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The Nonlinear Stability of Density Fronts in the Ocean

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Density and temperature fronts are common features of the ocean. However, frontal dynamics are not quasi-geostrophic because the isopycnal deflections associated with fronts are large compared with the scale height of the hydrostatic geopotential. The frontal geostrophic model, developed by Cushman-Roisin et al.(1992) is generally used for describing the dynamics of surface-density ocean fronts, whereas the two-layer frontal geostrophic model is used for fronts on a sloping continental shelf.

This paper investigates the baroclinic nonlinear stability of surface-density ocean fronts and fronts on a sloping continental shelf using the two-layer frontal geostrophic model mentioned above. Nonlinear stability criteria for the two kinds of fronts are obtained using Arnol'd's (1965; 1969) variational principle and a prior estimate method. This is the first time a nonlinear stability criterion for surface ocean fronts has been established, furthermore, the results obtained for fronts on a sloping bottom are superior to any previous ones.

Key words : ocean fronts, frontal geostrophic model, nonlinear stability, variational principle .

1. Introduction

Fronts, which are important natural barriers in the ocean, are regions of large horizontal gradients with certain properties that usually include density. Through hindering the horizontal transfer of heat, momentum, and other properties, they play a crucial role in enhancing vertical exchanges. They are also large reservoirs of potential energy, which are often released in the form of eddies and rings. Density or temperature fronts and isolated eddies are com mon features in many coastal regions of the world's oceans. Examples of currents associated with density fronts include the Denmark Strait overflow (Smith, 1976), Antarctic Bottom Water formed in the Weddel Sea (Whitehead and Worthington, 1982), deep water formation in the Adriatic Sea (Zoccolotti and Salusti, 1987), and deep water replacement in the Strait of Georgia (LeBlond et al., 1991), among many others. The instability of these currents can lead to the formation of deep cold coherent eddies. These flows play an important role in the mesoscale, physical, and biological dynamics of the benthic boundaries on continental shelves. In particular, the correct parameterization of the many fluxes associated with density-driven currents and solitary eddies is crucial in order to accurately ascertain their contribution to longer term climate variations.

There are two important kinematic characteristics that must be accounted for in a complete dynamic description of these flows. The first centers on the fact that the isopcynal deflections associated with density currents and solitary cold eddies are substantial when compared with the scale height of the hydrostatic geopotential. This is an important fact because it implies that the space and time derivatives of the density field in the mass continuity equation can not be neglected in contrast to the horizontal divergence of the velocity field. This implies that, at least in principle, a quasi-geostrophic model is inappropriate as a dynamic theory. The second characteristic is the fact that these flows are strongly baroclinical. This fact is expressed theoretically through the Stern integral constraint (Mory, 1983; 1985), and experiments have consistently shown (Mory et al., 1987; Whitehead et al.,1990) that the Eulerian velocity field in the overlying surrounding fluid is at least comparable (if not larger than) to the co-moving velocities in the eddy interior.

Notwithstanding these points, Smith (1976) was able to use the baroclinic quasi-geostrophic model to describe certain aspects of the instability observed in the density-current associated with the Denmark Strait overflow, plus Griffiths et al. (1982) studied the stability of a couple of density fronts using a long wavelength perturbation analysis of the ageostrophic barotropic instability of the gravity current on a sloping bottom. However, there are substantial differences between their results and laboratory simulations of the instability of a buoyant coupled density front. Accordingly, the establishment of suitable dynamics for studying the instability of ocean fronts remains a problem. In order to construct a theory that can address the deficiencies of the earlier models, Flierl (1980) and Whitehead et al. (1990) developed a model that focused directly on the physics of the baroclinical and finite-amplitude isopcynal deflectional aspects. This model is a kind of simplified two-layer mixed model, i.e., for fronts in coastal regions, the upper layer is quasi-geostrophic, and the lower level is a simplified shallow water model, whereas for surface ocean fronts, the upper layer is a simplified shallow water model, and the lower level is quasi-geostrophic. Swaters (1991) and Swaters and Flierl (1991) discussed the linear instability of ocean and coastal region fronts using this model, and Swaters (1993) studied the nonlinear stability of an intermediate baroclinic flow on a sloping bottom.

