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Testing Homogeneity of Diagonal Covariance Matrices of K Multivariate Normal Populations

Hea-Jung Kim[1)]

Abstract

We propose a criterion for testing homogeneity of diagonal covariance matrices of K multivariate normal populations. It is based on a factorization of usual likelihood ratio, intended to propose and develop a criterion that makes use of properties of structures of the diagonal covariance matrices. The criterion then leads to a simple test as well as to an accurate asymptotic distribution of the test statistic via general result by Box (1949).

Key Words : Diagonal Covariance Matrices; Homogeneity Test; Modified Likelihood Ratio; Box's Approximation.

1. Introduction

Suppose that $X_1(i), \ldots, X_{N_i}(i)$ denote N_i independent p-variate observations from $N_p(\theta_i, \Sigma_i)$, $i=1,\ldots,K$. Then the general multivariate linear model for the observations is

$$EX_j(i) = z_{j1}(i)\beta_1 + z_{j2}(i)\beta_2 + \cdots + z_{jq}(i)\beta_q, \quad j=1,\ldots,N_i, \quad i=1,\ldots,K, \tag{1.1}$$

where the $z_{i\ell}(i)$'s are known constants and β_ℓ's are unknown p-component parameter vectors. Let assume $\Sigma_1 = \cdots = \Sigma_K = \Sigma$ and let a $p\times q$ matrix $\boldsymbol{B}' = (\beta_1, \beta_2, \ldots, \beta_q)$, a $p\times N$ matrix $\boldsymbol{Y}' = (X_1(1), X_2(1), \ldots, X_{N_K}(K))$ and a $N\times q$ matrix $\boldsymbol{Z} = \{z_{j\ell}(i)\}$. Then the model (1.1) can be written as

$$E\boldsymbol{Y} = \boldsymbol{Z}\boldsymbol{B}, \quad \text{where} \quad V(\boldsymbol{Y}) = I_N \otimes \Sigma, \tag{1.2}$$

where $N = \sum_{i=1}^{K} N_i$ and the symbol $\otimes$ represents the direct Kronecker product of two matrices. The combination of the formulas in (1.2) are referred to as the multivariate Gauss-Markov setup. The model (1.2) has broad application, especially in the life and social sciences, and hence bears inference problems due to various patterns of the correlation among response variables(see

1) Department of Statistics, Dongguk University, Seoul, 100-715, Korea

e.g. Kshirsagar and Smith 1995, Krzanowski and Marriott 1995, Weissfield and Kshirsagar 1992 and Timm 1980).

On the other hand, if a hypothesis $H: \Sigma_1 = \cdots = \Sigma_K = D$, a diagonal matrix, is true, we no longer need the general multivariate model to analyze the p-dimensional data. Instead we need K univariate linear models

$$EY_i = ZB_i, \quad \text{where} \quad V(Y_i) = \sigma_i^2 I_N, \quad i = 1, \ldots, K, \tag{1.3}$$

where Y_i and B_i are respective ith columns of $\boldsymbol{Y}$ and $\boldsymbol{B}$, and σ_i^2 represents ith diagonal element of the diagonal matrix D. Thus the test of H is necessary for certain procedures in MANOVA, repeated measures method, MDA(multiple discriminant analysis), multivariate regression analysis and so on where linear models are designed to take into account the presence of p correlated response variables (cf. Krzanowski and Marriott 1995). In other words, when H is true, various problems involved in analyzing the multivariate model (1.1) disappear, because we can simply adopt well defined univariate linear models analyzing each response variable separately.

A criterion for testing the hypothesis of $\Sigma_1 = \cdots = \Sigma_K$ has been obtained by Box(1949), see Rencher(1995) for an example. However an accurate criterion for testing H has not been seen yet. This is due to the complex distributional theory involved in the likelihood ratio for H. Although usual Wilks's approximation(1946) is available for the test statistic of H, however in small samples, it may tend to have the actual significance level greater than the nominal significance level (cf. Greenstreet and Conner 1974). Present paper considers an alternative criterion for testing H. Section 2 shows that by means of the Lemma 10.3.1 of Anderson (1984), an exact likelihood ratio criterion for testing H can be derived. Then Section 3 suggests a modified likelihood ratio criterion that leads to a simple test as well as to accurate asymptotic distribution of the test statistic. Finally, some concluding remarks are given in Section 4.

2. Test Criterion

Let $X_j(i)$ denote jth independent observations from ith population $N_p(\theta_i, \Sigma_i)$, $i = 1, \ldots, K$, and $j = 1, \ldots, N_i$. If it is desired to test the hypothesis $H: \Sigma_1 = \cdots = \Sigma_K = D$ based upon $X_j(i)'$ s, we cannot use the standard Box M-test (cf. Box 1949, Anderson 1984). However the following theorem enables us to derive a test criterion for H.

