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Semiprime and Semiprimary Fuzzy Ideals

Kyong Ah Oh, Tae Eun Jeong, Youn-Mi Nam Koong and Young Hee Kim

Department of Mathematics, Chungbuk National University

ABSTRACT

We study semiprime fuzzy ideals, semiprimary fuzzy ideals, and their properties. We investigate that if a fuzzy ideal is semiprime and semiprimary, then it is prime.

1. Introduction

Much interest has been shown in fuzzification of the concepts of prime and primary ideals[8,10,12], and the same attempts have been made to fuzzify the concept of semiprime ideals[11].

Kumbhojkar[10,11] has shown that every prime fuzzy ideal is primary and every prime fuzzy ideal is semiprime, but the converses are not true. The purpose of this paper is to find a condition that a fuzzy ideal is prime. Also, we find that every prime fuzzy ideal is weak primary and is semiprime. And then, we show that if a fuzzy ideal is weak primary and semiprime, then it is prime.

In particular, we observe some characterization of semiprimary fuzzy ideals. We show that if a fuzzy ideal is semiprime and semiprimary, then it is prime.

Throughout this paper, R stands for a commutative ring with identity and L stands for a lattice with the least element 0 and the greatest element 1. Unless otherwise stated, L is complete and completely distributive in the sense that it satisfies the following law:

$(\vee\{a_i \mid i \in I\}) \wedge (\vee b_j \mid j \in J) = \vee\{a_i \wedge b_j \mid i \in I, j \in J\}$ for all $a_i,\ b_j \in L$.

By[1,4], every distributive upper continuous lattice has this property, and every complete totally ordered set has this property. In particular the closed unit interval [0,1] has this property.

2. Preliminaries

In this section, we recall basic definitions and known results which we shall need in the later sections.

Definition 2.1[11] An L-fuzzy ideal is a function $J : R \rightarrow L$ satisfying the following axioms:

(1) $J(x+y) \geq J(x) \wedge J(y)$,

(2) $J(-x) = J(\xi)$,

(3) $J(xy) \geq J(x) \vee J(y)$.

Since we are considering L-fuzzy ideals over a fixed lattice L, we shall call them by fuzzy ideals only. We shall call J by a properly fuzzy ideal if it takes more than two distinct values.

Definition 2.2[2] Let $J : R \rightarrow L$ be a fuzzy ideal and $\alpha \in L$. Then the set $J_\alpha = x \in R \mid J(x) \geq \alpha\}$ is called a level cut of R with respect to J.

Proposition 2.3[11] (1) A function $J : R \rightarrow L$ is a fuzzy ideal if and only if $J(x-y) \geq J(x) \wedge J(y)$ and $J(xy) \geq J(x) \vee J(y)$.

(2) If $J : R \rightarrow L$ is a fuzzy ideal, then

(a) $J(0) \geq J(x) \geq J(1)$, for all $x \in R$;

(b) $J(x-y) = J(0)$ implies $J(x) = J(y)$ for all $x, y \in R$;

(c) the level cuts $J_\alpha = \{x \in R \mid J(x) \geq \alpha\}$ are ideals of R for all $\alpha \in L$, conversely, if each J_α is an ideal, then J is a fuzzy ideal.

Definition 2.4[10] A fuzzy ideal $P : R \rightarrow L$ is called prime if for all $x, y \in R$,

$P(xy) = P(x)$ or $P(xy) = P(y)$.

In[8,10], there are several definitions of prime fuzzy ideals. And the relation among the definitions is shown in[10], but we will use the above definition of prime fuzzy ideals.

Proposition 2.5[10] A fuzzy ideal $P : R \rightarrow L$ is prime if and only if every level cut $P_\alpha = \{x \in R \mid P(x) \geq \alpha\}$ is prime for all $\alpha > P(1)$.

For $\alpha = P(1)$, $P_\alpha = R$.

Definition 2.6[10] A fuzzy ideal $Q : R \rightarrow L$ is called primary if Q is nonconstant and for all $x, y \in R$,
$Q(xy) = Q(x)$ or $Q(y^n)$, for some positive integer n.

Proposition 2.7[10] Every prime fuzzy ideal is primary.

Definition 2.8 [11] If $J : R \rightarrow L$ is a fuzzy ideal, then the fuzzy set $\sqrt{J} : R \rightarrow L$, defined as $\sqrt{J}(x) = \vee \{J(x^n) \mid n>0\}$ is called the fuzzy (nil)radical of J.

