \leq x_1 < x_2 \leq \sqrt{T},\ 0 \leq y_1 < y_2 \leq \sqrt{T})$$

for which $\lambda(R) = (x_2 - x_1)(y_2 - y_1) \leq a_T$. Let $R_T^* = R^*(a_T) \subset R_T$ be the set of those elements R of R_T for which $\lambda(R) = a_T$.

Theorem A. (Csörgő and Révész (1981)) *Let* $\{W(x,y),\ 0 \leq x, y < \infty\}$ *be a two-parameter Wiener process, and let* a_T *be a nondecreasing function of* T *satisfying above conditions* (a) *and* (b). *Then*

$$\limsup_{T\to\infty} \sup_{R \in R_T} \beta_T |W(R)| = \limsup_{T\to\infty} \sup_{R \in R_T^*} \beta_T |W(R)| = 1 \qquad \text{a.s.},$$

where $\beta_T = (2a_T(\log(T/a_T) + \log\log T))^{-1/2}$.

If a_T *also satisfies the condition*

(c) $\lim_{T\to\infty} \log(T/a_T) \big/ \log\log T = \infty$, *then*

$$\lim_{T\to\infty} \sup_{R \in R_T} \beta_T |W(R)| = \lim_{T\to\infty} \sup_{R \in R_T^*} \beta_T |W(R)| = 1 \qquad \text{a.s.}$$

In this paper we intend to study such a limit theorem as Theorem A type for the increments of a two-parameter fractional Lévy Brownian motion with more general sets of rectangles. Instead of R_T, we consider the set $\mathbb{C}_T$ of rectangles $R = [x_1, x_2] \times [y_1, y_2]$ with $0 \le x_1 < x_2 \le a_T$ and $0 \le y_1 < y_2 \le h_T$, where a_T and h_T are nondecreasing functions of T.

Throughout this paper we shall always assume the following statements: Let $\{X(x,y),\ 0 \le x, y < \infty\}$ be a two-parameter fractional Lévy Brownian motion of order 2α with $0 < \alpha < 1$ on $(\Omega, \mathcal{S}, \mathcal{P})$, that is, let $\{X(x,y),\ 0 \le x, y < \infty\}$ be an almost surely continuous, real-valued Gaussian process on $(\Omega, \mathcal{S}, \mathcal{P})$ with mean zero, $X(0,0) = 0$ and stationary increments

$$E\{X(x_1, y_1) - X(x_2, y_2)\}^2 = \{(x_1 - x_2)^2 + (y_1 - y_2)^2\}^\alpha \tag{1.1}$$

for all distinct two time points (x_1, y_1) and (x_2, y_2) in $\mathbb{R}^2_+$. Then the process $\{X(x,y)\}$ is a general form of the two-parameter Lévy Brownian motion with $\alpha = 1/2$, which is defined as follows: A *two-parameter Lévy Brownian motion* on $\mathbb{R}^2_+$ is a stochastic process $L : \Omega \times \mathbb{R}^2_+ \to \mathbb{R}$ ($\mathbb{R}$ is a set of real numbers) such that

(a) $L(0,0) = 0$,

(b) $L(x,y)$ has a centered Gaussian distribution,

(c) $E\{L(x_1, y_1) - L(x_2, y_2)\}^2 = \{(x_1 - x_2)^2 + (y_1 - y_2)^2\}^{1/2}$,

(d) the sample path $(x,y) \mapsto L(\omega; x, y)$ is almost surely continuous in $(x,y) \in \mathbb{R}^2_+$, where $\omega \in \Omega$ is fixed.

Let us consider the rectangle $R := R(s,t,u,v) := [s, s+t] \times [u, u+v] \subset \mathbb{R}^2_+$ for all $s, u \ge 0$ and $t, v > 0$, and define the increment $X(R)$ of a two-parameter fractional Lévy Brownian motion on R by

$$\begin{aligned} X(R) &:= X(R(s,t,u,v)) \\ &:= X(s+t, u+v) - X(s, u+v) - X(s+t, u) + X(s,u). \end{aligned}$$

Using the property of (1.1), it is easy to see that the standard deviation of $X(R)$ has the translation invariant with respect to s and u, that is,

$$S(t,v) := \{E\big(X(R(s,t,u,v))\big)^2\}^{1/2} = \{E\big(X(R(s+h_1, t, u+h_2, v))\big)^2\}^{1/2}$$

for all $h_1, h_2 \ge 0$.

For $0 < T < \infty$, let a_T and h_T be nondecreasing functions of T for which

(i) $0 < h_T \le a_T \le T$,

(ii) T/a_T and T/h_T are nondecreasing on T,

(iii) there exists $0 < \mu \le 1$ such that $\lim_{T\to\infty} h_T/a_T = \mu$,

(iv) $\lim_{T\to\infty} h_T = \infty$.

For convenience, we denote:

$$B_T = \Big(\frac{T-a_T}{h_T} \vee 1\Big)\Big(\frac{T-h_T}{h_T} \vee 1\Big)$$

where $m \vee n = \max\{m, n\}$, and for $e < T < \infty$

$$\beta_T = \{2(\log B_T + \log\log T)\}^{1/2}.$$

Set

$$\mathbf{D}(a_T, h_T) = \sup_{0\le s\le T-a_T} \sup_{0\le u\le T-h_T} \frac{|X(R(s, a_T, u, h_T))|}{S(a_T, h_T)\beta_T}.$$

The main results are as follows:

Theorem 1.1. *Let $X(R)$ and $S(t, v)$ be as in the above statements. For $0 < T < \infty$, let a_T and h_T be nondecreasing functions of T for which*

(i) $0 < h_T \le a_T \le T$,

(ii) *T/a_T and T/h_T are nondecreasing on T,*

(iii) *there exists $0 < \mu \le 1$ such that $\lim_{T\to\infty} h_T/a_T = \mu$,*

(iv) $\lim_{T\to\infty} h_T = \infty$.

Then we have

$$\limsup_{T\to\infty} \mathbf{D}(a_T, h_T) \le 1 \quad \text{a.s.} \tag{1.2}$$

Theorem 1.2. *Let the assumptions of Theorem 1.1 be satisfied. Further assume that*

(v) $\lim_{T\to\infty} \log B_T / \log\log T = \infty$.

Then we have

$$\liminf_{T\to\infty} \mathbf{D}(a_T, h_T) \ge 1 \quad \text{a.s.} \tag{1.3}$$

Combining Theorems 1.1 and 1.2, we immediately obtain the following limit theorem:

Corollary 1.1. *Under the assumptions of Theorem 1.2, we have*

$$\lim_{T\to\infty} \sup_{0\le s\le T-a_T} \sup_{0\le u\le T-h_T} \frac{|X(R(s,a_T,u,h_T))|}{S(a_T,h_T)\beta_T} = 1 \qquad \text{a.s.}$$

Remark 1.1. The most important difference point between Theorem A and Corollary 1.1 is that the variance $\lambda(R)$ of $W(R)$ in Theorem A represents an area of R itself, but the variance $S^2(a_T, h_T)$ of $X(R)$ in Corollary 1.1 is not its area but $2[{a_T}^{2\alpha} + {h_T}^{2\alpha} - ({a_T}^2 + {h_T}^2)^\alpha]$, because our construction of $X(R)$ was originated from the two-parameter fractional Lévy Brownian motion $\{X(x,y)\}$ of order 2α.

