한국 퍼지 및 지능시스템학회 논문지 1999, Vol. 9, No. 5 (1999)550~554

Fuzzy closure spaces and fuzzy quasi-proximity spaces

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ABSTRACT

We will define a fuzzy quasi-proximity space and give some examples of it. We show that the family $M(X, C)$ of all fuzzy quasi-proximities on X which induce C is nonempty. Moreover, we will study the relationship between the category of fuzzy closure spaces and that of fuzzy quasi-proximity spaces.

1. Introduction and Preliminaries

A.S. Mashhour and M.H. Ghanim [9] introduced fuzzy closure spaces as a generalization of closure spaces. On the other hand, A. Kandil and M.E. El-Shafee[4] introduced the concept of fuzzy proximity spaces and investigated some properties of them.

In this paper, we define a fuzzy quasi-proximity space in a sense of [4]. It is weaker than the definition of A.K. Katsaras and C.G. Petalas [6]. We give some examples of fuzzy quasi-proximity spaces. We show that the family $M(X, C)$ of all fuzzy quasi-proximities on X which induce C is nonempty. Moreover, we study the relationship between the category of fuzzy closure spaces and that of fuzzy quasi-proximity spaces.

Throughout this paper, I denotes the unit interval. A member μ of I^X is called a fuzzy set. $\tilde{0}$ and $\tilde{1}$ denote constant fuzzy sets taking the values 0 and 1 on X, respectively. A *fuzzy point* x_t for $0<t\leq 1$ is an element of I^X such that

$$x_t(y) = \begin{cases} t, & \text{if } y=x, \\ 0, & \text{if } y\neq x. \end{cases}$$

A fuzzy point $x_t \in \lambda$ iff $t \leq \lambda(x)$. For $\lambda, \mu \in I^X$, the fuzzy set λ is *quasi-coincident* with μ, denoted by $\lambda\, q\, \mu$, if there exists $x \in X$ such that $\lambda(x) + \mu(x) > 1$. If λ is not quasi-coincident with μ, we denote $\lambda\, \bar{q}\, \mu$. All other notations and definitions are standard in fuzzy set theory.

Lemma 1.1 [4,8] For $\lambda, \mu, \mu_i \in I^X$, we have the following properties.

(1) If $\lambda\, q\, \mu$, then $\lambda \wedge \mu \neq \tilde{0}$.

(2) $\lambda\, \bar{q}\, \mu$ iff $\lambda \leq \tilde{1} - \mu$.

(3) $\lambda\leq\mu$ iff $x_t\, q\, \lambda$ implies $x_t\, q\, \mu$ iff $x_t\, \bar{q}\, \mu$ implies $x_t\, \bar{q}\, \lambda$.

(4) $x_t\, q \bigvee_{i\in\Gamma} \mu_i$ iff there exists $i_0 \in \Gamma$ such that $x_t\, q\, \mu_{i_0}$.

(5) If $f: X\to Y$ is a function and $\lambda\, q\, \mu$, then $f(\lambda)\, q\, f(\mu)$.

Definition 1.2 [2] A subset T of I^X is called a *fuzzy topology* on X if it satisfies the following conditions:

(O1) $\tilde{0}, \tilde{1} \in T$.

(O2) If $\mu_1, \mu_2 \in T$, then $\mu_1 \wedge \mu_2 \in T$.

(O3) If $\mu_i\in T$ for each $i\in\Gamma$, then $\bigvee_{i\in\Gamma} \mu_i \in T$.

The pair (X, T) is called a *fuzzy topological* space.

Let (X, T_1) and (Y, T_2) be fuzzy topological spaces. A function $f: (X, T_1) \to (Y, T_2)$ is called *fuzzy continuous* if $f^{-1}(\mu)\in T_1$ for all $\mu\in T_2$.

Definition 1.3 [9] A function $C: I^X\to I^X$ is called a *fuzzy closure operator* on X if it satisfies the following conditions:

(C1) $C(\tilde{0}) = \tilde{0}$.

(C2) $C(\lambda)\geq\lambda$, for all $\lambda\in I^X$.