Cushman-Roisin (1986) initiated the one-layer frontal geostrophic model for describing the dynamics of surface-density ocean fronts. His main idea was using a primitive equation, similar to that used in establishing the quasi-geostrophic model, to obtain a frontal-geostrophic model, however, in the derivation of the frontal geostrophic equations the Rossby number is made small by requiring that the length scale be larger than the deformation radius due to the characteristics of the finite interfacial displacements. The formalisms of both the frontal geostrophic model and the quasi-geostrophic model are based on the assumption of a small Rossby number so that the velocity field can be split into an easily calculable geostrophic component and a much smaller ageostrophic correction. It is assumed that the interfacial displacement scale is δH (not necessarily equal to the mean depth layer, H) and the motion length scale is L (not necessarily the deformation radius, L_R), then, the f-plane geostrophic requirement provides that the velocity scale is $U=\frac{g'\delta H}{f_0 L}$, thereafter, it follows that the Rossby number is $\varepsilon=\frac{U}{f_0 L}=\frac{g'\delta H}{f_0^2 L^2}=(\frac{\delta H}{H})(\frac{L_R}{L})^2$. The Rossby number, therefore, depends on two scales that characterize the motion: δH and L. In the case of $\delta H \ll H$ and thus $\varepsilon \ll 1$, the dynamics are quasi-geostrophic. In contrast, frontal motions with finite displacements ($\delta H=H$) are ageostrophic unless the length scale is larger than the deformation radius ($L^2 \gg L_R^2$ and thus $\varepsilon \gg 1$), in which case the dynamics are frontal geostrophic. In fact, it turns out that the Rossby number is then the square of the ratio of the deformation radius over the length scale and, for practical purposes, is sufficiently small when the length scale is three or more times the deformation radius. Overall, it is recognized that the quasi-geostrophic and frontal geostrophic formalisms are both established by parallel mathematical developments and yet are based on opposite and complementary physical processes, respectively.

Due to the baroclinical nature of ocean fronts, Cushman-Roisin et al.(1992) established a two-layer frontal geostrophic model based on the one-layer model. This two-layer model overcomes the disadvantages of the former models, and is suitable for both the baroclinic and finite interfacial displacements of ocean fronts Accordingly, it plays an important role in the study of the occurence, evolution, destruction, and bursting of ocean fronts.

The principal purpose of this paper is to demonstrate the nonlinear stability of surface ocean fronts and fronts on a sloping bottom by employing Arnol'd's (1965; 1969) variational principle and a prior-estimate method. Aronl'd's method is essentially a generalization of the Lyapunov stabilitymethod for analysing finite-dimensional dynamic systems in infinite-dimensional ones. Arnol'd actually studied the nonlinear stability of 2-D incompressible ideal fluid motion using this method. Ever since the 1980's many scientists have studied the nonlinear stability of various fluids using Arnol'd's method including Holm et al.(1985), McIntyre and Shepherd(1987), Zeng (1989), Mu et al (1995), Li and Mu(1996a,b), and Li et al.(1999a, b). Swaters (1993) also studied the nonlinear stability of fronts on a sloping bottom using a simple model. Accordingly, it is very important to study the nonlinear stability of ocean fronts using frontal geostrophic models.

This paper initially identifies two-layer frontal-geostrophic equations for coastal region density fronts by considering the bottom topography as a sloping continental shelf. Next, the nonlinear stability of surface ocean fronts and fronts on a sloping bottom are studied using frontal-geostrophic models. Thereafter, nonlinear stability criteria for the two kinds of fronts are obtained using Arnol'd's (1965 ; 1969) variational principle and a prior estimate method. This is the first time a nonlinear stability criterion for surface ocean fronts has been obtained, plus the results for fronts on a sloping bottom are superior to any previous ones.

