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Fuzzy quasi-uniform bases

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ABSTRACT

We will define a base of a fuzzy (quasi-)uniform space and investigate some properties of bases. In particular, for the family $\{\beta_i\}_{i\in\Gamma}$ of fuzzy (quasi-)uniform bases on X, there exists the coarsest fuzzy (quasi-)uniformity on X which is finer than fuzzy (quasi-)uniformity Φ_{β_i} generated by β_i for each $i\in\Gamma$.

1. Introduction and preliminaries

R. Badard *et al.* [1] introduced the gradation of entourages called smooth fuzzy preuniform structure. Moreover, S.K. Samanta [6] defined the fuzzy uniformity as a generalization of that of B. Hutton [3].

In this paper, we define the fuzzy uniformity in a somewhat different view of S.K. Samanta [6]. Moreover, we will define a base of a fuzzy (quasi-) uniform space and investigate some properties of bases. In particular, for the family $\{\beta_i\}_{i\in\Gamma}$ of fuzzy (quasi-) uniform bases on X, there exists the coarsest fuzzy (quasi-)uniformity on X which is finer than fuzzy (quasi-)uniformity Φ_{β_i} generated by β_i for each $i\in\Gamma$. Throughout this paper, all the notations and the definitions are standard in fuzzy set theory.

Let Ω_X denote the family of all functions $U: I^X \to I^X$ with the following properties:

(1) $U(\tilde{0}) = \tilde{0}$, $\mu\leq U(\mu)$, for every $\mu\in I^X$,
(2) $U(\vee_{i\in\Gamma}\mu_i) = \vee_{i\in\Gamma} U(\mu_i)$, for $\{\mu_i\}_{i\in\Gamma}\subset I^X$.

For U, $V\in\Omega_X$, we define, for all $\mu\in I^X$,

$U^{-1}(\mu)=\wedge\{\rho \mid U(\tilde{1}-\rho)\leq \tilde{1}-\mu\}$,
$(U\sqcap V)(\mu)=\wedge\{U(\mu_1)\vee V(\mu_2) \mid \mu_1\vee \mu_2=\mu\}$,
$U\circ V(\mu)= U(V(\mu))$.

Then $U\sqcap V$, $U\circ V$ and $U^{-1}\in\Omega_X$ from Lemma 2 of [3].

For every U, $V\in\Omega_X$, we define $U\leq V$ iff $U(\mu)\leq V(\mu)$ for each $\mu\in I^X$.

Lemma 1.1. [5] For every U, V, W, U_1, $V_1\in \Omega_X$, the following properties hold:

(1) If $U\leq U_1$ and $V\leq V_1$, then $U\sqcap V \leq U_1\sqcap V_1$.
(2) $U\sqcap V\leq U$, $U\sqcap V\leq V$ and $U\sqcap U=U$.
(3) $(U^{-1})^{-1}=U$.
(4) $U\leq V$ iff $U^{-1}\leq V^{-1}$.
(5) Let a function $U_{\tilde{1}}: I^X\to I^X$ be defined by

$$U_{\tilde{1}}(\mu) = \begin{cases} \tilde{1} & \text{if } \mu\neq\tilde{0}, \\ \tilde{0} & \text{if } \mu=\tilde{0}. \end{cases}$$

Then $U_{\tilde{1}} =U^{-1}_{\tilde{1}} \in \Omega_X$ and $U\sqcap U_{\tilde{1}} =U$.
(6) $(V\circ U)^{-1}=U^{-1}\circ V^{-1}$.
(7) $(U\sqcap V)^{-1}=U^{-1}\sqcap V^{-1}$.
(8) $(U\sqcap V)\sqcap W=U\sqcap(V\sqcap W)$.

We define the fuzzy (quasi-)uniformity in which the conditions (FQU3) and (FU) are defined in a somewhat different view of S.K. Samanta [6].

Definition 1.2. A function $\Phi: \Omega_X\to I$ is said to be a *fuzzy quasi-uniformity* on X if it satisfies the following conditions:

(FQU1) for U, $V\in\Omega_X$, we have $\Phi(U\sqcap V)\geq \Phi(U)\wedge\Phi(V)$,
(FQU2) if $V\leq U$, then $\Phi(V)\leq\Phi(U)$,
(FQU3) for $U\in\Omega_X$, we have $sup\{\Phi(V) \mid V\circ V\leq U\}\geq\Phi(U)$,
(FQU4) there exists $U\in\Omega_X$ such that $\Phi(U)=1$.