(C3) $C(\lambda\vee\mu) = C(\lambda)\vee C(\mu)$ for all $\lambda, \mu\in I^X$.

The pair (X, C) is called *fuzzy closure space.*

A fuzzy closure space (X, C) is called *topological* provided that

(C4) $C(C(\lambda)) = C(\lambda)$, for all $\lambda\in I^X$.

Let (X, C_1) and (Y, C_2) be fuzzy closure spaces. A function $f: (X, C_1) \to (Y, C_2)$ is called a *fuzzy closure map* (for short C-map) if $f(C_1(\lambda)) \leq C_2(f(\lambda))$, for all $\lambda \in I^X$.

Theorem 1.4 [8] Let (X, T) be a fuzzy topological space. We define an operator $C_T: I^X\to I^X$ as follows: for each $\lambda\in I^X$,

$$C_T(\lambda) = \wedge\{\mu \mid \mu \geq \lambda, \tilde{1}-\mu \in T\}.$$

Then (X, C_T) is a topological fuzzy closure space.

Theorem 1.5 [8] Let (X, C) be a fuzzy closure space. Define T_C on X by

$$T_C = \{\tilde{1} - \lambda \mid C(\lambda) = \lambda\}.$$

Then:

(1) T_C is a fuzzy topology on X.

(2) $C = C_{T_C}$ iff (X, C) is topological.

2. Fuzzy quasi-proximity and fuzzy topological space

From the definition of A. Kandil *et al.*[4], we can define a fuzzy quasi-proximity.

Definition 2.1. A binary relation δ on I^X is said to be a *fuzzy quasi-proximity* on X if it satisfies the following conditions: for $\lambda, \mu, \rho \in I^X$,

(FQP1) $(\tilde{0}, \tilde{1}) \notin \delta$ and $(\tilde{1}, \tilde{0}) \notin \delta$.

(FQP2) $(\lambda \vee \rho, \mu) \in \delta$ iff $(\lambda, \mu) \in \delta$ or $(\rho, \mu) \in \delta$ and $(\mu, \lambda \vee \rho) \in \delta$ iff $(\mu, \lambda) \in \delta$ or $(\mu, \rho) \in \delta$.

(FQP3) If $(\lambda, \mu) \notin \delta$, then $\lambda \bar{q} \mu$.

The pair (X, δ) is called a *fuzzy quasi-proximity space*.

A fuzzy quasi-proximity space (X, δ) is called a *fuzzy proximity space* if it satisfies:

(FP) If $(\lambda, \mu) \in \delta$ for $\lambda, \mu \in I^X$, then $(\mu, \lambda) \in \delta$.

Let δ_1 and δ_2 be fuzzy quasi-proximities on X. We say δ_2 is finer than δ_1 (δ_1 is *coarser* than δ_2) if $(\lambda, \mu) \in \delta_2$ implies $(\lambda, \mu) \in \delta_1$.

Remark 1. Let (X, δ) be a fuzzy quasi-proximity space.

(1) If $(\lambda, \nu) \in \delta$ and $\lambda \le \mu$, then, by (FQP2), we have $(\mu, \nu) \in \delta$.

(2) We define a binary relation δ^{-1} on I^X if for any $\lambda, \mu \in I^X$, $(\lambda, \mu) \in \delta^{-1}$ iff $(\mu, \lambda) \in \delta$. Then (X, δ^{-1}) is a fuzzy quasi-proximity space.

Theorem 2.2 [4] Let (X, δ) be a fuzzy quasi-proximity space. For each $\lambda \in I^X$, we define operators $C_\delta, C^* : I^X \to I^X$ as follows:

$$C_\delta(\lambda) = \wedge\{\tilde{1} - \rho \mid (\rho, \lambda) \notin \delta\}.$$

and

$$x_t q C^*(\lambda) \text{ iff } (x_t, \lambda) \in \delta.$$

Then:

(1) $C_\delta = C^*$.

(2) (X, C_δ) is a fuzzy closure space.

Example 1. For any $\lambda, \mu \in I^X$, we define binary relations δ_0 and δ_1 on I^X by

$(\lambda, \mu) \notin \delta_0$ iff $\lambda = \tilde{0}$ or $\mu = \tilde{0}$

and

$(\lambda, \mu) \notin \delta_1$ iff $\lambda \bar{q} \mu$.