2. Derivation of Governing Equations

2.1 Model for fronts on a sloping continental shelf

In order to maintain a certain degree of clarity in the derivations, the analysis starts with a series of assumptions aimed at limiting the number of parameters. These assumptions include that there are two layers, an f-plane, sloping bottom, and rigid lid, and that the upper layer is at least as deep as the lower layer (Fig.1). The equations that govern an oceanic system with one interfacial degree of freedom are those of the two-layer shallow water model in an f-plane

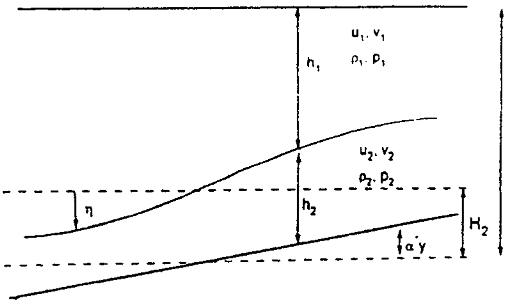

Fig.1. Geometry of two-layer model for fronts on a sloping bottom.

$$\vec{V}_{it}+(\vec{V}_i\cdot\nabla)\vec{V}_i+f_0\vec{k}\times\vec{V}_i=-\nabla p_i,\quad i=1,2 \qquad (1)$$

$$h_{it}+\nabla\cdot(h_i\vec{V}_i)=0,\quad i=1,2 \qquad (2)$$

where f_0 is the Coriolis parameter(constant), $\vec{V}_i=(u_i, v_i)$, $i=1,2$ is the horizontal velocity vector, $\nabla=(\partial_x, \partial_y)$ with (x, y) as the horizontal coordinates, and t is time. The subscripts with respect to (x, y, t) indicate partial differentiation, $\vec{k}$ is the unit vertical vector, and h_i is the ith layer depth. According to these assumptions, the pressure p_i can be written as

$$p_1=\pi,\quad p_2=\pi+g'(\eta-\alpha^* y) \qquad (3)$$

where π is the pressure in the upper layer and η the upward interfacial displacement, $g'=g(\rho_2-\rho_1)/\rho_2$ is the reduced gravity (stable stratification), ρ_1 and ρ_2 are the densities in the upper and lower layers, respectively, and α^* is the slope parameter.

$$h_1=H-H_2-\eta,\quad h_2=H_2+\eta-\alpha^* y \qquad (4)$$

where H is the total depth of the fluid and H_2 is the lower layer depth.

Suppose that L is the horizontal length-scale $L_R=\frac{\sqrt{g'H_2}}{f_0}$ is the internal deformation radius, and the scales of p_1, η, p_2 and t are P, ΔH, $g'\Delta H$ and T, respectively, then the velocity scale of u_1, v_1 is P/f_0L, and that of u_2, v_2 is $g'\Delta H/f_0L$. Therefore, the dimensionless numbers of the layer-thickness ratio and the slope parameter can be defined as

$$\delta=\frac{H_2}{H} \tag{5}$$

$$\alpha=\frac{\alpha^* L}{H} \tag{6}$$

The dimensionless numbers that arise naturally from the scaling of these equations are

$$s=\frac{g' H_2}{f^2 L^2}=(\frac{L_R}{L})^2 \tag{7}$$

$$\varepsilon=\frac{g'\Delta H}{f^2 L^2}=(\frac{\Delta H}{H_2})(\frac{L_R}{L})^2 \tag{8}$$

where the Rossby number, ε can be considered as a measure of the motion amplitude, ΔH, once L is specified. Meanwhile, these unknown scales, T and P, generate two last dimensionless numbers

$$\omega=\frac{1}{f_0 T} \quad \gamma=\frac{P}{g'\Delta H} \tag{9}$$

Based on the above-mentioned assumptions, the relationships of these dimensionless numbers for fronts in coastal regions can be expressed as follows

$$s=\varepsilon,\ \ \alpha=\delta,\ \ \omega=\varepsilon^2,\ \gamma=\varepsilon,\ \ \delta\le\varepsilon\le\delta^{1/2}\ll 1 \tag{10}$$

By using the relationships in Eq.(10), Eqs.(1) and (2) can be nondimentionalized and the symbols of the variables remain unchanged

$$\varepsilon^2\overrightarrow{V}_{1t}+\varepsilon^2(\overrightarrow{V_1}\cdot\nabla)\overrightarrow{V_1}+\vec{k}\times\overrightarrow{V_1}=-\nabla p_1 \tag{11}$$

$$-\varepsilon^2\eta_t-\varepsilon^2\nabla(\eta\overrightarrow{V_1})+(\frac{\varepsilon^2}{\delta}-\varepsilon^2)\nabla\overrightarrow{V_1}=0 \tag{12}$$

$$\varepsilon^2\overrightarrow{V}_{2t}+\varepsilon(\overrightarrow{V_2}\cdot\nabla)\overrightarrow{V_2}+\vec{k}\times\overrightarrow{V_2}=-\nabla p_2 \tag{13}$$