Proposition 2.9[11] If $J : R \rightarrow L$ is a fuzzy ideal, then $\sqrt{J}$ is a fuzzy ideal.

Definition 2.10[9] Let $A: X \rightarrow L$ be a fuzzy subset of a set X, where L is a complete lattice. A is said to have the supremum property if for every subset S of X, there exists $x_0 \in S$, such that $\vee \{A(x) \mid x \in S\} = A(x_0)$.

Proposition 2.11[11] If $J : R \rightarrow L$ is a fuzzy ideal with the supremum property, then $(\sqrt{J})_\alpha = \sqrt{J_\alpha}$.

Proposition 2.12[11] For any $0 \leq \alpha < 1$, and a fuzzy ideal $J : R \rightarrow L$, $(\sqrt{J})_\alpha = \sqrt{J_\alpha}$ where L is totally odered set, $J_\alpha = \{x \in R \mid J(x) > \alpha\}$ and $(\sqrt{J})_\alpha = \{x \in R \mid \sqrt{J}(x) > \alpha\}$.

Proposition 2.13[11] If $P : R \rightarrow L$ is prime, then $\sqrt{P} = P$.

3. Weak Primary and Semiprime Fuzzy Ideals

In this section, we study the concepts of weak primary fuzzy ideals and semiprime fuzzy ideals, and their properties. We investigate that if a fuzzy ideal is weak primary and semiprime, then it is prime.

Definition 3.1[10] A fuzzy ideal $J : R \rightarrow L$ is said to be weak primary or in short w-primary if , for all $x, y \in R$, $J(xy) = J(x)$ or $J(xy) \leq J(y^n)$ for some integer $n > 0$.

Proposition 3.2[10] Every primary fuzzy ideal is w-primary. In particular, every prime fuzzy ideal is w-primary.

Proposition 3.3[11] A fuzzy ideal is w-primary if and only if each of its level cuts is primary.

The fuzzy ideal given below is w-primary but not primary.

Example 3.4 Let $J : Z \rightarrow [0,1]$ be defined as follows

$$J(x) = \begin{cases} 0 & x \notin \langle p \rangle, \\ \dfrac{n}{n+1} & x \in \langle p^n \rangle - \langle p^{n+1} \rangle,\ n = 1,2,\ldots,5, \\ 1 & x \in \langle p^6 \rangle. \end{cases}$$

This fuzzy ideal is finite-valued, it may be observed that $x \in \langle p^n \rangle - \langle p^{n+1} \rangle$ if and only if $x = ap^n$ where $p \times \mid a$, $\alpha \in Z$, $n = 1, 2, \cdots, 5$. Let $x = ap^n$ and $y = bp^m$, where $a, b, m, n \in Z$, $p \times a$, $p \times b$, $1 \leq n \leq 5$, $1 \leq m \leq 5$. Let $xy = abp^{n+m}$, then $J(xy) = J(abp^{n+m})$.

Case1. If $1 \leq m+n \leq$, then
$J(xy) = J(abp^{n+m}) = \dfrac{m+n}{m+n+1} \leq J(y^6)J(b^6p^{6m}) = 1$.

Case2. If $m+n \geq 6$, then $J(xy) = 1 = J(y^6)$. Hence J is w-primary. But J is not a primary. For, $J(p^2p^3) \neq J(p^2)$ and $J(p^2p^3) \neq J(p^{3n})$ for any $n = 1, 2, \cdots$.

Definition 3.5[11] A fuzzy ideal $S : R \rightarrow L$ is called semiprime if $S(x^2) = S(x)$ for all $x \in R$.

Proposition 3.6[11] Every prime fuzzy ideal is a semiprime fuzzy ideal.

Theorem 3.7[11] If $S : R \rightarrow L$ is a fuzzy ideal, then the following are equivalent:

(1) S is semiprime,
(2) each level cut of S is semiprime,
(3) $S(x^n) = S(x)$ for all integers $n>0$ and $x \in R$,
(4) $J^2 \subseteq S \Rightarrow J \subseteq S$ for all fuzzy ideals $J : R \rightarrow L$,
(5) $J^n \subseteq S$ for $n > 0 \Rightarrow J \subseteq S$, for all fuzzy ideals $J : R \rightarrow L$,
(6) $S = \sqrt{S}$ where $\sqrt{S}$ is the fuzzy nilradical of S, when L is totally ordered, each of the above statements is equivalent to the following:
(7) S coincides with its prime fuzzy radical,
(8) $S = \cap \{P \mid P \in \mathcal{C}$, where $\mathcal{C}$ is a class of prime fuzzy ideals.