Now we shall show a simple illustration of Corollary 1.1 :

Example 1.1. Let $\{X(x,y),\ 0\le x,y<\infty\}$ be a two-parameter Lévy Brownian motion with $\alpha = 1/2$ in (1.1). For $T\ge e$, let $a_T = \sqrt{T}/2$ and $h_T = a_T - \frac{1}{T}$. Then all conditions of the above Corollary 1.1 are satisfied, and $\mu = 1$ in (iii). Thus we have

$$\lim_{T\to\infty} \sup_{0\le s\le T-a_T} \sup_{0\le u\le T-h_T} \frac{|X(R(s,a_T,u,h_T))|}{2S(a_T,h_T)\sqrt{\log(2\sqrt{T})}} = 1 \qquad \text{a.s.}$$

2. PROOFS

For proving the theorems, we first need the following Lemma 2.1. Let $\mathbb{D} = \{\mathbf{t} : \mathbf{t} = (t_1,\cdots,t_N),\ a_i\le t_i\le b_i,\ i=1,2,\cdots,N\}$ be a real N-dimensional time parameter space. We assume that the space $\mathbb{D}$ has the usual Euclidean norm $\|\cdot\|$ such that $\|\mathbf{t}-\mathbf{s}\|^2 = \sum_{i=1}^N (t_i-s_i)^2$. Let $\{X(\mathbf{t}),\ \mathbf{t}\in\mathbb{D}\}$ be a real-valued separable Gaussian process with $E\,X(\mathbf{t}) = 0$. Suppose that $0 < \sup_{\mathbf{t}\in\mathbb{D}} E\{X(\mathbf{t})\}^2 := \Gamma^2 < \infty$, $\Gamma > 0$, and $E\{X(\mathbf{t}) - X(\mathbf{s})\}^2 \le \varphi^2(\|\mathbf{t}-\mathbf{s}\|)$, where $\varphi(\cdot)$ is a nondecreasing continuous function which satisfies $\int_0^\infty \varphi(e^{-y^2})\,dy < \infty$.

Lemma 2.1. (Fernique, 1964) *Let $\{X(\mathbf{t}),\ \mathbf{t}\in\mathbb{D}\}$ be given as the above statements. Then, for $\lambda>0$, $x\ge 1$ and $A>\sqrt{2N\log 2}$, we have*

$$\begin{aligned} P\Big\{\sup_{\mathbf{t}\in\mathbb{D}} X(\mathbf{t}) \ &\ge\ x\Big(\Gamma + (2\sqrt{2}+2)A\int_0^\infty \varphi(\sqrt{N}\,\lambda 2^{-y^2})\,dy\Big)\Big\} \\ &\le\ (2^N+\psi)\Big(\prod_{i=1}^N\Big(\frac{b_i-a_i}{\lambda}\vee 1\Big)\Big)e^{-x^2/2}, \end{aligned}$$

where $\psi = \sum_{n=1}^{\infty} \exp\{-2^{n-1}(A^2 - 2N\log 2)\} < \infty$.

For some $0 < p < 1$, set $T_k = \exp(k^p), k \in \mathbb{N}$, where $\mathbb{N}$ is the set of positive integers. Throughout this paper, we shall simply denote as follows:

$$\begin{aligned} a_k &= a_{T_k}, \quad h_k = h_{T_k}, \quad \beta_k = \beta_{T_k}, \\ \bar{S}_k &= \sup_{\substack{a_{k-1} \le t \le a_k \\ h_{k-1} \le v \le h_k}} S(t,v), \quad \underline{S}_k = \inf_{\substack{a_{k-1} \le t \le a_k \\ h_{k-1} \le v \le h_k}} S(t,v), \\ \bar{S}_k^\star &= \sup_{h_{k-1} \le t,v \le h_k} S(t,v), \quad \underline{S}_k^\star = \inf_{h_{k-1} \le t,v \le h_k} S(t,v), \\ \bar{\beta}_k &= \sup_{T_{k-1} \le T \le T_k} \beta_T, \quad \underline{\beta}_k = \inf_{T_{k-1} \le T \le T_k} \beta_T. \end{aligned}$$

In the proofs we shall let c and C denote positive constants changing in lines if necessary.

2.1. Proof of Theorem 1.1.

The proof of (1.2) is mainly based on the following Lemmas 2.2~ 2.4:

Lemma 2.2. *Under the assumptions of Theorem 1.1, we have*

$$\lim_{k\to\infty} \frac{T_k}{T_{k-1}} = \lim_{k\to\infty} \frac{a_k}{a_{k-1}} = \lim_{k\to\infty} \frac{h_k}{h_{k-1}} = 1, \tag{2.1}$$

$$\lim_{k\to\infty} \frac{\bar{S}_k^2}{(h_k)^{2\alpha}} = \lim_{k\to\infty} \frac{\underline{S}_k^2}{(h_k)^{2\alpha}} = 2C_{\mu,\alpha} \quad \text{and} \quad 0 < C_{\mu,\alpha} \le 1, \tag{2.2}$$

where $C_{\mu,\alpha} = 1 - \{(1+\mu^2)^\alpha - 1\}\mu^{-2\alpha}$ *for* $0 < \mu \le 1$, *and* $C_{0,\alpha} = 1$, *also we have*

$$\lim_{k\to\infty} \frac{\bar{S}_k^2}{(a_k)^{2\alpha}} = \lim_{k\to\infty} \frac{\underline{S}_k^2}{(a_k)^{2\alpha}} = 2\{1 + \mu^{2\alpha} - (1+\mu^2)^\alpha\}, \quad 0 \le \mu \le 1, \tag{2.3}$$

$$\lim_{k\to\infty} \frac{\bar{S}_k}{\underline{S}_k} = \lim_{k\to\infty} \frac{\bar{S}_k^\star}{\underline{S}_k^\star} = \lim_{k\to\infty} \frac{\bar{\beta}_k}{\underline{\beta}_k} = 1. \tag{2.4}$$

Proof: Using the condition (ii) and the mean-value theorem, it follows that

$$1 \le \frac{a_k}{a_{k-1}} \le \frac{T_k}{T_{k-1}} \le \exp\big(p(k-1)^{p-1}\big) \longrightarrow 1 \quad \text{as} \quad k \to \infty,$$

$$1 \le \frac{h_k}{h_{k-1}} \le \exp\big(p(k-1)^{p-1}\big) \longrightarrow 1 \quad \text{as} \quad k \to \infty.$$

So we get (2.1). As for (2.2), using the relation $ab = \frac{1}{2}(a^2 + b^2 - (a-b)^2)$, it is easy to see that, for all $t, v > 0$,

$$S^2(t,v) = 2\big[t^{2\alpha} + v^{2\alpha} - (t^2+v^2)^\alpha\big] \; > \; 0. \tag{2.5}$$

