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A Note on the Modified Scheme for Nonlinear Shallow-Water Equations
비선형 천수방정식의 보정차분기법

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Abstract □ An extension of the modified leap-frog scheme is made to solve the nonlinear shallow-water equations. In the extended model, the physical dispersion of the Boussinesq equations is replaced by the numerical dispersion resulted from the leap-frog finite difference scheme. The model is used to simulate propagations of a solitary wave over a constant water depth and a linearly varying water depth. Obtained numerical results are compared with available analytical and other numerical solutions. A reasonable agreement is observed.

Keywords : nonlinear shallow-water equations, modified leap-frog scheme, Boussinesq equations, dispersion, solitary wave

要　旨 : 비선형 천수방정식을 해석하기 위하여 보정 leap-frog 기법을 확장하였다. 차분화 과정에서 발생하는 수치분산을 조정하여 Boussinesq 방정식의 분산을 대치하도록 하였다. 새로이 개발된 보정 leap-frog 기법을 이용하여 일정수심 및 경사면을 진행하는 고립파를 모의하였다. 새로운 확장기법에 의해 계산된 자유수면변위는 기존의 해석해 및 수치해와 잘 일치한다.

핵심용어 : 비선형 천수방정식, 보정 leap-frog 기법, Boussinesq 방정식, 분산, 고립파

1. INTRODUCTION

In this study, an extension of the corrected modified finite difference approximations described in Cho and Yoon (1998) is made. The corrected modified finite difference scheme is extended to include nonlinear effects by applying the leap-frog scheme to the nonlinear shallow-water equations. The numerical dispersion is manipulated to play an equivalent role of the physical dispersion of the Boussinesq equations by choosing spatial grid and time-step sizes properly.

The one-dimensional nonlinear shallow-water equations over a constant water depth is considered for simplicity in this note. The bottom frictional effects are excluded in this note. Then, the continuity and momentum equations can be written as

$$\frac{\partial \zeta}{\partial t}+\frac{\partial P}{\partial x}=0 \tag{1}$$

$$\frac{\partial P}{\partial t}+\frac{\partial}{\partial x}\left(\frac{P^2}{H}\right)+gH\frac{\partial \zeta}{\partial x}=0 \tag{2}$$

In (1) and (2), ζ denotes the free surface displacement, P is the volume flux in x-direction, h is the still water depth, H is the total water depth defined as $H=h+\zeta$, and g is the gravitational acceleration. Hereinafter, the notation $P^2/H=U$ is used only for simplicity.

2. MODIFIED EQUATIONS

The nonlinear shallow-water equations (1) and (2) are discretized with the leap-frog finite difference scheme as

$$\frac{\zeta_i^{n+1/2}-\zeta_i^{n-1/2}}{\Delta t}+\frac{P_{i+1/2}^n-P_{i-1/2}^n}{\Delta x}=0 \tag{3}$$

$$\frac{P_{i+1/2}^{n+1}-P_{i+1/2}^n}{\Delta t}+\frac{U_{i+1}^{n+1/2}-U_i^{n+1/2}}{\Delta x}+gH\frac{\zeta_{i+1}^{n+1/2}-\zeta_i^{n+1/2}}{\Delta x}=0 \tag{4}$$

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In finite difference approximations (3) and (4), a staggered mesh system has been used and the free surface displacement ζ is defined at a node (i) and the volume flux component in x-direction P is defined at a node $(i+1/2)$. The staggered mesh system is also used in time (Cho and Yoon, 1998).

By applying Taylor series expansions of variables ζ and P at a node (i, n) to (3) and (4) and arranging them, the following equations can be obtained

$$\frac{\partial\zeta}{\partial t}+\frac{\partial P}{\partial x}-C_o^2\frac{(\Delta t)^2}{24}\frac{\partial^3\zeta}{\partial t^3}+\frac{(\Delta t)^2}{24}\frac{\partial^3 P}{\partial x^3}=0 \quad (5)$$

$$\frac{\partial P}{\partial t}+\frac{\partial U}{\partial x}+C_o^2\frac{\partial\zeta}{\partial x}-C_o^4\frac{(\Delta t)^2}{24}\frac{\partial^3\zeta}{\partial x^3}-\frac{\Delta x}{2}\frac{\partial^2 U}{\partial x^2}-\frac{\Delta t}{2}\frac{\partial^2 U}{\partial x\partial t}=0 \quad (6)$$

in which $C_o=\sqrt{gH}$, the truncation errors of (5) and (6) are $O[(\Delta x)^4, (\Delta t)^4]$ and $O[(\Delta x)^3, (\Delta t)^3, (\Delta x)^2, \Delta t, \Delta x\ (\Delta t)^2]$, respectively. To derive (5) and (6) the following leading order relationships have been used

$$\frac{\partial\zeta}{\partial t}+\frac{\partial P}{\partial x}=O[(\Delta x)^2,(\Delta t)^2] \quad (7)$$

$$\frac{\partial P}{\partial t}+\frac{\partial U}{\partial x}+C_o^2\frac{\partial\zeta}{\partial x}=O[(\Delta x)^2,\Delta x\Delta t,(\Delta t)^2] \quad (8)$$

Furthermore, all the second and higher order time derivatives are replaced by the corresponding spatial derivatives similar to the linear case, and $O(\Delta t)\approx O(h)$ and $O(\Delta t)\approx O(h\sqrt{gh})$ are used in (5) and (6) as also used in Cho and Yoon (1998).