$$\varepsilon^2\eta_t+\varepsilon\nabla(\eta\overrightarrow{V_2})+\varepsilon\nabla(y\overrightarrow{V_2})+\varepsilon\nabla\overrightarrow{V_2}=0 \tag{14}$$

and Eq.(3) becomes

$$p_1=\pi,\ \ p_2=\varepsilon\pi+\eta-y \tag{15}$$

When combining Eq.(11) and the geostrophic balance $\overrightarrow{V_1}=\vec{k}\times\nabla p_1$, the following is produced

$$\overrightarrow{V_1}=\vec{k}\times\nabla p_1-\varepsilon^2\nabla p_{1t}-\varepsilon^2 J(p_1,\nabla p_1) \tag{16a}$$

$$\nabla\overrightarrow{V_1}=\nabla(\vec{k}\times\nabla p_1)-\varepsilon^2\nabla^2 p_{1t}-\varepsilon J(p_1,\nabla^2 p_1) \tag{16b}$$

where $J(f,g)=f_x g_y-f_y g_x$ is the horizontal Jacobian operator and ∇^2 is the two-dimensional Laplacian. When Eq.(16) is substituted by Eq.(15) in Eq.(11) the following is the result

$$(\eta+\frac{\varepsilon^2}{\delta}\nabla^2\pi)_t+J(\pi,\frac{\varepsilon^2}{\delta}\nabla^2\pi+\eta)=0 \tag{17}$$

Similarly, when combining Eq.(13) and the geostrophic balance $\overrightarrow{V_2}=\vec{k}\times\nabla p_2$, the following is produced

$$\overrightarrow{V_2}=\vec{k}\times\nabla p_2-\varepsilon^2\nabla p_{2t}-\varepsilon J(p_2,\nabla p_2) \tag{18}$$

When Eq.(18) is substituted into Eq.(14), this eliminates the high items of ε (since $\varepsilon\ll 1$), then the following is obtained

$$\eta_t-J(\eta-y,\pi+\nabla^2(\eta-y)+(\eta-y)\nabla^2(\eta-y)+\frac{1}{2}\nabla(\eta-y)\cdot\nabla(\eta-y))=0 \tag{19}$$

When the upper depth is denoted by $h=1+\eta$, consider the case of $\delta=\varepsilon^2$ and eliminate the high items (ε^2 items) in Eqs.(17) and (19). Thereafter, the flowing equations below can be obtained

$$(\nabla^2\pi+h)_t+J(\pi,\nabla^2\pi+h)=0 \tag{20a}$$

$$h_t-J(h-y,\pi+(h-y)\nabla^2(h-y)+\frac{1}{2}\nabla(h-y)\cdot\nabla(h-y))=0 \tag{20b}$$

Eq.(20) is the two-layer frontal geostrophic model for fronts on a sloping continental shelf. With this model, the conservations can be obtained by using the boundary conditions

$$\frac{d}{dt}\int\int F(\frac{\varepsilon^2}{\delta}\nabla^2\pi+h)dxdy=0 \tag{21a}$$

$$\frac{d}{dt}\int\int G(h-y)dxdy=0 \tag{21b}$$

where F and G are two arbitrary integrable functions.

2.2 Model for surface-density ocean fronts

This case considers the β-plane i.e., $f=f_0+\beta y$ plus the lower boundary is flat (Fig.2). As in Section 2.1, the two-layer frontal geostrophic model for surface-density ocean fronts can be obtained as follows

$$h_t+J(\pi+h\nabla^2 h+\frac{1}{2}\nabla h\cdot\nabla h,h)=0 \tag{22a}$$

$$(\nabla^2\pi+h+y)_t+J(\pi,\nabla^2\pi+h+y)=0 \tag{22b}$$

and the conservations can be determined as follows

$$\frac{d}{dt}\int\int F^*(\nabla^2\pi+h+y)dxdy=0 \tag{23a}$$

$$\frac{d}{dt}\int\int G^*(h)dxdy=0 \tag{23b}$$

where F^* and G^* are two arbitrary integrable functions.