First consider the case $\mu = 0$. Then

$$\begin{aligned}
(t^2+v^2)^\alpha - t^{2\alpha} &= \int_t^{\sqrt{t^2+v^2}} \frac{d(x^{2\alpha})}{dx}\, dx \\
&= \int_t^{\sqrt{t^2+v^2}} 2\alpha \frac{x^{2\alpha}}{x}\, dx \;\le\; 2\alpha \frac{(t^2+v^2)^\alpha}{t}\big(\sqrt{t^2+v^2} - t\big) \\
&= 2\alpha \frac{(t^2+v^2)^\alpha}{t} \frac{v^2}{\sqrt{t^2+v^2}+t} \;\le\; \alpha\Big(\frac{vt^\alpha}{t}\Big)^2 \frac{(t^2+v^2)^\alpha}{t^{2\alpha}},
\end{aligned}$$

and by (iii) and (2.1)

$$\begin{aligned}
&\sup_{\substack{a_{k-1}\le t\le a_k \\ h_{k-1}\le v\le h_k}} \frac{(t^2+v^2)^\alpha - t^{2\alpha}}{v^{2\alpha}} \le \sup_{\substack{a_{k-1}\le t\le a_k \\ h_{k-1}\le v\le h_k}} \alpha\Big(\frac{vt^\alpha}{tv^\alpha}\Big)^2 \frac{(t^2+v^2)^\alpha}{t^{2\alpha}} \\
&\le \alpha\Big(\frac{h_k}{a_{k-1}}\Big)^{2-2\alpha}\Big(1 + \Big(\frac{h_k}{a_{k-1}}\Big)^2\Big)^\alpha \\
&= \alpha\Big(\frac{h_{k-1}}{a_{k-1}}\frac{h_k}{h_{k-1}}\Big)^{2-2\alpha}\Big(1 + \Big(\frac{h_{k-1}}{a_{k-1}}\frac{h_k}{h_{k-1}}\Big)^2\Big)^\alpha \\
&\longrightarrow 0 \quad \text{as} \quad k \to \infty.
\end{aligned} \tag{2.6}$$

From (2.5), we note that

$$\frac{\bar{S}_k{}^2}{(h_k)^{2\alpha}} = 2 + \sup_{\substack{a_{k-1}\le t\le a_k \\ h_{k-1}\le v\le h_k}} \frac{2\big\{t^{2\alpha} + v^{2\alpha} - (t^2+v^2)^\alpha - (h_k)^{2\alpha}\big\}}{(h_k)^{2\alpha}}.$$

It follows from (2.1) and (2.6) that, for $\mu = 0$,

$$\begin{aligned}
&\sup_{\substack{a_{k-1}\le t\le a_k \\ h_{k-1}\le v\le h_k}} \frac{\big|t^{2\alpha} + v^{2\alpha} - (t^2+v^2)^\alpha - (h_k)^{2\alpha}\big|}{(h_k)^{2\alpha}} \\
&\quad \le \sup_{\substack{a_{k-1}\le t\le a_k \\ h_{k-1}\le v\le h_k}} \frac{(t^2+v^2)^\alpha - t^{2\alpha}}{v^{2\alpha}} + \frac{(h_k)^{2\alpha} - (h_{k-1})^{2\alpha}}{(h_k)^{2\alpha}} \\
&\quad \longrightarrow 0 \quad \text{as} \quad k \to \infty.
\end{aligned}$$

Thus $\lim_{k\to\infty} \bar{S}_k{}^2/(h_k)^{2\alpha} = 2$ for $\mu = 0$. Next suppose that $0 < \mu \le 1$. Then by (iii), (iv) and (2.1)

$$\begin{aligned}
&\Big|\frac{\bar{S}_k{}^2}{(h_k)^{2\alpha}} - 2\frac{(a_k)^{2\alpha} + (h_k)^{2\alpha} - (a_k^2 + h_k^2)^\alpha}{(h_k)^{2\alpha}}\Big| \\
&\quad \le 2\Big(\frac{(a_k)^{2\alpha} - (a_{k-1})^{2\alpha}}{(h_k)^{2\alpha}} + \frac{(h_k)^{2\alpha} - (h_{k-1})^{2\alpha}}{(h_k)^{2\alpha}} \\
&\qquad\qquad + \frac{(a_k{}^2 + h_k{}^2)^\alpha - (a_{k-1}{}^2 + h_{k-1}{}^2)^\alpha}{(h_k)^{2\alpha}}\Big) \\
&\quad \longrightarrow 0 \quad \text{as} \quad k \to \infty,
\end{aligned} \tag{2.7}$$

$$\begin{aligned}
&\lim_{k\to\infty} \frac{(h_k)^{2\alpha} + (a_k)^{2\alpha} - (a_k{}^2 + h_k{}^2)^\alpha}{(h_k)^{2\alpha}} \\
&\quad = \lim_{k\to\infty}\Big\{1 - \frac{(a_k)^{2\alpha}\big(1 + (h_k/a_k)^2\big)^\alpha - (a_k)^{2\alpha}}{(a_k)^{2\alpha}}\Big(\frac{a_k}{h_k}\Big)^{2\alpha}\Big\} \\
&\quad = 1 - \big\{(1+\mu^2)^\alpha - 1\big\}\,\mu^{-2\alpha}.
\end{aligned} \tag{2.8}$$

Thus (2.7) and (2.8) imply (2.2) for $0 < \mu \le 1$. Using the mean-value theorem, it is easy to prove that $0 < C_{\mu,\alpha} \le 1$. (2.3) follows by the same way as in (2.7) and (2.8). As for (2.4), we shall verify only two limits:

$$\Big(\frac{\bar{S}_k^\star}{\underline{S}_k^\star}\Big)^2 = \frac{2(2-2^\alpha)(h_k)^{2\alpha}}{2(2-2^\alpha)(h_{k-1})^{2\alpha}} \longrightarrow 1 \quad \text{as} \quad k \to \infty,$$

$$1 \le \frac{\bar{\beta}_k}{\underline{\beta}_k} \le \left(\frac{\frac{T_k}{h_k}\big(\frac{T_k - a_k}{h_k} \vee 1\big)T_k}{\frac{T_{k-1}}{h_{k-1}}\big(\frac{T_{k-1}-a_{k-1}}{h_{k-1}} \vee 1\big)T_{k-1}}\right)^{1/2} \longrightarrow 1 \quad \text{as} \quad k \to \infty. \qquad \square$$

Lemma 2.3. *Under the assumptions of Theorem 1.1, we have*

$$\limsup_{T\to\infty} \mathbf{D}(a_T, h_T) \le \limsup_{k\to\infty} \mathbf{D}(a_k, h_k) \qquad \text{a.s.} \tag{2.9}$$

Let the assumptions of Theorem 1.2 be satisfied. Then we have

$$\liminf_{T\to\infty} \mathbf{D}(a_T, h_T) \ge \liminf_{k\to\infty} \mathbf{D}(a_k, h_k) \qquad \text{a.s.} \tag{2.10}$$

Proof: Let us first prove (2.9). Let T be in $T_{k-1} \le T \le T_k$, $k \in \mathbb{N}$, where $T_k = \exp(k^p)$ for some $0 < p < 1$. Then the condition (ii) implies that

$$T - a_T \le T_k - a_k \quad \text{and} \quad T - h_T \le T_k - h_k.$$

We denote $\mathbb{A}_k = \{(s,t,u,v) : 0 \le s \le T_k - a_k,\ a_{k-1} \le t \le a_k,\ 0 \le u \le T_k - h_k,\ h_{k-1} \le v \le h_k\}$. Note that, for $T_{k-1} \le T \le T_k$,

$$\begin{aligned}&\mathbf{D}(a_T, h_T)\\ &\le \Big\{\mathbf{D}(a_k,h_k) + \sup_{(s,t,u,v)\in\mathbb{A}_k} \frac{|X(R(s,t,u,v)) - X(R(s,a_k,u,h_k))|}{S(a_k,h_k)\beta_k}\Big\}\frac{\bar{S}_k\bar{\beta}_k}{\underline{S}_k\underline{\beta}_k}.\end{aligned}$$