Then δ_0 and δ_1 are fuzzy proximities on X.

We can obtain C_{δ_0} and C_{δ_1} from Theorem 2.2 as follows:

$$C_{\delta_0}(\lambda) = \begin{cases} \tilde{0}, & \text{if } \lambda = \tilde{0}, \\ \tilde{1}, & \text{otherwise} \end{cases} \quad \text{and} \quad C_{\delta_1}(\lambda) = \lambda.$$

From Theorem 1.5, $T_{C_{\delta_0}} = \{\tilde{0}, \tilde{1}\}$ and $T_{C_{\delta_1}} = I^X$ are fuzzy topologies on X. □

Example 2. Let $X = \{x, y, z\}$ be a set. Define a binary relation δ on I^X as follows:

$$(\lambda, \mu) \notin \delta \begin{cases} \text{if } \lambda = \tilde{0} \text{ or } \mu = \tilde{0}, \\ \text{if } \lambda \le \chi_{\{x\}},\ \mu \le \chi_{\{y,z\}} \end{cases}$$

where χ is a characteristic function. Then (X, δ) is a fuzzy quasi-proximity space from the followings:

(FQP1) and (FQP3) are immediate from the definition of δ.

(FQP2) Since $\lambda \vee \rho \le \chi_{\{y,z\}}$ iff $\lambda \le \chi_{\{y,z\}}$ and $\rho \le \chi_{\{y,z\}}$, we have $(\mu, \lambda \vee \rho) \notin \delta$ iff $(\mu, \lambda) \notin \delta$ and $(\mu, \rho) \notin \delta$. Similarly, $(\lambda \vee \rho, \mu) \notin \delta$ iff $(\lambda, \mu) \notin \delta$ and $(\rho, \mu) \notin \delta$.

Since $(\chi_{\{x\}}, \chi_{\{y,z\}}) \notin \delta$, but $(\chi_{\{y,z\}}, \chi_{\{x\}}) \in \delta$, then δ is not a fuzzy proximity on X. From Remark 1(2), δ^{-1} is defined as follows:

$$(\lambda, \mu) \notin \delta^{-1} \begin{cases} \text{if } \lambda = \tilde{0} \text{ or } \mu = \tilde{0} \\ \text{if } \lambda \le \chi_{\{y,z\}},\ \mu \le \chi_{\{x\}}. \end{cases}$$

We can obtain C_δ and $C_{\delta^{-1}}$ from Theorem 2.2 as follows:

$$C_\delta(\lambda) = \begin{cases} \tilde{0}, & \text{if } \lambda = \tilde{0}, \\ \chi_{\{y,z\}}, & \text{if } \tilde{0} \ne \lambda \le \chi_{\{y,z\}}, \\ \tilde{1}, & \text{otherwise} \end{cases}$$

and

$$C_{\delta^{-1}}(\lambda) = \begin{cases} \tilde{0}, & \text{if } \lambda = \tilde{0}, \\ \chi_{\{x\}}, & \text{if } \tilde{0} \ne \lambda \le \chi_{\{x\}}, \\ \tilde{1}, & \text{otherwise.} \end{cases}$$

From Theorem 1.5, $T_{C_\delta} = \{\tilde{0}, \tilde{1}, \chi_{\{x\}}\}$ and $T_{C_{\delta^{-1}}} = \{\tilde{0}, \tilde{1}, \chi_{\{y,z\}}\}$. □

Definition 2.3. Let δ be a fuzzy quasi-proximity and C a fuzzy closure operator on X. A fuzzy quasi-proximity δ on X is said to be *compatible* with C if $C_\delta = C$.

Let $M(X, C)$ be the family of all fuzzy quasi-proximities on X compatible with a given fuzzy closure space (X, C).

Theorem 2.4. Let (X, C) be a fuzzy closure space. We define a binary relation δ_C on as follows: for $\lambda, \mu \in I^X$

$$(\lambda, \mu) \in \delta_C \text{ iff } \lambda q C(\mu).$$

Then:

(1) $\delta_C \in M(X, C)$.