The pair(X, Φ) is said to be a *fuzzy quasi-uniform space*.

A fuzzy quasi-uniform space (X, Φ) is called a *fuzzy uniform space* if (FU) for $U\in\Omega_X$, we have $\sup\{\Phi(V) \mid V\leq U^{-1}\}\geq\Phi(U)$.

In the above definition, (FQU3) and (FU) are defined in the sense of R. Badard *et. al*[1].

Let Φ_1 and Φ_2 be fuzzy (quasi-)uniformities on X. Φ_1 is finer than Φ_2 (or Φ_2 is *coarser* than Φ_1), denoted by $\Phi_2\leq\Phi_1$, iff for any $U\in\Omega_X$, $\Phi_2(U)\leq\Phi_1(U)$.

Remark 1. (1) The definition of S.K. Samanta [6] is that of our sense.

(2) Let (X, Φ) be a fuzzy (quasi-)uniform space. By (FQU1), (FQU2) and Lemma 1.1 (2), we have $\Phi(U \sqcap V)=\Phi(U) \wedge \Phi(V)$.

(3) Let (X, Φ) be a fuzzy (quasi-)uniform space. By Lemma 1.1 (5) and (FQU4), since $U \leq U_{\top}$ for all $U \in \Omega_X$, we have $\Phi(U_{\top})=1$.

(4) If (X, Φ) is a fuzzy uniform space, then, by (FU) and (FQU3), we have $sup\{\Phi(V) \mid V \leq U^{-1}\}=\Phi(U^{-1})$. From Lemma 1.1(3), we have $\Phi(U^{-1})=\Phi(U)$.

2. Fuzzy quasi-uniform base

Definition 2.1. Let Θ_X be a subset of Ω_X. A function $\beta : \Theta_X \rightarrow I$ is said to be a *base* for a fuzzy quasi-uniformity on X if it satisfies the following conditions:

(FQB1) $\beta(U_1 \sqcap U_2) \geq \beta(U_1) \wedge \beta(U_2)$,

(FQB2) for $U \in \Theta_X$, we have $sup\{\beta(V) \mid V \circ V \leq U\} \geq \beta(U)$,

(FQB3) there exists $U \in \Theta_X$ such that $\beta(U)=1$.

The pair (X, β) is called a *fuzzy quasi-uniform base*.

A fuzzy quasi-uniform base (X, β) is called a *fuzzy uniform base* if

(FB) for $U \in \Theta_X$, we have $sup\{\beta(V) \mid V \leq U^{-1}\} \geq \beta(U)$.

Remark 2. Every fuzzy (quasi-)uniform space (X, Φ) is a fuzzy (quasi-)uniform base in the sense of $\Theta_X=\Omega_X$.

A fuzzy (quasi-)uniform base β always *generates* a fuzzy (quasi-)uniformity Φ_β on X in following theorem.

Theorem 2.2. Let (X, β) a fuzzy (quasi-)uniform base. Define, for every $U \in \Omega_X$,

$$\Phi_\beta(U) = \begin{cases} sup\{\beta V \mid V \leq U\} & if \{V \in \Theta_X \mid V \leq U\} \neq \varnothing, \\ 0 & \text{otherwise.} \end{cases}$$

Then Φ_β is a fuzzy (quasi-)uniformity on X.

Proof. (FQU1) For any $U_1, U_2 \in \Omega_X$, we will show that

$\Phi_\beta(U_1 \sqcap U_2) \geq \Phi_\beta(U_1) \wedge \Phi_\beta(U_2)$.

If $\Phi_\beta(U_1)=0$ or $\Phi_\beta(U_2)=0$, it is trivial.

If $\Phi_\beta(U_1) \neq 0$ and $\Phi_\beta(U_2) \neq 0$, for ε such that $\Phi_\beta(U_1) \wedge \Phi_\beta(U_2) > \varepsilon > 0$, there exist $V_1, V_2 \in \Theta_X$ such that

$\beta(V_1) \geq \Phi_\beta(U_1) - \varepsilon,\ V_1 \leq U_1$,

$\beta(V_2) \geq \Phi_\beta(U_2) - \varepsilon,\ V_2 \leq U_2$.