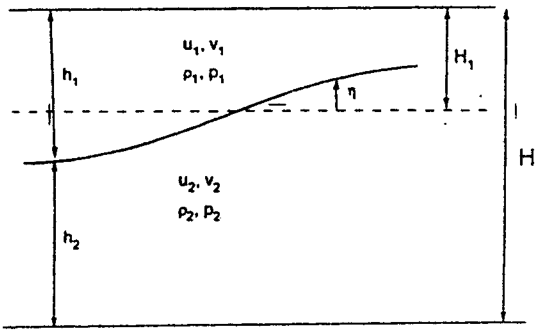

Fig. 2. Geometry of two-layer model for surface ocean fronts.

3. Nonlinear stability of ocean fronts

3.1 Nonlinear stability of surface-density fronts

For the governing equation, the horizontal area is a zonal period channel

$$D=\{-1\leq x\leq 1,\ -1\leq y\leq 1\} \tag{24}$$

The horizontal boundary conditions are that there is no normal flow and circulation is conserved at each level

$$\pi_x=0,\ \frac{d}{dt}\left\{\int_{-1}^{1}\pi_y dx|_{y=-1,1}\right\}=0,\ y=-1,1 \tag{25a}$$

$$h_x=0,\ \frac{d}{dt}\left\{\int_{-1}^{1}h_y dx|_{y=-1,1}\right\}=0,\ y=-1,1 \tag{25b}$$

In this case, Eq. (23) holds.

Suppose $(\pi, h)=(\pi_0, h_0)$ is a steady solution to the frontal geostrophic equation, a disturbance $(\delta h, \delta\pi)$ to this steady basic state can thus be defined as follows

$$h(x,y,t)=h_0(x,y)+\delta h(x,y,t), \quad \pi(x,y,t)=\pi_0(x,y)+\delta\pi(x,y,t) \tag{26}$$

plus the boundary conditions can be satisfied as follows

$$\delta\pi_x=0,\ \int_{-1}^{1}\delta\pi_y dx|_{y=-1,1}=0,\ y=-1,1 \tag{27a}$$

$$\delta h_x=0,\ \int_{-1}^{1}\delta h_y dx|_{y=-1,1}=0,\ y=-1,1 \tag{27b}$$

As a result

$$J(\pi_0+h_0\nabla^2 h_0+\frac{1}{2}\nabla h_0\cdot\nabla h_0, h_0)=0 \tag{28a}$$

$$J(\pi_0, \nabla^2\pi_0+h_0+y)=0 \tag{28b}$$

A sufficient condition for Eq.(28) is

$$\pi_0+h_0\nabla^2 h_0+\frac{1}{2}\nabla h_0\cdot\nabla h_0=F_1(h_0) \tag{29a}$$

$$\pi_0=F_2(\nabla^2\pi_0+h_0+y) \tag{29b}$$

where F_1 and F_2 are two arbitrary differentiable functions.

According to the conservations in Eq.(23) and the boundary conditions of Eqs.(25) and (27), the following conservations are obtained

$$H(\vec{q})=\frac{1}{2}\iint_D(|\nabla\pi|^2+h|\nabla h|^2+h^2\nabla^2 h)dxdy \tag{30}$$

$$C(\vec{q})=\iint_D\Phi_1(h)dxdy+\iint_D\Phi_2(\nabla^2\pi+h+y)dxdy \tag{31}$$

where $\vec{q}=(q_1, q_2)=(h, \nabla^2\pi+h+y)$, Φ_1 and Φ_2 are arbitrary first order continuously differentiable functions.

According to the study of Arnol'd (1965; 1969), suppose there are positive constants $C_i>0, i=1,\ldots,4$, such that

$$C_1<F_1'(\xi)<C_2 \tag{32a}$$

$$C_3<-F_2'(\xi)<C_4 \tag{32b}$$

where F_1 and F_2 are defined in Eq.(29).