(2) For any fuzzy quasi-proximity δ on X, δ_{C_δ} is finer than δ.

Proof. (1) First, we show that δ_C is a fuzzy quasi-proximity X.

(FQP1) It is trivial.

(FQP2) We have it from the following:

$(\mu, \lambda \vee \rho) \in \delta_C$ iff $\mu q C(\lambda \vee \rho)$

iff $\mu q C(\lambda)$ or $\mu q C(\rho)$ (by Lemma 1.1(4))

iff $(\mu, \lambda) \in \delta_C$ or $(\mu, \rho) \in \delta_C$.

Similarly, $(\mu \vee \lambda, \rho) \in \delta_C$ iff $(\mu, \rho) \in \delta_C$ or $(\mu, \lambda) \in \delta_C$.

(FQP3) If $(\mu, \lambda) \notin \delta_C$, then $\mu \bar{q}\ C(\lambda)$. Hence $\mu \bar{q}\ \lambda$.

Finally, since

$$x_t \bar{q} C(\lambda) \Leftrightarrow (x_t, \lambda) \notin \delta_C$$
$$\Leftrightarrow x_t \bar{q} C_{\delta_C}(\lambda),$$

by Lemma 1.1 (3), we have $C = C_{\delta_C}$.

(2) Since $C_\delta(\lambda) = \wedge\{\tilde{1} - \rho \mid (\rho, \lambda) \notin \delta\}$, we have

$$(\mu, \lambda) \notin \delta \Rightarrow C_\delta(\lambda) \leq \tilde{1} - \mu$$
$$\Rightarrow \mu \bar{q}\ C_\delta(\lambda)$$
$$\Rightarrow (\mu, \lambda) \notin \delta_{C_\delta}$$

Hence δ_{C_δ} is finer than δ. □

Example 3. Let $X = \{x, y, z\}$ be a set. Define $C : I^X \to I^X$ as follows:

$$C(\lambda) = \begin{cases} \tilde{0}, & \text{if } \lambda = \tilde{0}, \\ \chi_{\{x,y\}}, & \text{if } \lambda = x_t, \\ \chi_{\{z\}}, & \text{if } \lambda = z_s, \\ \tilde{1}, & \text{otherwise.} \end{cases}$$

Then C is a fuzzy closure space such that

$$\tilde{1} = C(C(x_t)) \neq C(x_t) = \chi_{\{x, y\}}.$$

We obtain δ_C from Theorem 2.4 as follows:

$$(\lambda, \mu) \notin \delta_C \begin{cases} \text{if } & \lambda = \tilde{0} \text{ or } \mu = \tilde{0}, \\ \text{if } & \lambda = z_s, \quad \mu = x_t, \\ \text{if } & \lambda \leq \chi_{\{x,y\}}, \quad \mu = z_s. \end{cases}$$

From Theorem 2.2, we have

$$C_{\delta_C}(\lambda) = \begin{cases} \tilde{0}, & \text{if } \lambda = \tilde{0}, \\ \chi_{\{x,y\}}, & \text{if } \lambda = x_t, \\ \chi_{\{z\}}, & \text{if } \lambda = z_s, \\ \tilde{1}, & \text{otherwise.} \end{cases}$$

Hence $\delta_C \in M(X, C)$. □

Example 4. Let N be a natural number set. Define a quasi-proximity δ on N by

$$(\lambda, \mu) \notin \delta \begin{cases} \text{if } \lambda = \tilde{0} \text{ or } \mu = \tilde{0}, \\ \text{if there exist nonempty finite subsets} \\ F_1, F_2 \text{ of N such that } \lambda \leq \chi_{F_1}, \mu \leq \chi_{F_2}. \end{cases}$$

If there exists a nonempty finite subset F of N such that $\tilde{0} \neq \lambda \leq \chi_F$, then $C_\delta(\lambda) = \chi_{F_2}$, where F_2 is the minimal nonempty finite set satisfying $\tilde{0} \neq \lambda \leq \chi_F$ from the following (A) and (B).