Since $V_1 \sqcap V_2 \leq U_1 \sqcap U_2$, we have

$$\begin{aligned} \Phi_\beta(U_1 \sqcap U_2) &\geq \beta(V_1 \sqcap V_2) \\ &\geq \beta(U_1) \wedge \beta(U_2) \\ &\geq \Phi_\beta(U_1) \wedge \Phi_\beta(U_2) - \varepsilon. \end{aligned}$$

Since ε is arbitrary, this gives the desired result.

(FQU2) It is easily proved from the definition of Φ_β.

(FQU3) If $\Phi_\beta(U)=0$, then there exists the identity function $E \in \Omega_X$ with $E \circ E \leq U$ such that $\Phi_\beta(E) \geq 0$.

Let $\Phi_\beta(U) \neq 0$. Suppose that there exist $U \in \Omega_X$ and $r \in (0, 1)$ such that

$sup\{\Phi_\beta(U_1) \mid U_1 \circ U_1 \leq U\} < r < \Phi_\beta(U)$.

By the definition of Φ_β, there exists $V \leq U$ such that $\Phi_\beta(U) \geq \beta(V) > r$.

Since $sup\{\beta(V_1) \mid V_1 \circ V_1 \leq V\} \geq \beta(V) > r$ from (FQB2), there exists $W \in \Theta_X$ such that $W \circ W \leq V$ and $\beta(W) > r$.

From $W \circ W \leq V \leq U$, it follows

$sup\{\Phi_\beta(U_1) \mid U_1 \circ U_1 \leq U\} \geq \beta(W) > r$.

It is a contradiction. Hence $sup\{\Phi_\beta(U_1) \mid U_1 \circ U_1 \leq U\} \geq \Phi_\beta(U)$.

(FU) It is similar to (FQU3).

Definition 2.3. A function β is said to be a *base* for a fuzzy (quasi-)uniformity Φ on X if $\Phi_\beta = \Phi$.

Definition 2.4. Let β_1 and β_2 be fuzzy (quasi-)uniform bases on X. We say β_2 is finer than β_1, denoted by $\beta_2 > \beta_1$, iff for any $\beta_1(U) > 0$ and $\varepsilon > 0$, there exists $V \leq U$ such that $\beta_2(V) \geq \beta_1(U) - \varepsilon$.

Theorem 2.5. Let (X, β_1) and (X, β_2) be fuzzy (quasi-)uniform bases for (X, Φ_1) and (X, Φ_2), respectively. Then $\Phi_1 \leq \Phi_2$ iff $\beta_1 < \beta_2$.

Proof. For any $\beta_1(U) > 0$, since $\Phi_1 \leq \Phi_2$ we have $\Phi_2(U) \geq \Phi_1(U) \geq \beta_1(U)$.

From the definition of Φ_2, for $\varepsilon > 0$ there exists $V \subset U$ such that

$\beta_2(V) \geq \Phi_2(U) - \varepsilon \geq \beta_1(U) - \varepsilon$.

Hence $\beta_1 < \beta_2$.

Conversely, suppose that there exist $U \in \Omega_X$ and $r \in (0, 1)$ such that

$\Phi_1(U) > r > \Phi_2(U)$.

By the definition of Φ_1, there exists $V \leq U$ such that $\Phi_1(U) \geq \beta_1(V) > r > \Phi_2(U)$.

Since $\beta_1 < \beta_2$, for $\beta_1(V) > r$ and $\varepsilon = \beta_1(V) - r$, there exists $W \leq V$ such that

$\beta_2(W) \geq (\beta_1(V) - \varepsilon) = r$.

Hence $\Phi_2(U) \geq \Phi_2(W) \geq \beta_2(W) \geq r$. It is a contradiction. Therefore $\Phi_1 \leq \Phi_2$.