Using Lemma 2.2 (2.3), the inequality (2.9) immediately follows if we only show that

$$\lim_{k\to\infty}\sup_{(s,t,u,v)\in\mathbb{A}_k} \frac{|X(R(s,t,u,v)) - X(R(s,a_k,u,h_k))|}{S(a_k,h_k)\beta_k} = 0 \quad \text{a.s.} \tag{2.11}$$

In order to prove (2.11), we shall make use of Lemma 2.1. Set

$$Z(s,t,u,v) = \frac{X(R(s,t,u,v)) - X(R(s,a_k,u,h_k))}{S(a_k,h_k)}, \quad (s,t,u,v)\in\mathbb{A}_k,$$

$$\varphi(w) = \frac{8(\sqrt{2}w)^\alpha}{S(a_k,h_k)}, \quad w>0.$$

Clearly, $E\,Z(s,t,u,v) = 0$. Using the relation $(A+B-C)^2 \le 4(A^2+B^2+C^2)$, it follows from (iii), (iv), (2.1) and (2.3) that, for all large k,

$$\begin{aligned}\Gamma^2 &:= \sup_{(s,t,u,v)\in\mathbb{A}_k} E\big(Z(s,t,u,v)\big)^2\\ &= \sup_{(s,t,u,v)\in\mathbb{A}_k} \frac{1}{S^2(a_k,h_k)} E\big\{[X(s+t,u+v) - X(s+a_k,u+h_k)]\\ &\qquad +[X(s+a_k,u) - X(s+t,u)] - [X(s,u+v) - X(s,u+h_k)]\big\}^2\\ &\le \sup_{(s,t,u,v)\in\mathbb{A}_k} \frac{4}{S^2(a_k,h_k)}\big\{E[X(s+t,u+v) - X(s+a_k,u+h_k)]^2\\ &\qquad +E[X(s+a_k,u) - X(s+t,u)]^2 + E[X(s,u+v) - X(s,u+h_k)]^2\big\}\\ &= \sup_{(s,t,u,v)\in\mathbb{A}_k} \frac{4}{S^2(a_k,h_k)}\big\{\big((t-a_k)^2 + (v-h_k)^2\big)^\alpha\\ &\qquad +(a_k-t)^{2\alpha} + (v-h_k)^{2\alpha}\big\}\\ &\le \frac{12}{S^2(a_k,h_k)}\big\{(a_k-a_{k-1})^2 + (h_k-h_{k-1})^2\big\}^\alpha\\ &= \frac{12}{\underline{S}_k{}^2}(a_k)^{2\alpha}\Big\{\Big(1-\frac{a_{k-1}}{a_k}\Big)^2 + \Big(\frac{h_k}{a_k}\Big)^2\Big(1-\frac{h_{k-1}}{h_k}\Big)^2\Big\}^\alpha\\ &\le c\Big\{\Big(1-\frac{a_{k-1}}{a_k}\Big)^2 + \Big(1-\frac{h_{k-1}}{h_k}\Big)^2\Big\}^\alpha < (\varepsilon')^2, \quad \forall\,\varepsilon'>0,\end{aligned}$$

where $c > 0$ is a constant. Setting $\mathbf{t} = (s_1, t_1, u_1, v_1)$ and $\mathbf{s} = (s_2, t_2, u_2, v_2)$ in $\mathbb{A}_k$, we analogously get

$$\begin{aligned}
&E\{Z(\mathbf{t}) - Z(\mathbf{s})\}^2 \\
&\le \frac{2}{S^2(a_k, h_k)} E\big\{\big[X(R(s_1, t_1, u_1, v_1)) - X(R(s_2, t_2, u_2, v_2))\big]^2 \\
&\qquad + \big[X(R(s_2, a_k, u_2, h_k)) - X(R(s_1, a_k, u_1, h_k))\big]^2\big\} \\
&\le \frac{64}{S^2(a_k, h_k)} \big((s_1 - s_2 + t_1 - t_2)^2 + (u_1 - u_2 + v_1 - v_2)^2\big)^{\alpha} \\
&\le \frac{64}{S^2(a_k, h_k)} \big(\sqrt{2}\, \|\mathbf{t} - \mathbf{s}\|\big)^{2\alpha} = \varphi^2(\|\mathbf{t} - \mathbf{s}\|).
\end{aligned} \tag{2.12}$$

On the other hand, it follows from (2.2) that for any $\varepsilon' > 0$ there exists a small $c = c(\varepsilon') > 0$ depending only on ε' such that

$$\begin{aligned}
&(2\sqrt{2} + 2) A \int_0^{\infty} \varphi(2ch_k 2^{-y^2})\, dy \\
&\qquad \le 4(2\sqrt{2} + 2) A \frac{(2\sqrt{2}\, ch_k)^{\alpha}}{\underline{S}_k} \sqrt{\frac{\pi}{\alpha \log 2}} < \varepsilon'
\end{aligned}$$

for all large k, where A is a constant such that $A > 2\sqrt{2 \log 2}$. For any given $\varepsilon > 0$ and large $k > 0$, we set

$$x = \varepsilon\big\{2(\log B_{T_k} + \log\log T_k)\big\}^{1/2} / (\Gamma + \varepsilon').$$

Choosing $\varepsilon' = \varepsilon^2$, we have

$$x \ge \big\{2(\log B_{T_k} + \log\log T_k)\big\}^{1/2} / (2\varepsilon).$$

Now applying Lemma 2.1, it follows from (iii)$\sim$(v) and (2.1) that there exists a constant $C(\varepsilon) > 0$ such that

$$\begin{aligned}
&P\Big\{\sup_{(s,t,u,v)\in\mathbb{A}_k} \frac{|X(R(s,t,u,v)) - X(R(s, a_k, u, h_k))|}{S(a_k, h_k)\beta_k} > \varepsilon\Big\} \\
&\le P\Big\{\sup_{(s,t,u,v)\in\mathbb{A}_k} \frac{|X(R(s,t,u,v)) - X(R(s, a_k, u, h_k))|}{S(a_k, h_k)} \\
&\qquad > x\Big(\Gamma + (2\sqrt{2} + 2) A \int_0^{\infty} \varphi(2ch_k 2^{-y^2})\, dy\Big)\Big\} \\
&\le C(\varepsilon) B_{T_k} \exp\Big(-\frac{1}{4\varepsilon^2} \log(B_{T_k} \log T_k)\Big) \le C(\varepsilon)\, k^{-p/(4\varepsilon)},
\end{aligned}$$

provided k is big enough. Hence the series

$$\sum_k P\Big\{\sup_{(s,t,u,v)\in\mathbb{A}_k} \frac{|X(R(s,t,u,v)) - X(R(s, a_k, u, h_k))|}{S(a_k, h_k)\beta_k} > \varepsilon\Big\}$$

is convergent, and using the Borel-Cantelli lemma, we obtain (2.11). This proves (2.9).

