여러개의 two-state MMPP 입력을 갖는 대기체계에 대한 계산방법

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Analysis of MMPP/M/1 Queue with several homogeneous two-state MMPP sources

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ABSTRACT

In this paper, we suggest a simple computational algorithm to obtain the queue length distribution in the finite queue, where the input process consists of several homogeneous two-state Markov modulated Poisson processes. With comparison to the conventional algorithm, our algorithm is more practical, in particular, when a large number of input sources are loaded to the system.

I. Introduction

Integrated service communication systems usually have very complicated input streams. A typical example is a statistical multiplexer, whose input consists of a superposition of packetized voice sou- rces together with data traffic[5].

The number of packet arrivals in adjacent time intervals can be highly correlated, which turns the input process into a complex non-renewal process and significantly affects queueing performance of the system.

Thus, a great interest has recently arisen in the modeling of the superposition of traffic streams and in the analysis of the resulting queueing model.

Within this framework, various input processes have been studied. A particularly interesting point process is a well-known Markov modulated Poisson process (MMPP). It possesses an important property which makes it suitable for approximation of complicated non-renewal processes. By using a multiple-state MMPP or a superposition of several homogeneous two-state MMPP as an arrival proc- ess, various computer and communication systems have modeled, and then solved by the matrix- geometric algorithm [6][3][1], or the folding algo- rithm [4].

However, these algorithms are computationally intensive and impractical, especially when state space of the aggregated arrival process of several homogeneous MMPP sources is large [2]. This is usually the case in communication networks since we may expect to have a large number of source being served by a single statistical multiplexer. Thus, we study a simple computational algorithm solving the queueing model, where the input process consists of several homogeneous two-state MMPP sources.

This paper is organized as follows. Section 2 present a simple algorithm to analyze the system

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loaded with a single two-state MMPP source. In Section 3, we extend the proposed algorithm to the sys tem where the input process consists of a large number of homogeneous two-state MMPP sources.

Ⅱ. An two-state MMPP/M/1 Queue

In this section, we consider a single sever queu- eing system where customers arrive in accordance with a two-state MMPP. Upon arrival, they can enter the system only if there are less than K customers in the system. Service time distribution is exponential with rate μ.

Before analyzing the system, let us briefly describe a two-state MMPP. It is a double ly stochastic Poisson process, whose mean arrival rate changes according to the state of a underlying two-state Markov process. The generator of the underlying Markov process and the mean arrival rate matrix shall be denoted by

$$Q=\begin{pmatrix} -a & a \\ b & -b \end{pmatrix} \quad \text{and} \quad \Lambda=\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

respectively. We easily observe that the stationary vector θ of the stocha stic matrix Q is equal to $(a+b)^{-1}(b,a)$ and the traffic intensity ρ of this queueing system is equal to $\mu^{-1}\,\theta\Lambda\,e$, where $e=(1,1)^t$.

Now we derive the queue length distribution of the system. Let $X(t)$ and $J(t)$ denote the queue length and the state of the underlying Markov process at time t. Then the couplet $(J(t), X(t))$ is a two-dimensional Markov process with the following infinitesimal generator

$$Q_s=\begin{pmatrix} Q-\Lambda & \Lambda & & & \\ \mu I & Q-\Lambda-\mu I & \Lambda & & \\ & & \ddots & & \\ & & & Q-\Lambda-\mu I & \Lambda \\ & & & \mu I & Q-\mu I \end{pmatrix}$$

where I is an identity matrix of order 2. Note that Q_s is a matrix of size $2K\times 2K$. Our aim is to find the following stationary joint distribution

$$\pi_{i,n}=\lim_{t\to\infty}\Pr(J(t)=i,\, X(t)=n)$$

for all $i=1,2,\ 0\le n\le K$. For the sake of notational convenience, we set $\mu=1$ and write $\boldsymbol{\pi}_n=(\pi_{1,n}, \pi_{2,n})$ for all $0\le n\le K$. Then it is well known that $(\boldsymbol{\pi}_0,\cdots,\boldsymbol{\pi}_K)Q_s=\mathbf{0}$ and

$$\begin{aligned}(1-z)[\,\boldsymbol{\pi}_0-z^{K+1}\,\boldsymbol{\pi}_K\Lambda] \\ = \boldsymbol{\pi}(z)[I+z(Q-\Lambda-I)+z^2\Lambda]\end{aligned} \tag{1}$$

where $\mathbf{0}=(0,0)$ and $\boldsymbol{\pi}(z)=\sum_{n=0}^{K}\boldsymbol{\pi}_n z^n$. To solve the above equation, let us again define the matrix $\Phi(z)=\Lambda+z(Q-\Lambda-I)+z^2I$ and its determinant $\phi(z)$. Since the determinant $\phi(z)=[1-z(a+\lambda_1+1)+z^2\lambda_1]+1)\,[1-z(b+\lambda_2+z^2\lambda_2]-abz^2$, we directly derive the follow- ing lemma. See [3] for the details.