By eliminating P from (5) and (6), a combined modified equation for the nonlinear shallow-water equations can be obtained as

$$\frac{\partial^2\zeta}{\partial t^2}-\left[\frac{\partial^2 U}{\partial x^2}+gH\frac{\partial^2\zeta}{\partial x^2}\right]-C_o^2\frac{(\Delta t)^2}{12}(1-C_r^2)\frac{\partial^4\zeta}{\partial x^4}$$
$$+\left[\frac{\Delta x}{2}\frac{\partial^3 U}{\partial x^3}+\frac{\Delta t}{2}\frac{\partial^3 U}{\partial x^2\partial t}\right]=0 \quad (9)$$

in which $Cr(=C_o\Delta t/\Delta x)$ is the Courant number. If the nonlinear terms vanish, (9) reduces to the one-dimensional form of the combined modified equation for the linear shallow-water equations derived by Cho and Yoon (1998).

The Boussinesq equation over a constant depth can be written as (Mei, 1989)

$$\frac{\partial^2\zeta}{\partial t^2}-\left[\frac{\partial^2 U}{\partial x^2}+gH\frac{\partial^2\zeta}{\partial x^2}\right]=\frac{gh^3}{3}\frac{\partial^4\zeta}{\partial x^4} \quad (10)$$

in which the right-had side term represents the frequency dispersion results from the vertical acceleration. If this is ignored, the Boussinesq equation is simplified to the nonlinear shallow-water equation.

By comparing (9) with the Boussinesq equation (10), it is found that two equations are identical if the last term of (9) is ignored and the following condition is satisfied

$$(\Delta x)^2=4h^2+gh(\Delta t)^2 \quad (11)$$

The condition (11) is also derived in the modified equations for the linear shallow-water equations (Cho and Yoon, 1998).

Since it is difficult to remove the last term of equation (9), the corrected modified finite difference approximations are proposed as (Abbott *et al.*, 1981; Cho and Yoon, 1998)

$$\frac{\zeta_i^{n+1/2}-\zeta_i^{n-1/2}}{\Delta t}+\frac{P_{i+1/2}^n-P_{i-1/2}^n}{\Delta x}=0 \quad (12)$$

$$\frac{P_{i+1/2}^{n+1}-P_{i+1/2}^n}{\Delta x}+\frac{U_{i+1}^{n+1/2}-U_i^{n+1/2}}{\Delta x}+gH\frac{\zeta_{i+1}^{n+1/2}-\zeta_i^{n+1/2}}{\Delta x}$$
$$+\frac{1}{2\Delta x}(U_{i+3/2}^n-U_{i+1/2}^n-U_{i+1/2}^{n-1}+U_{i-1/2}^{n-1})=0 \quad (13)$$

After applying the same procedure to (12) and (13), a combined modified equation for ζ can be derived.

3. Numerical Example

The corrected modified finite difference equations, (12) and (13), are tested by applying to the propagation of a solitary wave over a constant water depth and a linearly varying water depth. Obtained numerical solutions are compared with available analytical and other numerical solutions.

In Fig. 1, a comparison is made for the solitary waves propagation over a constant water depth. In numerical computations, $\Delta x=2h$ for $A=0.05h$ ($\varepsilon=0.05$), $\Delta x=1.75h$ for $A=0.1h$ ($\varepsilon=0.1$) and $\Delta x=1.75h$ for $A=0.2h$ ($\varepsilon=0.2$) have been used. The time-step size is fixed as $\Delta t=0.02$ s. Although the numerical model produces slightly deviated solutions from the analytical solutions, the overall agreement is excellent. The extended model produces the numerical dispersion almost equivalent to the physical dispersion of the Boussinesq equations.

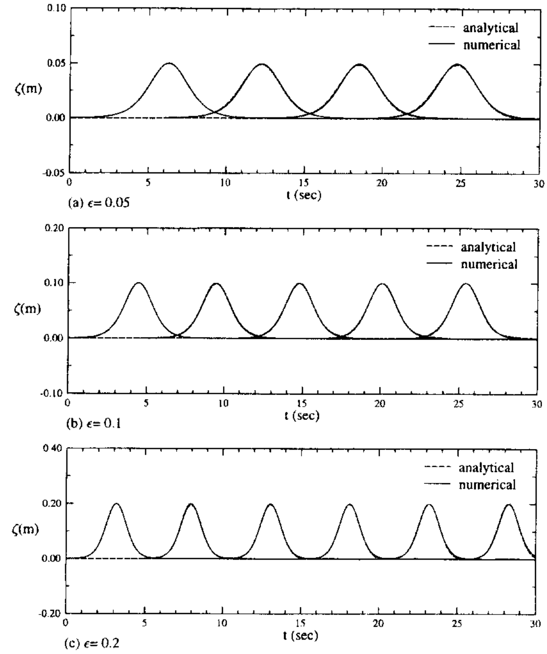

Fig. 1. Time histories of free surface profiles of solitary waves: (a) x=0, 10h, 20h, 30h; (b) x=0, 10h, 20h, 30h, 40h; (c) x=0, 10h, 20h, 30h, 40h, 50h.