The proof of (2.10) is the same technique as that of (2.9): For $T_k \le T \le T_{k+1}$, we denote:

$$\begin{aligned}\mathbb{B}_k = \{(s,t,u,v) : \quad & 0 \le s \le T_{k+1} - a_{k+1},\ a_k \le t \le a_{k+1},\\ & 0 \le u \le T_k - h_{k+1},\ h_k \le v \le h_{k+1}\}.\end{aligned}$$

Then we have

$$\begin{aligned}&\mathbf{D}(a_T, h_T)\\ &\ge \Big\{\mathbf{D}(a_k,h_k) - \sup_{(s,t,u,v)\in\mathbb{B}_k} \frac{|X(R(s,t,u,v)) - X(R(s,a_k,u,h_k))|}{S(a_k,h_k)\beta_k}\Big\} \frac{S_k\beta_k}{\bar{S}_k\bar{\beta}_k}.\end{aligned}$$

In terms of Lemma 2.2 (2.4), the proof of (2.10) is completed only if we show that

$$\lim_{k\to\infty} \sup_{(s,t,u,v)\in\mathbb{B}_k} \frac{|X(R(s,t,u,v)) - X(R(s,a_k,u,h_k))|}{S(a_k,h_k)\beta_k} = 0 \quad \text{a.s.},$$

and this proof is the exactly same as that of (2.11). Thus we have (2.10). □

Lemma 2.4. *Let the assumptions of Theorem 1.1 be satisfied. Then for any* $\varepsilon' > 0$ *there exists a positive constant* $C_{\varepsilon'}$ *depending on* ε' *such that the inequality*

$$P\Big\{\sup_{0\le s\le T_k - a_k}\ \sup_{0\le u\le T_k - h_k} \frac{|X(R(s,a_k,u,h_k))|}{S(a_k,h_k)} \ge w\Big\} \le C_{\varepsilon'} B_{T_k} e^{-w^2/(2+\varepsilon')}$$

holds for all $w \ge 0$ *and large* $k > 0$.

Proof: Let $\mathbb{C}_k = \{(s,u) : 0 \le s \le T_k - a_k,\ 0 \le u \le T_k - h_k\}$ be a two-dimensional space. In order to apply Lemma 2.1, we set

$$Y(s,u) = \frac{X(R(s,a_k,u,h_k))}{S(a_k,h_k)}, \quad (s,u) \in \mathbb{C}_k,$$

$$\varphi(z) = \frac{4z^\alpha}{S(a_k,h_k)}, \quad z > 0.$$

Clearly, $EY(s,u) = 0$, $\Gamma^2 := \sup_{(s,u)\in\mathbb{C}_k} E\{Y(s,u)\}^2 = 1$ and

$$\begin{aligned}E\{Y(s_1,u_1) - Y(s_2,u_2)\}^2 &\le \frac{16\{\sqrt{(s_1-s_2)^2 + (u_1-u_2)^2}\}^{2\alpha}}{S^2(a_k,h_k)}\\ &= \varphi^2\big(\sqrt{(s_1-s_2)^2 + (u_1-u_2)^2}\big).\end{aligned}$$

On the other hand, it follows from (2.2) that for any given $\varepsilon' > 0$ there exists a small constant $c = c_{\varepsilon'} > 0$ such that

$$\begin{aligned} &(2\sqrt{2}+2)A\int_0^\infty \varphi(\sqrt{2}\,ch_k\,2^{-y^2})\,dy \\ &\leq \ 4(\sqrt{2}+1)A\frac{(\sqrt{2}\,c\,h_k)^\alpha}{\underline{S}_k}\sqrt{\frac{\pi}{\alpha\log 2}} < \frac{\varepsilon'}{8} \end{aligned}$$

for all large k, where A is a constant such that $A > 2\sqrt{\log 2}$. Let $w = x(1 + (\varepsilon'/8))$, $x \geq 1$. Then it follows from Lemma 2.1 that there exists a constant $C_{\varepsilon'} > 0$ such that

$$\begin{aligned} &P\Big\{\sup_{0\leq s\leq T_k-a_k}\ \sup_{0\leq u\leq T_k-h_k}\frac{|X(R(s,a_k,u,h_k))|}{S(a_k,h_k)} \geq w\Big\} \\ &\leq \ 2P\Big\{\sup_{(s,u)\in\mathbb{C}_k} Y(s,u) \geq x\Big[1+(2\sqrt{2}+2)A\int_0^\infty \varphi(\sqrt{2}\,c\,h_k\,2^{-y^2})\,dy\Big]\Big\} \\ &\leq \ 2(2^4+\psi)\Big(\frac{T_k-a_k}{c\,h_k}\vee 1\Big)\Big(\frac{T_k-h_k}{c\,h_k}\vee 1\Big)e^{-x^2/2} \\ &\leq \ C_{\varepsilon'}\,B_{T_k}e^{-w^2/(2+\varepsilon')} \end{aligned}$$

for all large $k > 0$. In case $0 \leq w \leq 1$, Lemma 2.4 immediately follows if we take $C_{\varepsilon'}$ big enough. □

Proof of (1.2): For any given small $\varepsilon > 0$, take $\varepsilon' < 2\varepsilon$. Then it follows from Lemma 2.4 that

$$\begin{aligned} P\{\mathbf{D}(a_k,h_k) > \sqrt{1+\varepsilon}\} \ &\leq \ C_{\varepsilon'}B_{T_k}\exp\Big(-\frac{2+2\varepsilon}{2+\varepsilon'}(\log B_{T_k}+\log\log T_k)\Big) \\ &\leq \ C_{\varepsilon'}(\log T_k)^{-\frac{2+2\varepsilon}{2+\varepsilon'}}\,. \end{aligned}$$

As in the earlier statements of section 2, we set $T_k = \exp(k^p)$, where $(2+\varepsilon')/(2+2\varepsilon) < p < 1$. Then

$$P\{\mathbf{D}(a_k,h_k) > \sqrt{1+\varepsilon}\} \leq C_{\varepsilon'}k^{-p\left(\frac{2+2\varepsilon}{2+\varepsilon'}\right)}\,.$$

The Borel-Cantelli lemma implies that

$$\limsup_{k\to\infty}\mathbf{D}(a_k,h_k) \ \leq \ 1 \quad \text{a.s.}$$

Lemma 2.3 (2.9) completes the proof of (1.2). □

2.2. Proof of Theorem 1.2.

The proof of (1.3) is mainly due to the following Lemmas 2.5∼2.7:

Lemma 2.5. *Let $p > 0$ and let m, N be positive integers and a $(\neq 0)$ be a real number. Then there exists a constant $c > 0$ such that*

$$\begin{aligned}&\Big|\int_{\sqrt{a^2+(Nm)^2p^2}}^{\sqrt{a^2+(Nm+1)^2p^2}} d(x^{2\alpha}) - \int_{\sqrt{a^2+(Nm-1)^2p^2}}^{\sqrt{a^2+(Nm)^2p^2}} d(x^{2\alpha})\Big| \\ \leq\ & c\frac{\{a^2+(Nm+1)^2p^2\}^{\alpha}p^2}{a^2+(Nm-1)^2p^2}.\end{aligned}$$