Example 3.8 An infinite valued semiprime fuzzy ideal is defined as follows: Let R be the ring of integers and $L = [0,1]$. Let $\mathcal{C}$ be an ordered set of distinct prime numbers say $\mathcal{C} = \{p_1, p_2, \cdots, p_n, \cdots\}$. Let K^k be the ideal generated by the integer $p_1 p_2 \cdots p_k$. Then we have an infinite chain of ideals $K^1 \supset K^2 \supset \cdots \subset K^n \supset \cdots$. Define a fuzzy ideal $S : R \rightarrow L$ as follows:

$$S(x) = \begin{cases} 1 & x = 0, \\ \dfrac{n}{n+1} & x \in K^n - K^{n+1}, \\ 0 & x \notin K^1. \end{cases}$$

The level cuts of S are 0, R and $K^n = \langle p_1 p_2 \cdots p_n \rangle$, for $n = 1, 2, \cdots$. Let $p_1 p_2 \cdots p_n \mid x^2$ for all $x^2 \in K^n$ and then $p_1 \mid x^2$, $p_2 \mid x^2$, $\cdots$, $p_n \mid x^2$. Since R is a unique factorization domain, $p_1 \mid x$, $p_2 \mid x$, $\cdots$, $p_n \mid x$. Thus every level cut of S is semiprime, and then S is semiprime fuzzy ideal by Theorem 2.7. But since prime ideals of R are precisely the ideals generated by prime numbers, S is not prime.

Therefore every prime fuzzy ideal is semiprime, but the converse is not true.

4. Semiprimary Fuzzy Ideals

In this section, we study the concept of semiprimary fuzzy ideals and its properties. Also we investigate that if fuzzy ideal is semiprime and semiprimary, then it is prime.

Definition 4.1 [7] A fuzzy ideal $J : R \to L$ is called a semiprimary fuzzy ideal if, for all $x, y \in R$, either $J(xy) \geq J(x^n)$ for some integer $n > 0$ or else $J(xy) \leq J(y^m)$ for some integer $m > 0$.

Proposition 4.2 A fuzzy ideal $J : R \to L$ is semiprimary if and only if each of its level cuts $J_\alpha = \{x \in R \mid J(x) \geq \alpha\}$ is semiprimary.

Proof. Assume that J is semiprimary. Let $x, y \in R$ and $xy \in J_\alpha$. If $x^n \notin J_\alpha$ for all integer $n > 0$, then $J(x^n) < \alpha \leq J(xy)$ so that $J(xy) \leq J(y^m)$ for some $m > 0$, so $y^m \in J\alpha$. It show that the level cut J_α is semiprimary.

Conversely, suppose that J_α, $\alpha \in L$, is semiprimary. Let $x, y \in R$, choose $\alpha = J(xy)$ and let $J(xy) > J(x^n)$ for all integers $n > 0$. Since J_α is semiprimary and $xy \in J_\alpha$, it follows that $y^m \in J_\alpha$ for some integer $m > 0$. Hence $J(y^m) \geq \alpha = J(xy)$. Therefore J is semiprimary.

The fuzzy ideal given below is semiprimary, but it is not prime.

Example 4.3 Let R be the polynomial ring $F[x]$, where F is a field and $L = [0,1]$. Define a fuzzy ideal J: $R \to L$ as follow:

$$J(a) = \begin{cases} 0.8 & a \in \langle x^4 \rangle, \\ 0.6 & a \in \langle x^2 \rangle - \langle x^4 \rangle, \\ 0.2 & a \in R - \langle x^2 \rangle. \end{cases}$$

Clearly its level cuts are $\langle x^4 \rangle$, $\langle x^2 \rangle$, and R. Since $\sqrt{\langle x^4 \rangle} = \sqrt{\langle x^2 \rangle} = \langle x \rangle$ and $\langle x \rangle$ is prime, $\langle x^4 \rangle$, $\langle x^2 \rangle$ and R are semiprimary. Hence fuzzy ideal J: $R \to L$ is semiprimary. But J is not a prime fuzzy ideal, for its level cuts $\langle x^4 \rangle$ and $\langle x^2 \rangle$ are not prime.