Lemma 1 The determinant $\phi(z)$ of the matrix $\Phi(z)$ has four positive roots, denoted by α_1, α_2, β_1 and β_2 in order. If traffic intensity ρ is less than one, the roots satisfy $\alpha_2<\alpha_1<\beta_1=1<\beta_2$. Otherwise the roots satisfy $\alpha_2<\alpha_1=1<\beta_1<\beta_2$.

In similar way in [3] and [1], we can solve the equation (1) through the matrix-geometric algorithm. For this purpose, define the following four matrices

$$V_1=\begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha \end{pmatrix},\quad V_2=\begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix}$$

and

$$L_1=\begin{pmatrix} \boldsymbol{\alpha}_1 \\ \boldsymbol{\alpha}_2 \end{pmatrix},\quad L_2=\begin{pmatrix} \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_2 \end{pmatrix}$$

where $\boldsymbol{\alpha}_1$, $\boldsymbol{\alpha}_2$, $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$ are left eigen-vectors of the matrix $\Phi(z)$ corresponding to four roots α_1, α_2, β_1 and β. Murthy et. al. proved that $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ are eigen-vecors of the minimal solution of the following nonlinear matrix equation:

$$\Lambda + R(Q - \Lambda - I) + R^2 = 0$$

and that α_1 and α_2 are eigenvalues of the minimal solution R in [2]. Similarly $\boldsymbol{\beta}_1$, $\boldsymbol{\beta}_2$, β_1^{-1}, and β_2^{-1} are so with respect to $I + R(Q - \Lambda - I) + R^2\Lambda = 0$. Thus both L_1 and L_2 are invertible. Using these four matrices, we derives the following theorem for the queue length distribution.

Theorem 1 The queue length distribution is given by

$$\boldsymbol{\pi}_n = \boldsymbol{w}_{K1}(I - V_1)V_1^n L_1 + \boldsymbol{w}_{K2}(I - V_2)V_2^n L_2 \qquad (2)$$

where ($\boldsymbol{w}_{K1}$, $\boldsymbol{w}_{K2}$) is a left eigen-vector of matrix

$$\begin{pmatrix} L_1 Q & V_1^{K+1} L_1 Q \\ L_2 Q & V_2^{K+1} L_2 Q \end{pmatrix} \qquad (3)$$

for a zero eigen-value and normalized so that

$$1 = \boldsymbol{w}_{K1}(I - V_1^{K+1})L_1 \boldsymbol{e} + \boldsymbol{w}_{K2}(I - V_2^{K+1})L_2 \boldsymbol{e} \qquad (4)$$

Proof) By the simple algebric manipulation, it is easily shown that

$$L_i Q = (I - V_i)L_i(Q - \Lambda) + V_i^2 L_i$$

$$0 = \Lambda + V_i L_i(Q - \Lambda - I) + V_i^2 L_i$$

$$V_i^2 L_i Q = (I - V_i)L_i \Lambda + (I - V_i)V_i L_i(Q - I)$$

for all $i = 1, 2$. From this fact and the equation (3), we easily know that ($\boldsymbol{\pi}_0, \cdots, \boldsymbol{\pi}_K$) defined in (2) is a unique stationary vector of the stochastic matrix Q_s. So the proof is complete.

The method presented here unifies the finite and infinite queue system in a single frame work. In order to see this, let us look at coefficient vectors $\boldsymbol{w}_{K1}$ and $\boldsymbol{w}_{K2}$ defined in Theorem 1 when the queue size is infinite and traffic intensity ρ is less than one. Since $\alpha_1 < \alpha_2 < 1 < \beta_1 < \beta_2$, V_1^{K+1} goes to zero and V_2^{K+1} diverge as $K \to \infty$. Thus, in order to satisfy normalization equation (3), the coefficient $\boldsymbol{w}_{K2}$ becomes to be a zero vector. This fact derives that $\boldsymbol{w}_{\infty 1} = \boldsymbol{\theta} L_1^{-1}$, which is equal to results in [3].

Ⅲ. Several two-state MMPP/M/1 Queue

In this section, we extend results in Section 2 to the system where customers arrive in accordance with a superposition of several homogeneous two-state MMPP sources. To do this, let us first describe the input process. When m homogeneous two-state MMPP sources with parameters (Q, Λ) defined in Section 2 are superposed, the generator of the underlying Markov process and the mean arrival rate matrix of the superposed process are given by

$$Q_m = \begin{pmatrix} -ma & ma & & & \\ (m-1)a & -(m-1)a - b & & & \\ & & \ddots & & \\ & & & -a-(m-1)b & a \\ & & & mb & -mb \end{pmatrix}$$

and $\Lambda_m = diag(m\lambda_1, (m-1)\lambda_1 + \lambda_2, \cdots, m\lambda_2)$.