Proof: Set $b = (Nm-1)p$, $c = Nmp$ and $d = (Nm+1)p$. Then

$$\begin{aligned}&\int_{\sqrt{a^2+c^2}}^{\sqrt{a^2+d^2}} d(x^{2\alpha}) - \int_{\sqrt{a^2+b^2}}^{\sqrt{a^2+c^2}} d(x^{2\alpha}) \\
=\ & \int_{\sqrt{a^2+b^2}}^{\sqrt{a^2+d^2}+\sqrt{a^2+b^2}-\sqrt{a^2+c^2}} d((y+\sqrt{a^2+c^2}-\sqrt{a^2+b^2})^{2\alpha}) \\
&\qquad - \int_{\sqrt{a^2+b^2}}^{\sqrt{a^2+c^2}} d(x^{2\alpha}) \\
=\ & \int_{\sqrt{a^2+b^2}}^{\sqrt{a^2+d^2}+\sqrt{a^2+b^2}-\sqrt{a^2+c^2}} \Big(\frac{d((x+\sqrt{a^2+c^2}-\sqrt{a^2+b^2})^{2\alpha})}{dx} - \frac{d(x^{2\alpha})}{dx}\Big)\, dx \\
&\qquad + \int_{\sqrt{a^2+c^2}}^{\sqrt{a^2+d^2}+\sqrt{a^2+b^2}-\sqrt{a^2+c^2}} \frac{d(x^{2\alpha})}{dx}\, dx \\
=\ & \int_{\sqrt{a^2+b^2}}^{\sqrt{a^2+d^2}+\sqrt{a^2+b^2}-\sqrt{a^2+c^2}} \Big(\int_x^{x+\sqrt{a^2+c^2}-\sqrt{a^2+b^2}} \frac{d^2(y^{2\alpha})}{dy^2}\, dy\Big)\, dx \\
&\qquad + \int_{\sqrt{a^2+c^2}}^{\sqrt{a^2+d^2}+\sqrt{a^2+b^2}-\sqrt{a^2+c^2}} \frac{d(x^{2\alpha})}{dx}\, dx \\
=\ & \mathrm{I} + \mathrm{J}, \text{ say.}\end{aligned}$$

Let us estimate an upper bound for I. Then, for some constant $c_2 > 0$,

$$\begin{aligned}\mathrm{I}\ =\ & \int_{\sqrt{a^2+b^2}}^{\sqrt{a^2+d^2}+\sqrt{a^2+b^2}-\sqrt{a^2+c^2}} \Big(\int_x^{x+\sqrt{a^2+c^2}-\sqrt{a^2+b^2}} \big(2\alpha(2\alpha-1)\frac{y^{2\alpha}}{y^2}\big)\, dy\Big)\, dx \\
\leq\ & c_2 \int_{\sqrt{a^2+b^2}}^{\sqrt{a^2+d^2}+\sqrt{a^2+b^2}-\sqrt{a^2+c^2}} \frac{(x+\sqrt{a^2+c^2}-\sqrt{a^2+b^2})^{2\alpha}}{x^2} \\
&\qquad \times\big(\sqrt{a^2+c^2}-\sqrt{a^2+b^2}\big)\, dx\end{aligned}$$

$$\begin{aligned} &\le c_2 \frac{(a^2+d^2)^\alpha}{a^2+b^2}\big(\sqrt{a^2+d^2}-\sqrt{a^2+c^2}\big)\big(\sqrt{a^2+c^2}-\sqrt{a^2+b^2}\big) \\ &= c_2 \frac{(a^2+d^2)^\alpha (d^2-c^2)(c^2-b^2)}{(a^2+b^2)(\sqrt{a^2+d^2}+\sqrt{a^2+c^2})(\sqrt{a^2+c^2}+\sqrt{a^2+b^2})} \\ &\le c_2 \frac{(a^2+d^2)^\alpha}{a^2+b^2}(d-c)(c-b) = c_2 \frac{(a^2+(Nm+1)^2p^2)^\alpha p^2}{a^2+(Nm-1)^2p^2}. \end{aligned}$$

As for J, we have, for some constant $c_1 > 0$,

$$\begin{aligned} \mathrm{J} &= \int_{\sqrt{a^2+c^2}}^{\sqrt{a^2+d^2}+\sqrt{a^2+b^2}-\sqrt{a^2+c^2}} \Big(2\alpha \frac{x^{2\alpha}}{x}\Big)\, dx \\ &\le c_1 \frac{(\sqrt{a^2+d^2}\,)^{2\alpha}}{\sqrt{a^2+c^2}}\big(\sqrt{a^2+d^2}-\sqrt{a^2+c^2}-(\sqrt{a^2+c^2}-\sqrt{a^2+b^2})\big) \\ &= c_1 \frac{(a^2+d^2)^\alpha}{\sqrt{a^2+c^2}}\Big(\frac{2Nm+1}{\sqrt{a^2+d^2}+\sqrt{a^2+c^2}} - \frac{2Nm-1}{\sqrt{a^2+c^2}+\sqrt{a^2+b^2}}\Big)p^2 \\ &\le c_1 \frac{(a^2+d^2)^\alpha}{\sqrt{a^2+c^2}} \frac{2p^2}{\sqrt{a^2+c^2}} = 2c_1 \frac{(a^2+(Nm+1)^2p^2)^\alpha p^2}{a^2+N^2m^2p^2}. \end{aligned}$$

Comparing upper bounds of I and J, we have Lemma 2.5. □

Lemma 2.6. (Leadbetter et al. 1983, pp. 81-84) *Let* $\{\xi_{ij},\ i,j=1,2,\cdots,n\}$ *be jointly standardized normal random variables with* covariance $(\xi_{ij}, \xi_{i'j'}) = \Lambda_{ij}^{i'j'}$ *such that*

$$\delta := \max_{(i,j)\neq(i',j')} |\Lambda_{ij}^{i'j'}| < 1.$$

Then for any real number u *and integers* $1 \le l_1 < l_2 < \cdots < l_k \le n$ *and* $1 \le l_1 < l_2 < \cdots < l_g \le n$ *with* $g, k \le n$,

$$\begin{aligned} &P\Big\{\max_{1\le i\le k}\max_{1\le j\le g} \xi_{l_i l_j} \le u\Big\} \\ &\le \{\Phi(u)\}^{kg} + c \sum_{(i,j)\neq(i',j')} |\lambda_{ij}^{i'j'}| \exp\Big(-\frac{u^2}{1+|\lambda_{ij}^{i'j'}|}\Big), \end{aligned} \tag{2.13}$$

where $\lambda_{ij}^{i'j'} = \Lambda_{l_i l_j}^{l_{i'} l_{j'}}$ *and* $c = c(\delta)$ *is a constant independent of* n, u, g *and* k, *and* $\Phi(u) = \int_{-\infty}^{u} \frac{1}{\sqrt{2\pi}} \exp(-y^2/2) dy$.

In order to estimate the upper bound for the second term of the right hand side of the inequality (2.13), we establish the following lemma:

Lemma 2.7. *Let* $\{\xi_{ij}\}, \delta, g, k$ *and* $\lambda_{ij}^{i'j'}$ *be as in Lemma 2.6. Assume that*

$$|\lambda_{ij}^{i'j'}| < (|i-i'|\,|j-j'|)^{-\nu}, \qquad i \neq i',\ j \neq j',$$

and set $u = \sqrt{(2-\eta)\log(kg)}$, *where* ν *and* η *are positive constants such that* $0 < \eta < (1-\delta)\nu/\ (1+\nu+\delta)$. *Then we have*

$$\sum := \sum_{(i,j)\neq(i',j')} |\lambda_{ij}^{i'j'}| \exp\Big(-\frac{u^2}{1+|\lambda_{ij}^{i'j'}|}\Big) \leq c\,(kg)^{-\delta_0},$$

where $\delta_0 = \{\nu(1-\delta) - \eta(1+\delta+\nu)\}/\{(1+\nu)(1+\delta)\} > 0$ *and* c *is a positive constant independent of* g *and* k.