Lemma 2.6. Define $U_\rho: I^X \to I^X$ as follows.

$$U_\rho(\lambda) = \begin{cases} \tilde{0} & \text{if } \lambda = \tilde{0}, \\ \rho & \text{if } \tilde{0} \neq \lambda \leq \rho, \\ \tilde{1} & \text{otherwise.} \end{cases}$$

Then:
(1) $U_\rho \in \Omega_X$,
(2) $(U_\rho)^{-1} = U_{1-\rho}$,
(3) $U_\rho \circ U_\rho = U_\rho$ and $(U_\rho \sqcap U_\mu) \circ (U_\rho \sqcap U_\mu) = U_\rho \sqcap U_\mu$.

Proof. (1), (2) and $U_\rho \circ U_\rho = U_\rho$ of (3) are easily proved.

Since

$$U_\rho \sqcap \mu_\mu(\lambda) = \begin{cases} \tilde{0} & \text{if } \lambda = \tilde{0}, \\ \mu \wedge \rho & \text{if } \tilde{0} \neq \lambda \leq \mu \wedge \rho, \\ \mu & \text{if } \lambda \leq \mu, \lambda \not\leq \rho, \\ \rho & \text{if } \lambda \leq \rho, \lambda \leq \mu, \\ \mu \vee \rho & \text{if } \lambda \leq \mu \vee \rho, \lambda \not\leq \mu, \lambda \not\leq \rho, \\ \tilde{1} & \text{otherwise.} \end{cases}$$

we have $(U_\rho \sqcap U_\mu) \circ (U_\rho \sqcap U_\mu) = U_\rho \sqcap U_\mu$.

Example 1. Define β_1 and β_2 on X as follows:

$$\beta_1(U) = \begin{cases} 1 & \text{if } U = U_{\tilde{1}}, \\ \frac{1}{2} & \text{if } U = U_\rho \end{cases}$$

and

$$\beta_2(U) = \begin{cases} 1 & \text{if } U = U_{\tilde{1}}, \\ \frac{2}{3} & \text{if } U = U_\rho \sqcap U_\mu. \end{cases}$$

From Lemma 2.6, β_1 and β_2 are fuzzy quasi-uniform bases on X. From Definition 2.4 we have $\beta_1 < \beta_2$.

From Theorem 2.2 we obtain the followings:

$$\Phi_{\beta 1}(U) = \begin{cases} 1 & \text{if } U = U_{\tilde{1}}, \\ \frac{1}{2} & \text{if } U_\rho \leq U < U_{\tilde{1}}, \\ 0 & \text{otherwise} \end{cases}$$

and

$$\Phi_{\beta 2}(U) = \begin{cases} 1 & \text{if } U = U_{\tilde{1}}, \\ \frac{2}{3} & \text{if } U_\rho \sqcap U_\mu \leq U < U_{\tilde{1}}, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\Phi_{\beta 1} \leq \Phi_{\beta 2}$.

Theorem 2.7. Let (X, β_i) be a fuzzy (quasi-)uniform basis of (X, Φ_i) for each $i \in \Gamma$. Let

$\Theta_X = \{ \sqcap_{i=1}^n U_{ki} \mid \beta_{ki}(U_{ki}) > 0 \text{ for all } k_i \in K \}$,

$\Theta'_X = \{ \sqcap_{i=1}^n V_{ki} \mid \Phi_{ki}(V_{ki}) > 0 \text{ for all } k_i \in K \}$

for every finite index set $K = \{k_1, \ldots, k_n\} \subset \Gamma$. Define the function $\beta: \Theta_X \to I$ by

$$\beta(U) = sup\{ \bigwedge_{i=1}^n \beta_{ki}(U_i) \mid U = \sqcap_{i=1}^n U_{ki} \}$$

and the function $\beta' : \Theta'_X \to I$ by

$$\beta'(U) = sup\{ \bigwedge_{i=1}^n \Phi(V_{ki}) \mid U = \sqcap_{i=1}^n V_{ki} \}$$

for every finite index set $K = \{k_1, \ldots, k_n\} \subset \Gamma$. Then:

(1) The structures β and β' are fuzzy (quasi-)uniform bases on X.

(2) The fuzzy (quasi-)uniform structure Φ_β generated by β is the coarsest fuzzy (quasi-)uniform structure on X finer than Φ_i for all $i \in \Gamma$.

(3) $\Phi_\beta = \Phi_{\beta'}$.