Consider the following invariant functionals

$$L(\vec{q})\equiv H(\vec{q_0}+\delta\vec{q})-H(\vec{q_0})+C(\vec{q_0}+\delta\vec{q})-C(\vec{q_0}) \tag{33}$$

where H and C are the invariant functionals defined in Eqs.(30) and (31). As a result, Eq.(33) can be written as

$$\begin{aligned}L(\vec{q})=&\frac{1}{2}\iint_D\{\nabla\delta\pi\cdot\nabla\delta\pi-h\nabla\delta h\cdot\nabla\delta h+\nabla^2 h(\delta h)^2\}dxdy\\&-\iint_D\left\{\int_{q_{10}}^{q_{10}+\delta q_1}F_1(\xi)d\xi-F_1(q_{10})\delta q_1\right\}dxdy\\&+\iint_D\left\{\int_{q_{20}}^{q_{20}+\delta q_2}F_2(\xi)d\xi-F_2(q_{20})\delta q_2\right\}dxdy\end{aligned} \tag{34}$$

where $\delta\vec{q}=(\delta q_1, \delta q_2)=(\delta h, \nabla^2\delta\pi+\delta h)$ is the finite amplitude disturbance, $\vec{q}=\vec{q_0}+\delta\vec{q}$.

According to the assumption in Eq.(32), the following can be obtained

$$\frac{C_1(\delta q_1)^2}{2}<\int_{q_{10}}^{q_{10}+\delta q}F_1(\xi)d\xi-F_1(\xi)\delta q<\frac{C_2(\delta q_1)^2}{2} \tag{35a}$$

$$\frac{C_3(\delta q_2)^2}{2}<-\int_{q_{20}}^{q_{20}+\delta q}F_2(\xi)d\xi-F_2(\xi)\delta q<\frac{C_4(\delta q_2)^2}{2} \tag{35b}$$

When Eq.(35) is substituted into Eq.(34), the following inequality is obtained

$$\frac{1}{2}\iint_D\{\nabla\delta\pi\cdot\nabla\delta\pi-h\nabla\delta h\cdot\nabla\delta h+\nabla^2 h(\delta h)^2-C_2(\delta q_1)^2-C_4(\delta q_2)^2dxdy\}<L(\vec{q})$$

$$\frac{1}{2}\iint_D\{\nabla\delta\pi\cdot\nabla\delta\pi-h\nabla\delta h\cdot\nabla\delta h+\nabla^2 h(\delta h)^2-$$

$$C_1(\delta q_1)^2 - C_3(\delta q_2)^2 dxdy\} < L(\vec{q}) \quad (36)$$

In contrast, using the boundary conditions of Eqs. (27a) and (27b), two Poincacutere inequalities can be obtained (Ladyzhenskaya, 1966)

$$\int\int_D \nabla\delta\pi \cdot \nabla\delta\pi dxdy \le \tilde{C}\int\int_D (\nabla^2\delta\pi)^2 dxdy \quad (37)$$

$$\int\int_D \delta h \cdot \delta h dxdy \le \frac{1}{\tilde{C}}\int\int_D \nabla\delta h \cdot \nabla\delta h dxdy \quad (38)$$

where $\tilde{C}$ and $\tilde{C}$ are positive constants.

When Eqs.(37) and (38) are combined with Eq.(36), an important inequality is identified as follows

$$L(\vec{q}) < \frac{1}{2}\int\int_D (\tilde{C} - C_3)(\nabla^2\delta\pi + \delta h)^2 - 2\tilde{C}(\nabla^2\delta\pi + \delta h)\delta h + (\nabla^2 h_0 - \tilde{C}(\min_{(x,y)\in D} h_0) - \tilde{C} - C_1)(\delta h)^2 dxdy \quad (39)$$

Thus, the nonlinear stability criterion can be established as follows

Criterion 1. If the basic state (π_0, h_0) of the frontal geostrophic equation satisfies Eqs.(29) and (32), and

$$C_3 > \tilde{C} \quad (40a)$$

$$-C_1 - \tilde{C}(\min_{(x,y)\in D} h_0) + \nabla^2 h_0 < \frac{\tilde{C}C_3}{\tilde{C} - C_3} \quad (40b)$$

then the basic state is nonlinearly stable; i.e., for any $\varepsilon > 0$, there exists $\delta > 0$ such that, when

$$\|\vec{q}(x, y, 0) - \vec{q_0}(x, y)\| < \delta \quad (41)$$

then

$$\|\vec{q}(x, y, t) - \vec{q_0}(x, y)\| < \varepsilon \quad (42)$$

where the mode $\|\delta\vec{q}\|$ is defined as

$$\|\delta\vec{q}\| = \|\vec{q} - \vec{q_0}\| = [-L(\vec{q})]^{1/2} \quad (43)$$

Proof: Under the condition of Eq. (40), the use of Eq.(39) can prove that the invariant functional is a negative definite, i.e.

$$L(\vec{q}) \le -\Lambda\int\int_D |\nabla^2\delta\pi + \delta h|^2 + |\delta h|^2 dxdy$$

where Λ is a positive constant. Thus, by the definition of the mode $\|\delta\vec{q}\|$ in Eq.(43), it can be proved that the basic state (π_0, h_0) is nonlinearly stable.