Proof: Let a be such that $0 < a = (1+\eta\delta-\delta)/\{(1+\nu)(1+\delta)\} < 1$. We split the sum $\sum$ into four parts as follows:

$$\begin{aligned}
\sum &= \sum_{\substack{1\leq i,i'\leq k\\ 0<|i-i'|\leq[k^a]}} \sum_{\substack{1\leq j,j'\leq g\\ 0<|j-j'|\leq[g^a]}} |\lambda_{ij}^{i'j'}| \exp\Big(-\frac{u^2}{1+|\lambda_{ij}^{i'j'}|}\Big)\\
&\qquad + \sum_{\substack{1\leq i,i'\leq k\\ |i-i'|>[k^a]}} \sum_{\substack{1\leq j,j'\leq g\\ |j-j'|>[g^a]}} |\lambda_{ij}^{i'j'}| \exp\Big(-\frac{u^2}{1+|\lambda_{ij}^{i'j'}|}\Big)\\
&\qquad + \sum_{\substack{1\leq i,i'\leq k\\ 0<|i-i'|\leq[k^a]}} \sum_{\substack{1\leq j,j'\leq g\\ |j-j'|>[g^a]}} |\lambda_{ij}^{i'j'}| \exp\Big(-\frac{u^2}{1+|\lambda_{ij}^{i'j'}|}\Big)\\
&\qquad + \sum_{\substack{1\leq i,i'\leq k\\ |i-i'|>[k^a]}} \sum_{\substack{1\leq j,j'\leq g\\ 0<|j-j'|\leq[g^a]}} |\lambda_{ij}^{i'j'}| \exp\Big(-\frac{u^2}{1+|\lambda_{ij}^{i'j'}|}\Big)\\
&= \sum\nolimits^{(1)} + \sum\nolimits^{(2)} + \sum\nolimits^{(3)} + \sum\nolimits^{(4)}, \qquad \text{say,}
\end{aligned}$$

where $[\,\cdot\,]$ denotes the integer part. Now let us estimate each upper bound of the above four sums:

$$\begin{aligned}
\sum\nolimits^{(1)} &\leq c(kg)^{1+a} \exp\Big(-\frac{2-\eta}{1+\delta}\log(kg)\Big) = c(kg)^{1+a-\{(2-\eta)/(1+\delta)\}}\\
&= c(kg)^{\{\eta(1+\delta+\nu)-\nu(1-\delta)\}/\{(1+\nu)(1+\delta)\}} = c(kg)^{-\delta_0},
\end{aligned}$$

$$\begin{aligned}
\sum\nolimits^{(2)} &\leq c(kg)^{2-a\nu} \exp\big(-(1-|\lambda_{ij}^{i'j'}|)u^2\big)\\
&\leq c(kg)^{2-a\nu} \exp\big(-(2-\eta)\log(kg) + (kg)^{-a\nu}(2-\eta)\log(kg)\big)\\
&\leq c(kg)^{-a\nu+\eta} = c(kg)^{-\delta_0},
\end{aligned}$$

$$\begin{aligned}\sum\nolimits^{(3)} &\le k^{1+a} g^{2-a\nu} \exp\Big(-\frac{(2-\eta)\log g + (2-\eta)\log k}{1+|\lambda_{ij}^{i'j'}|}\Big)\\ &\le k^{1+a}\exp\Big(-\frac{2-\eta}{1+\delta}\log k\Big) g^{2-a\nu}\exp(-(1-g^{-a\nu})(2-\eta)\log g)\\ &\le ck^{1+a-\{(2-\eta)/(1+\delta)\}} g^{-a\nu+\eta} = c(kg)^{-\delta_0}\end{aligned}$$

and

$$\sum\nolimits^{(4)} \le cg^{1+a-\{(2-\eta)/(1+\delta)\}} k^{-a\nu+\eta} = c(kg)^{-\delta_0}. \qquad \Box$$

Proof of (1.3): The condition (v) of Theorem 1.2 implies that for any $\mathcal{B} > 0$ and one can find integer $N > 0$ large enough such that the inequality

$$\Big(\frac{T-a_T}{h_T}\Big)\Big(\frac{T-h_T}{h_T}\Big) > (\log T)^{\mathcal{B}} > N^2 \tag{2.14}$$

holds for all large $T > 0$. For such given $T > 0$, we define positive integers k_T and g_T by

$$k_T = \Big[\frac{T-a_T}{Nh_T}\Big] \quad\text{and}\quad g_T = \Big[\frac{T-h_T}{Nh_T}\Big],$$

where $[\,\cdot\,]$ denotes the integer part. By (2.14), it is clear that there is a constant $c > 0$ such that

$$k_T g_T > c(\log T)^{\mathcal{B}} \tag{2.15}$$

for large T. For $i = 0, 1, \cdots, k_T$ and $j = 0, 1, \cdots, g_T$, we also define incremental random variables on rectangle R such that

$$X_T(R_{ij}) := X(R((Ni-1)h_T, a_T, (Nj-1)h_T, h_T)).$$

Then $X_T(R_{ij})/S(a_T, h_T)$ are standard normal random variables. It follows from (v) that, for any $0 < \varepsilon' < \varepsilon < 1$ and large T,

$$\begin{aligned}&P\{\mathbf{D}(a_T,h_T) < \sqrt{1-\varepsilon}\}\\ &\le P\Big\{\sup_{0\le s\le T-a_T}\ \sup_{0\le u\le T-h_T} \frac{X(R(s,a_T,u,h_T))}{S(a_T,h_T)} < \{(2-2\varepsilon')\log(k_T g_T)\}^{1/2}\Big\}\\ &\le P\Big\{\max_{0\le i\le k_T}\ \max_{0\le j\le g_T} \frac{X_T(R_{ij})}{S(a_T,h_T)} < \{(2-2\varepsilon')\log(k_T g_T)\}^{1/2}\Big\}.\end{aligned} \tag{2.16}$$

Define the correlation function of $X_T(R_{ij})$ and $X_T(R_{i'j'})$ as follows:

$$\lambda_T(i,i',j,j') = \mathrm{Corr}\,(X_T(R_{ij}), X_T(R_{i'j'})),\quad i \ne i',\quad j \ne j',$$

and set $l = |i - i'| \geq 1$, $m = |j - j'| \geq 1$. By the relation $ab = \frac{1}{2}(a^2+b^2-(a-b)^2)$, it follows that

$$\begin{aligned}
&\operatorname{Cov}(X_T(R_{ij}), X_T(R_{i'j'})) \\
=\ & \frac{1}{2}\Big\{ - (\sqrt{l^2+m^2}\,Nh_T)^{2\alpha} + ((Nlh_T + a_T)^2 + N^2m^2{h_T}^2)^\alpha \\
+\ & (\sqrt{N^2l^2 + (Nm+1)^2}\,h_T)^{2\alpha} - ((Nlh_T + a_T)^2 + (Nm+1)^2{h_T}^2)^\alpha \\
+\ & ((Nlh_T - a_T)^2 + (Nmh_T)^2)^\alpha - (\sqrt{l^2+m^2}\,Nh_T)^{2\alpha} \\
-\ & ((Nlh_T - a_T)^2 + (Nm+1)^2{h_T}^2)^\alpha + (\sqrt{N^2l^2 + (Nm+1)^2}\,h_T)^{2\alpha} \\
+\ & (\sqrt{N^2l^2 + (Nm-1)^2}\,h_T)^{2\alpha} - ((Nlh_T + a_T)^2 + (Nm-1)^2{h_T}^2)^\alpha \\
-\ & (\sqrt{l^2+m^2}\,Nh_T)^{2\alpha} + ((Nlh_T + a_T)^2 + (Nmh_T)^2)^\alpha \\
-\ & ((Nlh_T - a_T)^2 + (Nm-1)^2{h_T}^2)^\alpha + (\sqrt{N^2l^2 + (Nm-1)^2}\,h_T)^{2\alpha} \\
+\ & ((Nlh_T - a_T)^2 + (Nmh_T)^2)^\alpha - (\sqrt{l^2+m^2}\,Nh_T)^{2\alpha}\Big\}.
\end{aligned}$$