Proof. (1) We will show that β is a fuzzy (quasi-) uniform basis on X.

(FQB1) For any $U, V \in \Theta_X$, we will show that $\beta(U \sqcap V) \geq \beta(U) \wedge \beta(V)$.

Since $\beta(U) > 0$ and $\beta(V) > 0$, for ε such that $\beta(U) \wedge \beta(V) > \varepsilon > 0$, there exist finite index sets $K = \{k_1, \ldots, k_n\}$, $L = \{l_1, \ldots, l_m\} \subset \Gamma$ such that

$$\bigwedge_{i=1}^n \beta_{ki}(U_{ki}) \geq \beta(U) - \varepsilon, \quad \sqcap_{i=1}^n U_{ki} = U,$$
$$\bigwedge_{j=1}^m \beta_{lj}(V_{lj}) \geq \beta(V) - \varepsilon, \quad \sqcap_{j=1}^m V_{lj} = V.$$

Since $U \sqcap V = (\sqcap_{i=1}^n U_{ki}) \sqcap (\sqcap_{j=1}^m V_{lj})$, we have

$$\beta(U \sqcap V) \geq (\bigwedge_{i=1}^n \beta_{ki}(U_{ki})) \wedge (\bigwedge_{i=1}^m \beta_{li}(V_{li})) \geq \beta(U) \wedge \beta(V) - \varepsilon.$$

(FQB2) Suppose that there exist $U \in \Theta_X$ and $r \in (0, 1)$ such that

$sup\{\beta(V) \mid V \circ V \leq U\} < r < \beta(U)$.

By the definition of β, there exists a finite index set $K = \{k_1, \ldots, k_n\} \subset \Gamma$ such that $\sqcap_{i=1}^n U_{ki} = U$ and

$$r < \bigwedge_{i=1}^n \beta_{ki}(U_{ki}) \leq \beta(U).$$

For each $k_i \in K$, since (X, β_{ki}) is a fuzzy (quasi-) uniform base, by (FQB2), we have

$sup\{\beta_{ki}(V) \mid V \circ V \leq U_{ki}\} \geq \beta_{ki}(U_{ki}) > r$.

Hence there exist $V_{ki} \in \Theta_X$ and $r_i \in (0,1)$ such that $V_{ki} \circ V_{ki} \leq U_{ki}$ and

$\beta_{ki}(V_{ki}) \geq r_i > r$.

Put $V = \sqcap_{i=1}^n V_{ki}$. For each $k_i \in K$, we have $V \circ V \leq V_{ki} \circ V_{ki}$.

From Lemma 1.1 (1) and (2), it follows

$V \circ V \leq \sqcap_{i=1}^{n} (V_{ki} \circ V_{ki})$
$\leq \sqcap_{i=1}^{n} U_{ki} \leq U.$

Then we have $V \circ V \leq U$ and
$\beta(V) \geq \bigwedge_{i=1}^{n} \beta_{ki}(V_{ki}) \geq \bigwedge_{i=1}^{n} r_i > r.$

Therefore $sup\{\beta(V) \mid V \circ V \leq U\} > r$. It is a contradiction. Hence for $U \in \Theta_X$,
$sup\{\beta(V) \mid V \circ V \leq U\} \geq \beta(U).$

(FQB3) Since (X, β_k) is a fuzzy (quasi-)uniform base, by (FQB3), there exists $U_k \in \Theta_X$ such that $\beta_k(U_k)=1$. For all finite index set $K=\{k_1,\ldots, k_n\} \subset \Gamma$, put $U = \sqcap_{i=1}^{n} U_{ki}$. Then there exists $U \in \Theta_X$ such that $\beta(U)=1$.