(A) For F_1 and F_2 are nonempty disjoint finite subsets of N and $\tilde{0} \neq \mu \leq \chi_{F_1}$,

$$C_\delta(\lambda) = \wedge\{\tilde{1} - \mu \mid (\mu, \lambda) \notin \delta\}$$
$$= \wedge\{\tilde{1} - \chi_{F_1} \mid (\chi_{F_1}, \lambda) \notin \delta\}$$
$$= \wedge\{\chi_{F_1^c} \mid (\chi_{F_1}, \lambda) \notin \delta\}$$
$$\geq \chi_{F_2}.$$

(B) We will show that $C_\delta(\lambda) \leq \chi_{F_2}$. We only show that $x \notin F_2$ implies $C_\delta(\lambda) = 0$. For each $x \notin F_2$, we have $(\chi_{\{x\}}, \lambda) \notin \delta$. Hence $C_\delta(\lambda) \leq \chi_{\{x\}^c}$. It implies $C_\delta(\lambda)(x) = 0$.

We obtain

$$C_\delta(\lambda) = \begin{cases} \tilde{0}, & \text{if } \lambda = \tilde{0}, \\ \chi_{F_2}, & \text{if there exists a nonempty finite} \\ & \text{set } F \text{ such that } \tilde{0} \neq \lambda \leq \chi_F \text{ and } F_2 \text{ is} \\ & \text{the minimal set satisfying } \tilde{0} \neq \lambda \leq \chi_F, \\ \tilde{1}, & \text{otherwise.} \end{cases}$$

Then δ_{C_δ} is defined from Theorem 2.4 as follows:

$$(\lambda, \mu) \notin \delta_{C_\delta} \begin{cases} \text{if } \lambda = \tilde{0} \text{ or } \mu = \tilde{0}, \\ \text{if there eixist a nonempty finite} \\ \quad \text{subset } F \text{ such that } \lambda \leq \chi_{F^C}, \mu \leq \chi_F. \end{cases}$$

For each F_1 and F_2 are nonempty disjoint finite subsets of N such that

$\tilde{0}\neq\lambda\leq\chi_{F_1},\tilde{0}\neq\mu\leq\chi_{F_2}$,

$(\lambda,\mu)\notin\delta$ implies $(\lambda,\mu)\notin\delta_{C\delta}$. On the other hand, $(\chi_{\{2\}^c},\chi_{\{2\}})\notin\delta_{C\delta}$ but $(\chi_{\{2\}^c},\chi_{\{2\}})\in\delta$. Hence $\delta_{C\delta}$ is strictly finer than δ. □

Definition 2.5. Let (X,δ_1) and (Y,δ_2) be fuzzy quasi-proximity spaces. A function $f:(X,\delta_1)\to(Y,\delta_2)$ is a *fuzzy quasi-proximity map* (P-map for short) if $(f(\mu), f(\nu))\in\delta_2$, for each $(\mu,\nu)\in\delta_1$.

Theorem 2.6. Let (X,δ_1) and (Y,δ_2) be fuzzy quasi-proximity spaces. If $f:(X,\delta_1)\to(Y,\delta_2)$ is a P-map, then:

(1) $f:(X,C_{\delta_1})\to(Y,C_{\delta_2})$ is a C-map.
(2) $C_{\delta_1}(f^{-1}(\mu))\leq f^{-1}(C_{\delta_2}(\mu))$, for each $\mu\in I^Y$.
(3) $f:(X,T_{C\delta_1})\to(Y,T_{C\delta_2})$ is a fuzzy continuous map.

Proof. (1) Let $y_tqf(C_{\delta_1}(\lambda))$, that is, $f(C_{\delta_1}(\lambda))(y)+t>1$. Then there exists $x\in X$ with $x\in f^{-1}(\{y\})$ such that $f(C_{\delta_1}(\lambda))(y)+t\geq C_{\delta_1}(\lambda)(x)+t>1$, that is, $x_tqC_{\delta_1}(\lambda)$. Since

$$x_tqC_{\delta_1}(\lambda)\Rightarrow(x_t,\lambda)\in\delta_1$$
$$\text{(since } f \text{ is a P-map,)}\Rightarrow(f(x_t),f(\lambda))\in\delta_2$$
$$\Rightarrow(f(x)_t=y_t)qC_{\delta_2}(f(\lambda)),$$

by Lemma 1.1(3), we have $f(C_{\delta_1}(\lambda))\leq C_{\delta_2}(f(\lambda))$.