Theorem 4.4 If a fuzzy ideal $J : R \to L$ has the supremum property, then J is semiprimary if and only if $\sqrt{J}$ is prime.

Proof. Let J be semiprimary. Then J_α is semiprimary by Proposition 3.2 and $\sqrt{J_\alpha}$ is prime for $\alpha \in L$.

Since J has the supremum property, $\sqrt{J_\alpha} = (\sqrt{J})_\alpha$ by Proposition 1.11. Hence $(\sqrt{J})_\alpha$ is prime for any $\alpha \in L$. Therefore $\sqrt{J}$ is prime.

Conversely if $\sqrt{J}$ is prime, then $(\sqrt{J})_\alpha$ is prime for any $\alpha \in$ L. And since $\sqrt{J_\alpha} = (\sqrt{J})_\alpha$, $\sqrt{J_\alpha}$ is prime so that J_α is semiprimary. Hence J is semiprimary.

Remark 4.5 By the definition of w-primary fuzzy ideal, we know that if a fuzzy ideal is w-primary, then it is semiprimary. Hence every prime fuzzy ideal is a semiprimary fuzzy ideal. But by Example 3.3, the converse is not true.

Theorem 4.6 If any fuzzy ideal is semiprime and semiprimary, then it is prime.

Proof. Let $J : R \to L$ be a fuzzy ideal which is semiprime and semiprimary, and $x, y \in R$. Since J is semiprime, $J(x^n) = J(x)$ and $J(y^m) = J(y)$ for all $n, m > 0$. And since J is semiprimary, either $J(xy) \leq J(x^n)$ for some $n > 0$ or $J(xy) \leq J(y^m)$ for some $m > 0$. Hence either $J(xy) \leq J(x)$ or $J(xy) \leq J(y)$. By the definition of fuzzy ideal, $J(xy) = J(x)$ or $J(xy) = J(y)$. Therefore J is prime.

Corollary 4.7 If any fuzzy ideal is semiprime and semiprimary, then it is w-primary.

Proof. By Theorem 3.6 and Proposition 2.2, it is obvious.

Corollary 4.8 If a fuzzy ideal is w-primary and semiprime, then it is prime.

Proof. Let $J : R \to L$ be a fuzzy ideal which is w-primary and semiprime, and $x, y \in R$. Since J is w-primary, $J(xy) = J(x)$ or $J(xy) \leq J(y^n)$ for some $n > 0$. And since J is semiprime, $J(y^n) = J(y)$. Hence $J(xy) = J(x)$ or

$J(xy) \leq J(y)$. But $J(xy) \geq J(y)$ for all $y \in R$ by definition of fuzzy ideal. Therefore J is prime.

Corollary 4.9 If a fuzzy ideal is primary and semiprime, then it is prime.

Proof. Since every primary fuzzy ideal is w-primary, every primary and semiprime fuzzy ideal is prime by the above corollary.

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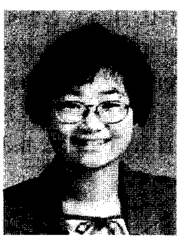
김 영 희 (Young-Hee Kim)

1975년 : 연세대학교 수학과 졸업
1986년 : 동대학원 졸업(이학박사)
1983년-현재 : 충북대 자연과학대학 수학과 교수
관심분야 : Fuzzy algebra

남궁윤미 (Youn-Mi Nam Koong) 정회원

1982년 : 연세대학교 수학과 졸업
1986년 : 동대학원 수학과 졸업(석사)
1998년 : 충북대학교 대학원 수학과 졸업(박사)
1994년~현재 : 충북대학교 수학과 시간강사
관심분야 : Fuzzy algebra



정 태 은 (Tae-Eun Jeong) 정회원

1992년 : 충북대학교 수학과 졸업
1996년 : 한남대학교 수학과 졸업(석사)
1996년-현재 : 충북대학교 대학원 수학과 박사과정
관심분야 : Fuzzy algebra

오 경 아 (Kyung-a Oh) 정회원

1997년 : 충북대학교 수학과 졸업
1999년 : 동대학원 수학과 졸업 (석사)