(FB) Suppose that there exist $U_k \in \Theta_X$ and $r \in (0, 1)$ such that
$sup\{\beta(V) \mid V \leq U^{-1}\} < r < \beta(U).$
By the definition of β, there exists a finite index set $K=\{k_1,\ldots,k_n\} \subset \Gamma$ such that $U = \sqcap_{i=1}^{n} U_{ki}$ and
$r < \bigwedge_{i=1}^{n} \beta_{ki}(U_{ki}) \leq \beta(U)$
For each $k_i \in K$, since (X, β_{ki}) is a fuzzy uniform base, by (FB), we have
$sup\{\beta_{ki}(V) \mid V \leq U_{ki}^{-1}\} \geq \beta_{ki}(U_{ki}) > r$.
Hence there exist $V_{ki} \in \Theta_X$ and $r_i \in (0,1]$ such that $V_{ki} \leq U_{ki}^{-1}$ and
$\beta_{ki}(V_{ki}) \geq r_i > r.$
Put $V = \sqcap_{i=1}^{n} V_{ki}$. For each $k_i \in K$, by Lemma 1.1 (1) and (7), we have
$V \leq \sqcap_{i=1}^{n} U_{ki}^{-1} = (\sqcap_{i=1}^{n} U_{ki})^{-1} = U^{-1}.$
Then we have $V \leq U^{-1}$ and
$\beta(V) \geq \bigwedge_{i=1}^{n} \beta_{ki}(V_{ki}) = \bigwedge_{i=1}^{n} r_i > r.$

Therefore $sup\{\beta(V) \mid V \leq U^{-1}\} > r$. It is a contradiction. Hence for $U \in \Theta_X$,
$sup\{\beta(V) \mid V^{-1} \leq U\} \geq \beta(U).$
Similarly, β' is a fuzzy (quasi-)uniform basis on X.

(2) First, we will show that for all $U \in \Omega_X$ and $i \in \Gamma$
$\Phi_i(U) \leq \Phi_\beta(U).$
Suppose there exist $U \in \Omega_X$ and $j \in \Gamma$ such that
$\Phi_j(U) > \Phi_\beta(U).$
Since β_j is a base of (X, Φ_j), there exists $V \in \Theta_X$ such that $V \leq U$ and
$\Phi_j(U) \geq \beta_j(V) > \Phi_\beta(U).$
On the other hand, from the definition of β, we have
$\Phi_\beta(U) \geq \Phi_\beta(V) \geq \beta(V) \geq \beta_j(V).$
It is a contradiction.
Second, if $\Phi_i(U) \leq \Phi^*(U)$ for all $U \in \Omega_X$, $i \in \Gamma$, we will show that
$\Phi_\beta(U) \leq \Phi^*(U).$
Suppose there exists $U \in \Omega_X$ such that
$\Phi_\beta(U) > \Phi^*(U).$

Then there exists $V \in \Theta_X$ such that $V \leq U$ and
$\Phi_\beta(U) \geq \beta(V) > \Phi^*(U).$
On the other hand, from the definition of β there exists a finite index set $K=\{k_1,\ldots,k_n\} \subset \Gamma$ such that $V = \sqcap_{i=1}^{n} U_{ki} \leq U$ and
$\beta(V) \geq \bigwedge_{i=1}^{n} \beta_{ki}(U_{ki}) > \Phi^*(U)$.

Since $\beta_{ki}(U_{ki}) \leq \Phi_{ki}(U_{ki}) \leq \Phi^*(U_{ki})$ for all $k_i \in K$, we have
$\bigwedge_{i=1}^{n} \beta_{ki}(U_{ki}) \leq \bigwedge_{i=1}^{n} \Phi^*(U_{ki})$
$\leq \Phi^*(\sqcap_{i=1}^{n} U_{ki})$ (by (FQU1))
$\leq \Phi^*(U).$
It is a contradiction.

(3) We will show that $\Phi_\beta(U) = \Phi_{\beta'}(U)$ for all it $U \in \Omega_X$.
Since $\beta_i(U_i) \leq \Phi_i(U_i)$, we have $\Phi_\beta \leq \Phi_{\beta'}$.
Suppose that $\Phi_\beta \not\geq \Phi_{\beta'}$. Then there exist $U \in \Omega_X$ and $r \in (0,1)$ such that
$\Phi_\beta(U) < r < \Phi_{\beta'}(U).$

From the definition of $\Phi_{\beta'}$, there exists a finite index set $K=\{k_1,\ldots,k_n\} \subset \Gamma$ such that $\sqcap_{i=1}^{n} U_{ki} \leq U$ and
$\bigwedge_{i=1}^{n} \Phi_{ki}(U_{ki}) > r.$

Since $\Phi_{ki}(U_{ki}) > r$ for all it $i=1,\ldots,n$, there exists $V_{ki} \in \Theta_X$ such that $V_{ki} \leq U_{ki}$ and
$\Phi_{ki}(U_{ki}) \geq \beta_{ki}(V_{ki}) > r.$

Hence $\Phi_\beta(U) \geq \bigwedge_{i=1}^{n} \beta_{ki}(V_{ki}) > r$. It is a contradiction.