(2) Since

$$\forall x_tqC_{\delta_1}(f^{-1}(\mu))\Rightarrow(x_t,f^{-1}(\mu))\in\delta_1$$
$$\text{(since } f \text{ is a P-map and } f(f^{-1}(\mu))\leq\mu,)$$
$$\Rightarrow(f(x_t),\mu)\in\delta_2$$
$$\Rightarrow f(x)_tqC_{\delta_2}(\mu)$$
$$\Rightarrow x_tqf^{-1}(C_{\delta_2}(\mu)),$$

then $C_{\delta_1}(f^{-1}(\mu))\leq f^{-1}(C_{\delta_2}(\mu))$.

(3) If $\mu\in T_{C\delta_2}$, by Theorem 1.5, we have $C_{\delta_2}(\tilde{1}-\mu)=\tilde{1}-\mu$. From (2), we have

$$C_{\delta_1}(f^{-1}(\tilde{1}-\mu))\leq f^{-1}(C_{\delta_2}(\tilde{1}-\mu))=f^{-1}(\tilde{1}-\mu).$$

Since $f^{-1}(\tilde{1}-\mu)=\tilde{1}-f^{-1}(\mu)$, by (C2) of Definition 1.3, we have

$$C_{\delta_1}(\tilde{1}-f^{-1}(\mu))=\tilde{1}-f^{-1}(\mu)$$

Hence $f^{1}(\mu)\in T_{C\delta_1}$. □

Example 5. Let $X=\{x,y,z\}$ be a set. Define fuzzy quasi-proximities δ_1 and δ_2 on X as follows:

$$(\lambda,\mu)\notin\delta_1\begin{cases}\text{if }\lambda=\tilde{0}\text{ or }\mu=\tilde{0},\\ \text{if }\lambda\leq\chi_{\{x,y\}},\ \mu=z_s.\end{cases}$$

and

$$(\lambda,\mu)\notin\delta_2\begin{cases}\text{if }\lambda=\tilde{0}\text{ or }\mu=\tilde{0},\\ \text{if }\lambda=z_s,\ \mu=x_t,\\ \text{if }\lambda\leq\chi_{\{x,y\}},\ \mu=z_s.\end{cases}$$

We can obtain C_{δ_1} and C_{δ_2} from Theorem 2.2 as follows:

$$C_{\delta_1}(\lambda)=\begin{cases}\tilde{0},\ \text{if }\ \lambda=\tilde{0},\\ \chi_{\{z\}},\text{if }\ \lambda=z_s,\\ \tilde{1},\ \text{otherwise.}\end{cases}$$

and

$$C_{\delta_2}(\lambda)=\begin{cases}\tilde{0},\ \text{if }\ \lambda=\tilde{0},\\ \chi_{\{x,y\}},\text{if }\ \lambda=x_t,\\ \chi_{\{z\}},\text{if }\ \lambda=z_s,\\ \tilde{1},\ \text{otherwise.}\end{cases}$$

The identity function $id_X:(X,\delta_1)\to(X,\delta_2)$ is not a P-map because $(z_s,x_t)\in\delta_1$ but $(z_s,x_t)\notin\delta_2$. Since $\tilde{1}=C_{\delta_1}(x_t)\not\leq C_{\delta_2}(x_t)=\chi_{\{x,y\}}$, id_X is not a C-map. On the other hand, since $T_{C\delta_1}=T_{C\delta_2}=\{\tilde{0},\tilde{1},\chi_{\{x,y\}}\}$ from Theorem 1.5, $id_X:(X,T_{C\delta_1})\to(Y,T_{C\delta_2})$ is fuzzy continuous. □