It is also well known that the stationary vector of the matrix $Q_{(m)}$ is given as follows;

$$\boldsymbol{\theta}_m = \frac{1}{(a+b)^m}\left(\binom{m}{0} b^m a^0, \cdots, \binom{m}{m} b^0 a^m\right)$$

and that the traffic intensity ρ_m of this system is equal to $m\rho$, where ρ is defined in Section 2.

Now we derive queue length distribution. In similar way as we did in Section 2, write

$$\pi_{n,i}=\lim_{t\to\infty}\Pr(X(t)=n, J(t)=i),$$

$1\le i\le m+1, 0\le n\le K$ and $\pi_n=(\pi(n,1),\cdots,\pi(n,m+1))$, $0\le n\le K$. Also we define.

$$\Phi(z)=\Lambda_m+z(Q_m-\Lambda_m-\mu I)+z^2 I$$

that is,

$$\Phi(z)=\begin{pmatrix} d_m(z) & maz & & & \\ bz & d_{m-1}(z) & \ddots & & \\ & 2bz & \ddots & & \\ & & \ddots & d_1(z) & az \\ & & & mbz & d_0(z)\end{pmatrix}$$

where $d_k(z)=k\lambda_1+(m-k)\lambda_2-z[k(a+\lambda_1)+(m-k)(b+\lambda_2)+1]+z^2$. Using the fact that $d_k(z)=\frac{k}{m}d_m(z)+\frac{m-k}{m}d_0(z)$, we can show that the determinant $\phi(z)$ of the matrix $\Phi(z)$ is given by

$$\phi(z)=\begin{cases}\prod_{k=0}^{(m+1)/2}s_k(z), & \text{if } m \text{ is odd}\\ d_{m/2}(z)\prod_{k=0}^{(m-1)/2}s_k(z), & \text{if } m \text{ is even}\end{cases}$$

where $s_k(z)=d_k(z)d_{m-k}(z)-(m-2k)^2abz^2$. From this fact, we derive the following lemma.

Lemma 2 The determinant $\phi(z)$ has $2m$ positive roots, denote by $\alpha_1,\cdots,\alpha_m$ and $\beta_1,\cdots,\beta_m$ in order. If traffic intensity ρ_m of the system is less than one, the roots satisfy $\alpha_1<\cdots<\alpha_m<\beta_1=1<\cdots<\beta_m$. Otherwise, the roots satisfy $\alpha_1<\cdots<\alpha_m=1<\beta_1<\cdots<\beta_m$.

Proof) We easily observe that $d_k(0)<0$ and $d_k(1)>0$ for all $k=0,1,\cdots,m$, so that $d_k(z)$ $d_{m-k}(z)$ has four positive roots $x_1<x_2<1<x_3<x_4$. It easily is shown that $s_k(1)=2k(m-k)(a+b)^2$. Thus we know that $s_k(z)$ has four positive roots satisfying $y_1<y_2<1<y_3<y_4$ for all $k\neq 0$ and that $s_0(z)$ has four positive root y_1, y_2, y_3 and y_4 satisfying $y_1<y_2=1<y_3<y_4$ or $y_1<y_2<y_3=1<y_4$.

Now, we shall prove that determinant $\phi(z)$ does not have multiple roots. If t_k is a root of $s_k(z)$, then $s_l(t_k)=(m-k-l)(l-k)([(\lambda_2-\lambda_1)(t_k-1)+(a-b)]^2+4ab]t_k^2$ for $l\neq k$. Thus determinant $\phi(z)$ has 2m distinct roots. So, the proof is complete.

Since $d_k(z)d_{m-k}(z)-(m-2k)^2abz^2=0$ are polynomial equation of order 4, we can easily derive all roots of the determinant $\phi(z)$. In similar way in Section 2, define L_1, L_2, V_1, and V_2 and then derive w_{K1} and w_{K2} satisfying

$$(w_{K1}, w_{K2})\begin{pmatrix}L_1Q & V_1^{K+1}L_1Q\\ L_2Q & V_2^{K+1}L_2Q\end{pmatrix}=0$$

and $w_{K1}(I-V_1^{K+1})L_1e+w_{K2}(I-V_2^{K+1})L_2e=1$. Then the queue length distribution can be obtained by equation (2) in Theorem 1. The above procedure above does not requires to compute inverse matrix as in [4] or to solve nonlinear matrix equation as in [1]. Consequently, it enables us to derive queue length distribution with less amount of computational work.

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