3.2 Nonlinear stability of fronts in coastal regions

Using the frontal geostrophic equation, the boundary conditions and nonlinear stability of fronts on a sloping continental shelf in the zonal period channel given in Eq.(24) and are considered as the same as those in Eq.(25). If $\tilde{h} = h - y$ is defined in Eq. (20), it will have the same form as in Eq.(22). Accordingly, the nonlinear stability criterion can be established in a similar manner to the case outlined in Section 3.1. Suppose (π_0, h_0) is the basic state of Eq. (20), then

$$\pi_0 + (h_0 - y)\nabla^2(h_0 - y) + \frac{1}{2}\nabla(h_0 - y)\cdot\nabla(h_0 - y) = \tilde{F}_1(h_0 - y) \quad (44a)$$

$$\pi_0 = \tilde{F}_2(\nabla^2\pi_0 + h_0) \quad (44b)$$

where $\tilde{F}_1$ and $\tilde{F}_2$ are arbitrary differentiable functions that can be denoted by

$$\tilde{\Phi}_1(q_1) = -\int^{q_1}\tilde{F}_1(\xi)d\xi \quad (45a)$$

$$\tilde{\Phi}_2(q_2) = -\int^{q_2}\tilde{F}_2(\xi)d\xi \quad (45b)$$

As a result, the nonlinear stability criterion for this model can be established

Criterion 2. If the basic state (π_0, h_0) of the dynamic system (Eq. (20)) satisfies Eq.(44), and

$$\tilde{C}_1 < \tilde{F}_1'(\xi) < \tilde{C}_2 \quad (46a)$$

$$\tilde{C}_3 < -\tilde{F}_2'(\xi) < \tilde{C}_4 \quad (46b)$$

where $\tilde{C}_i$, i=1,..,4 are positive constants, and

$$\tilde{C}_3 > \tilde{C} \quad (47a)$$

$$\nabla^2(h_0 - y) - \tilde{C}_1 - \tilde{C}(\min_{(x,y)\in D}(h_0 - y)) < \frac{\tilde{C}\tilde{C}_3}{\tilde{C} - \tilde{C}_3} \quad (47b)$$

then the basic state will be nonlinearly stable according to the meaning of **Criterion 1**.

Swaters (1993) produced a nonlinear stability criterion by using a simple two-layer model, the criterion was

$$\tilde{C}_3 > \tilde{C} \quad (48a)$$

$$-\tilde{C}_1 < \frac{\tilde{C}\tilde{C}_3}{\tilde{C} - \tilde{C}_3} \quad (48b)$$

When Eq.(47) was compared with Eq.(48), it was found that Eq.(47a) is the same as Eq.(48a); and Eq.(48b) is one special case (in the case of $\nabla^2(h_0 - y) - \tilde{C}(\min_{(x,y)\in D}(h_0 - y)) = 0$ of Eq. (47b). **Criterion 2** is, therefore, superior to the criterion proposed by Swaters(1993). Accordingly, the frontal geostrophic model established in this paper is more suitable for the study of ocean fronts.

4. Conclusion and Discussion

In this paper, the frontal geostrophic model, first developed by Cushman-Roisin (1986) and Cushman-Roisin et al. (1992), was extended to describe the dynamic system of density fronts on a sloping continent shelf. This new frontal geostrophic model has two-layers with a sloping bottom. The nonlinear stability of two kinds of ocean fronts, surface-density fronts and fronts in coastal regions, was discussed. Also nonlinear stability criteria for two kinds of two-layer frontal geostrophic models were obtained. Theoretical analyses demonstrate that for fronts with a sloping bottom, the proposed model is superior to that of Swaters(1993); plus a nonlinear stability criterion was established for surface fronts for the first time. It should be noted that this study only focused on theoretical analyses, accordingly, the application of these theoretical results to actual oceanic problems is the challenge for future studies.

Acknowledgements

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