Example 6. Let N be a natural number set. Define δ_1 and δ_2 as follows:

$$(\lambda,\mu)\notin\delta_1\begin{cases}\text{if }\ \lambda=\tilde{0}\ \text{ or }\ \mu=\tilde{0},\\ \text{if there exist nonempty finite subsets}\\ F_1,F_2\text{ of }N\text{ such that }\lambda\leq\chi_{F_1},\mu\leq\chi_{F_2}.\end{cases}$$

$$(\lambda,\mu)\notin\delta_2\begin{cases}\text{if }\ \lambda=\tilde{0}\ \text{ or }\ \mu=\tilde{0},\\ \text{if there exist nonempty finite subsets}\\ F\text{ such that }\lambda\leq\chi_{F^c},\mu\leq\chi_F.\end{cases}$$

From Example 4, we obtain $C_{\delta_1}=C_{\delta_2}$ as follows:

$$C_{\delta_1}(\lambda)=\begin{cases}\tilde{0},\ \text{if }\ \lambda=\tilde{0},\\ \chi_{F_2},\ \text{if there exists a nonempty finite}\\ \quad\text{set }F\text{ such that }\tilde{0}\neq\lambda\leq\chi_F\text{ and }F_2\text{ is}\\ \quad\text{the minimal set satisfying }\tilde{0}\neq\lambda\leq\chi_F\\ \tilde{1},\quad\text{otherwise.}\end{cases}$$

Hence $T_{C\delta_1}=T_{C\delta_2}$. The identity function $id_N:(N,\delta_1)\to(N,\delta_2)$ is not a P-map. But $id_N:(N,T_{C\delta_1})\to(N,T_{C\delta_2})$

is fuzzy continuous and $id_N : (N, c_{\delta_1}) \rightarrow (N, c_{\delta_2})$ is a C-map. □

Let **FC** be the category of fuzzy closure spaces and C-maps and **FQProx** the category of fuzzy quasi-proximity spaces and P-maps. From Theorem 2.5, we can prove the following theorem.

Theorem 2.6. Define F : **FQProx** → **FC** by
$F(X, \delta) = (X, C_\delta)$ and $F(f) = f$.
Then F is a functor.

Theorem 2.7. Define G : **FC**→**FQProx** by
$G(X, C) = (X, \delta_C)$ and $G(f) = f$.
Then G is a functor.

Proof. Let (X, C) be a fuzzy closure space. From Theorem 2.4, (X, δ_C) is a fuzzy quasi-proximity space. If $f : (X, C_1) \rightarrow (Y, C_2)$ is a C-map, then $f : (X, \delta_{C_1}) \rightarrow (Y, \delta_{C_2})$ is a P-map from the following:

$(\lambda, \mu) \in \delta_{C_1} \Rightarrow \lambda q C_1(\mu)$
$\Rightarrow f(\lambda) q f(C_1(\mu))$ (by Lemma 1.1(5))
$\Rightarrow f(\lambda) q C_2(f(\mu))$
(because $f(C_1(\mu)) \leq C_2(f(\mu))$)
$\Rightarrow (f(\lambda), f(\mu)) \in \delta_{C_2}$. □

Theorem 2.8. A functor G : **FC** → **FQProx** is a left adjoint of the functor F.

Proof. Let $(X, C) \in$ **FC**. Since $F \circ G(C) = C_{\delta_C} = C$ from Theorem 2.4, the identity map $id_X : (X, C) \rightarrow (X, F \circ G(C))$ is a C-map. For each (Y, δ) and each C-map $f : (X, C) \rightarrow F(Y, \delta)$ in **FC**, by Theorem 2.7, $G(f) : G(X, C) \rightarrow G \circ F(Y, \delta)$ is a P-map, that is, $f : (X, \delta_C) \rightarrow (Y, \delta_{C_\delta})$ is a P-map. Since δ_{C_δ} is finer than δ from Theorem 2.4, $id_Y : (Y, \delta_{C_\delta}) \rightarrow (Y, \delta)$ is a P-map. Hence $f : (X, \delta_C) \rightarrow (Y, \delta)$ is a P-map with $f = F(f) \circ id_X$. Therefore id_X is a F-universal map for (X, C). □

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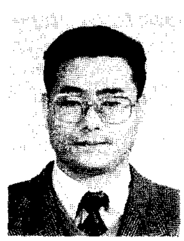

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