Hence we get

$$\begin{aligned}
&\big|\operatorname{Cov}(X_T(R_{ij}), X_T(R_{i'j'}))\big| \\
\leq\ & \big|(\sqrt{N^2l^2 + (Nm+1)^2}\,h_T)^{2\alpha} - (\sqrt{N^2l^2 + N^2m^2}\,h_T)^{2\alpha} \\
& -\{(\sqrt{N^2l^2 + N^2m^2}\,h_T)^{2\alpha} - (\sqrt{N^2l^2 + (Nm-1)^2}\,h_T)^{2\alpha}\}\big| \\
+\ & \frac{1}{2}\big|((Nlh_T - a_T)^2 + (Nm+1)^2{h_T}^2)^\alpha - ((Nlh_T - a_T)^2 + (Nmh_T)^2)^\alpha \\
& -\{((Nlh_T - a_T)^2 + (Nmh_T)^2)^\alpha - ((Nlh_T - a_T)^2 + (Nm-1)^2{h_T}^2)^\alpha\}\big| \\
+\ & \frac{1}{2}\big|((Nlh_T + a_T)^2 + (Nm+1)^2{h_T}^2)^\alpha - ((Nlh_T + a_T)^2 + (Nmh_T)^2)^\alpha \\
& -\{((Nlh_T + a_T)^2 + (Nmh_T)^2)^\alpha - ((Nlh_T + a_T)^2 + (Nm-1)^2{h_T}^2)^\alpha\}\big| \\
=\ & \Big|\int_{\sqrt{N^2l^2+N^2m^2}h_T}^{\sqrt{N^2l^2+(Nm+1)^2}h_T} d(x^{2\alpha}) - \int_{\sqrt{N^2l^2+(Nm-1)^2}h_T}^{\sqrt{N^2l^2+N^2m^2}h_T} d(x^{2\alpha})\Big| \\
+\ & \frac{1}{2}\Big|\int_{\sqrt{(Nlh_T-a_T)^2+(Nmh_T)^2}}^{\sqrt{(Nlh_T-a_T)^2+(Nm+1)^2{h_T}^2}} d(x^{2\alpha}) - \int_{\sqrt{(Nlh_T-a_T)^2+(Nm-1)^2{h_T}^2}}^{\sqrt{(Nlh_T-a_T)^2+(Nmh_T)^2}} d(x^{2\alpha})\Big| \\
+\ & \frac{1}{2}\Big|\int_{\sqrt{(Nlh_T+a_T)^2+(Nmh_T)^2}}^{\sqrt{(Nlh_T+a_T)^2+(Nm+1)^2{h_T}^2}} d(x^{2\alpha}) - \int_{\sqrt{(Nlh_T+a_T)^2+(Nm-1)^2{h_T}^2}}^{\sqrt{(Nlh_T+a_T)^2+(Nmh_T)^2}} d(x^{2\alpha})\Big|.
\end{aligned}$$

Applying Lemma 2.5 for $a = Nlh_T$, $Nlh_T - a_T$, $Nlh_T + a_T$ and $p = h_T$, respectively, we obtain

$$\big|\operatorname{Cov}(X_T(R_{ij}), X_T(R_{i'j'}))\big| \leq c\,\frac{((Nlh_T + a_T)^2 + (Nm+1)^2{h_T}^2)^\alpha {h_T}^2}{(Nlh_T - a_T)^2 + (Nm-1)^2{h_T}^2},$$

where $c > 0$ is a constant, and further by (iii), (iv) and (2.2) we have

$$\begin{aligned}
&|\lambda_T(i,j,i',j')| \\
&\le c\,\frac{((Nlh_T + a_T)^2 + (Nm+1)^2 {h_T}^2)^\alpha {h_T}^2}{\{(Nlh_T - a_T)^2 + (Nm-1)^2 {h_T}^2\} S^2(a_T, h_T)} \\
&\le c\,\frac{({a_T}^2\{((Nlh_T/a_T)+1)^2 + (Nm+1)^2 (h_T/a_T)^2\})^\alpha {h_T}^2}{{a_T}^2\{((Nlh_T/a_T)-1)^2 + (Nm-1)^2 (h_T/a_T)^2\}(h_T)^{2\alpha}} \\
&\le c\,\frac{(Nl+\frac{1}{\mu})^2 + (Nm+1)^2}{(Nl-\frac{1}{\mu})^2 + (Nm-1)^2}\{(Nl+\frac{1}{\mu})^2 + (Nm+1)^2\}^{-(1-\alpha)} \\
&\le c\,(N^2l^2 + N^2m^2)^{-(1-\alpha)} < (l^2+m^2)^{-(1-\alpha)} \\
&\le (2lm)^{-(1-\alpha)} < (lm)^{-\nu},
\end{aligned}$$

where $0 < \nu = 1 - \alpha < 1$ and N is big enough. To estimate the upper bound for the last inequality of (2.16), let us now apply Lemmas 2.6 and 2.7 for

$$\begin{aligned}
\xi_{l_i l_j} &= \frac{X_T(R_{ij})}{S(a_T, h_T)}, \quad i = 1, \ldots, k_T; \quad j = 1, \ldots, g_T, \\
\lambda_{ij}^{i'j'} &= \lambda_T(i,j,i',j') < (lm)^{-\nu}, \; l = |i-i'| \ge 1, \; m = |j-j'| \ge 1, \\
u &= u_T = \{(2-\eta)\log(k_T g_T)\}^{1/2}, \quad \eta = 2\epsilon' < \frac{(1-\delta)\nu}{1+\nu+\delta}, \\
g &= g_T \quad \text{and} \quad k = k_T.
\end{aligned}$$

Then the last inequality of (2.16) is less than or equal to $\{\Phi(u_T)\}^{k_T g_T} + c(k_T g_T)^{-\delta_0}$. Thus we have

$$\begin{aligned}
P\{\mathbf{D}(a_T, h_T) < \sqrt{1-\epsilon}\} &\le \exp(-c(k_T g_T)^\epsilon) + c(k_T g_T)^{-\delta_0} \\
&\le c(k_T g_T)^{-\delta_0}. \qquad (2.17)
\end{aligned}$$

For $k \in \mathbb{N}$, let $T_k = \exp(k^p)$, $0 < p < 1$. Note that the $\mathcal{B}$ in (2.15) can be taken sufficiently large so that $\mathcal{B} > 1/(p\delta_0)$ for given p and δ_0. Then, (2.17) and (2.15) yield

$$P\{\mathbf{D}(a_k, h_k) < \sqrt{1-\epsilon}\} \le c(\log T_k)^{-\mathcal{B}\delta_0} = ck^{-\mathcal{B}p\delta_0}.$$

By using the Borel-Cantelli lemma, we obtain

$$\liminf_{k\to\infty} \mathbf{D}(a_k, h_k) \ge \sqrt{1-\epsilon} \qquad \text{a.s.}$$

Thus the result (1.3) immediately follows from Lemma 2.3 (2.10). □

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