Theorem 2.8. Let $\beta: \Theta_X \to I$ be a fuzzy quasi-uniform base on X. Put $\Theta_X^{-1} = \{U \mid U^{-1} \in \Theta_X\}$.
We define for $U \in \Theta^{-1}_X$,
$\beta^{-1}(U) = \beta(U^{-1}).$
Then:
(1) A structure β^{-1} is a fuzzy quasi-uniform base on X.
(2) The coarsest fuzzy quasi-uniform base β^* on X finer than β and β^{-1} is a fuzzy uniform base.

Proof. (1) It is easily proved from Lemma 1.1.
(2) From Theorem 2.7, there exists the coarsest fuzzy quasi-uniform base β^* on X finer than β and β^{-1} where
$\Theta_X^* = \{U_1 \sqcap U_2 \mid \beta(U_1) > 0, \beta^{-1}(U_2) > 0\}.$
We only show the condition (FU).

(FU) Suppose that there exist $U \in \Theta_X^*$ and $r \in (0, 1)$ such that

$sup\{\beta^*(V) \mid V \leq U^{-1}\} < r < \beta^*(U).$
By the definition of β^*, there exist U_1, U_2 such that

$U=U_1 \sqcap U_2$ and

$\beta^*(U) \geq \beta(U_1) \wedge \beta^{-1}(U_2) > r$.

Since $U=U_1 \sqcap U_2$ iff $U^{-1}=U_1^{-1} \sqcap U_2^{-1}$ from Lemma 1.1(7), we have

$\beta(U_1) \wedge \beta^{-1}(U_2) = \beta^{-1}(U_1^{-1}) \wedge \beta(U_2^{-1}) > r$.

Hence $sup\{\beta^*(V) \mid V \leq U^{-1}\} \geq \beta^{-1}(U_1^{-1}) \wedge \beta(U_2^{-1}) > r$. It is a contradiction.

Example 2. Let β be a fuzzy quasi-uniform base on X as follows:

$$\beta(U) = \begin{cases} 1 & \text{if} \quad U = U_{\tilde{1}}, \\ \frac{1}{2} & \text{if} \quad U = U_{\rho}. \end{cases}$$

We obtain a fuzzy quasi-uniform base β^{-1} as follows:

$$\beta^{-1}(U) = \begin{cases} 1 & \text{if} \quad U = U_{\tilde{1}}^{-1} = U_{\tilde{1}}, \\ \frac{1}{2} & \text{if} \quad U = U_{\rho}^{-1} = U_{\tilde{1}-\rho}. \end{cases}$$

From Theorem 2.8, since $\Theta_X^* = \{U_1 \sqcap U_2 \mid \beta(U_1) > 0, \beta^{-1}(U_2) > 0\}$.

by Lemma 2.6 and Lemma 1.1(2,5), we have

$\Theta_X^* = \{U_{\tilde{1}}, U_{\rho}, U_{\rho}^{-1}, U_{\rho} \sqcap U_{\rho}^{-1}\}$.

Therefore we have

$$\beta^*(U) = \begin{cases} 1 & \text{if} \quad U = U_{\tilde{1}}^{-1} = U_{\tilde{1}}, \\ \frac{1}{2} & \text{if} \quad U = U_{\rho}, U_{\rho}^{-1}, U_{\rho} \sqcap U_{\rho}^{-1}. \end{cases}$$

Hence we obtain fuzzy uniformity Φ_{β}^* on X as follows

$$\Phi_{\beta^*}(U) = \begin{cases} 1 & \text{if} \quad U = U_{\tilde{1}}. \\ \frac{1}{2} & \text{if } U_{\rho} \sqcap U_{\rho}^{-1} \leq U < U_{\tilde{1}}. \\ 0 & \text{otherwise}. \end{cases}$$

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