Nextly, the propagation of a solitary wave over a linearly varying depth is simulated. Although the extended model is derived over a constant water depth, it also is applied to the mildly varying topography. For simplicity, the bottom topography is consisted of two constant depth regions (regions 1 and 3) and one sloping region (region 2): in region 1, h = 1.0 m and $-35 \le x \le 0$ m; in region 2, h varies 1.0 m to 0.5 m and $0 \le x \le 20$ m; in region 3, h=0.5 m and $20 \le x \le 50$ m. The angle of slope in region 2 is 1.432°. Since the water depth varies, a different spatial grid size is used in each region; Δx = 1.75 m in region 1, Δx = 1.33 m in region 2 and Δx = 0.875 m in region 3. The spatial grid size of each region is equal to 1.75h, where h is a local water depth in regions 1 and 3 and an averaged water depth in region 2. The grid size of $\Delta x = 1.75h$ produces the numerical dispersion, which is approximately equal to the physical dispersion of the Boussinesq equations. The time-step size is fixed as 0.02 s. The wave height of the incident solitary wave is 0.1 m. The initial condition is the horizontal velocity component of a solitary wave as prescribed in Liu and Cho (1994).

In Fig. 2, numerical solutions of time histories of free-

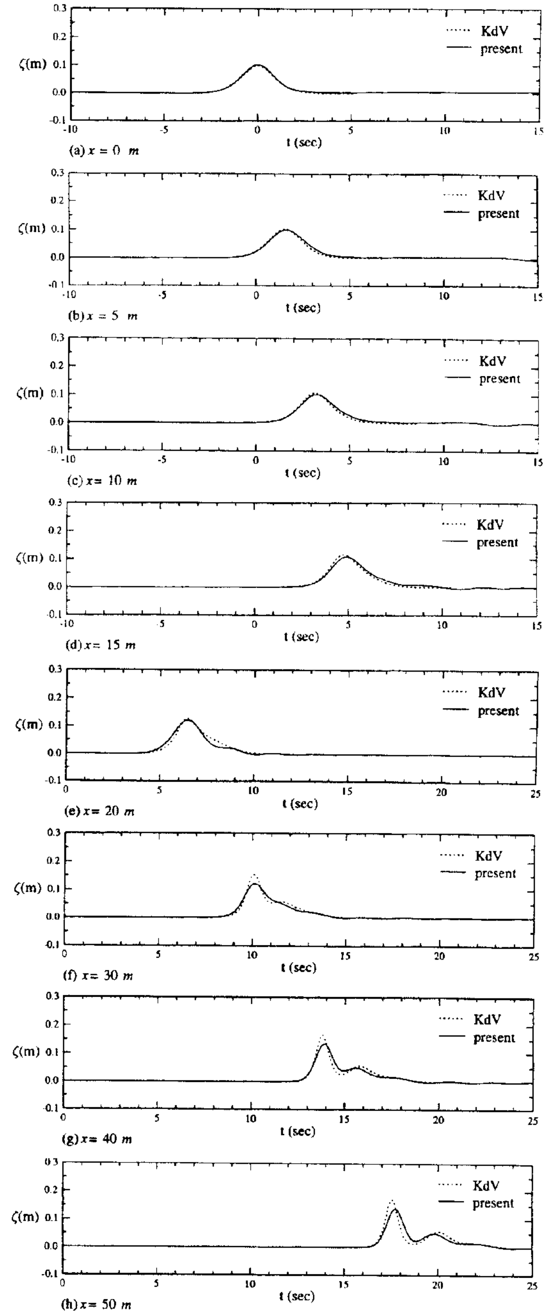

Fig. 2. Time histories of free surface profiles over a sloping topography.

surface displacements at several locations are compared to numerical solutions obtained by solving a variable coefficient KdV equation with the finite element method (Chen, 1995). Both numerical solutions agree well each other in region 2, while the present model underestimates free-surface displacements in comparison with numerical solutions of the finite element method in region 3. This is probably due to the frequency dispersion. However, the overall accuracy of the extended model is promising.

4. CONCLUDING REMARKS

In this study, the corrected modified leap-frog scheme is extended to solve the nonlinear shallow-water equations. A numerical model is developed by discretizing the nonlinear shallow-water equations. The physical dispersion of the Boussinesq equations is replaced by the numerical dispersion resulted from the leap-frog scheme. The developed model is used to simulate propagations of a solitary wave over a constant water depth and a linearly varying depth. Obtained numerical results are compared with analytical solutions. A reasonable agreement is observed.

Although the extension is made only for one-dimensional case, the numerical solutions are promising. The one-dimensional model itself is also valuable to simulate the propagation of tsunami near shoreline where the nonlinear effects are probably dominating the whole system.

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