Lemma 2.1 (Anderson 1984). Let $\boldsymbol{Z}$ be an observation vector on a random vector with density $f(Y, \delta, \psi)$, where δ and ψ are vectors of variances and means in spaces Ω and Ψ. Let H_a be the hypothesis $\delta \in \Omega_a \subset \Omega$, let H_β be the hypothesis $\delta \in \Omega_b \subset \Omega_a$, given $\delta \in \Omega_a$. Then the likelihood ratio of the hypothesis, $H_{a\beta}$, that $\delta \in \Omega_b$, given $\delta \in \Omega$ is

$$\lambda_{\alpha\beta}=\frac{\max_{\delta\in\Omega_b,\psi\in\Psi} f(z,\delta,\psi)}{\max_{\delta\in\Omega,\psi\in\Psi} f(z,\delta,\psi)}=\lambda_\alpha\times\lambda_\beta,$$

where

$$\lambda_\alpha=\left(\frac{\max_{\delta\in\Omega_a,\psi\in\Psi} f(z,\delta,\psi)}{\max_{\delta\in\Omega,\psi\in\Psi} f(z,\delta,\psi)}\right) \text{ and } \lambda_\beta=\left(\frac{\max_{\delta\in\Omega_b,\psi\in\Psi} f(z,\delta,\psi)}{\max_{\delta\in\Omega_a,\psi\in\Psi} f(z,\delta,\psi)}\right).$$

Since the hypothesis H is as a combination of the hypothesis H_A : Covariance matrix of $X(i)$ is diagonal or the components of $X(i)$ are independent for $i=1,\dots,K$, and H_B: $D_1=\cdots=D_K$, i.e. $X(i)$'s have common diagonal covariance matrix or each component of $X(i)$ have common variance for $i=1,\dots,K$. Thus by Lemma 2.1, the likelihood ratio criterion λ for H is the product of the criterion λ_{H_A} for H_A and λ_{H_B} for H_B. We can easily see that the criterion for H_A is

$$\lambda_{H_A}=\prod_i^K\frac{|V(i)|^{N_i/2}}{\prod_{j=1}^{p} v_{\ell\ell}(i)^{N_i/2}}, \tag{2.1}$$

where $V(i)=\sum_{j=1}^{N_i}(X_j(i)-\overline{X}(i))(X_j(i)-\overline{X}(i))'=\{v_{\ell\ell'}(i)\}$.

Likelihood ratio criterion for H_B given H_A is true can be viewed as a combination of the hypothesis H_1: The first diagonal elements of D_i's are equal, H_2: The second diagonal elements are equal given H_1 is true, up to H_p: The pth diagonal elements are equal given other hypotheses are true. Let $X_j(i)=(X_{1j}(i),\dots,X_{pj}(i))'$ and let the mean vector $\theta_i=(\mu_1(i),\dots,\mu_p(i))'$, $i=1,\dots,K$, $j=1,\dots,N_i$, so that $X_{1j}(i)$ be an observation from the ith univariate normal population $N(\mu_1(i),\sigma_1^2(i))$. We wish to test the hypothesis

$$H_1: \sigma_1^2(1)=\cdots=\sigma_1^2(K).$$

which is equivalent to the hypothesis of equality of the first diagonal elements in Σ_i's.

For the several sample case, various procedures have been proposed(cf. Rencher 1995). The likelihood ratio criterion obtained from the marginal distributions of $X_{1j}(i)'$ s is

$$\lambda_1=\frac{\prod_{i=1}^K v_{11}(i)^{N_i/2}}{v_{11}^{N/2}}\frac{N^{N/2}}{\prod_{i=1}^K N_i^{N_i/2}}, \tag{2.2}$$

where $v_{11}(1),\dots,v_{11}(K)$ are independent mean corrected sum of squares of $X_{1j}(i)'$ s, $v_{11}=\sum_{i=1}^K v_{11}(i)$ and $N=\sum_{i=1}^K N_i$.

Similarly, we consider testing the equality of variances of $X_{\ell j}(i)$'s, $\ell=2,\dots p$, when we assume the variances of $X_{\ell' j}(i)$'s are the same, $\ell'\le\ell$; that is, we test

$$H_\ell:\ \sigma^2_\ell(1)=\cdots=\sigma^2_\ell(K)\ \text{ given }\ \sigma^2_{\ell'}(1)=\cdots=\sigma^2_{\ell'}(K),\quad \ell'=1,\ldots,\ell-1.$$

We can see that the hypothesis is equivalent to that of homogeneity in the ℓ's diagonal elements of Σ_i's. From the joint distribution of $X_{\ell j}(i)$'s, we can easily derive that the likelihood ratio criterion for testing H_ℓ that is

$$\lambda_\ell=\frac{\prod_{i=1}^{K} v_{\ell\ell}(i)^{N_i/2}}{v_{\ell\ell}^{N/2}}\ \frac{N^{N/2}}{\prod_{i=1}^{K} N_i^{N_i/2}}, \tag{2.3}$$

where $v_{\ell\ell}(1),\ldots,v_{\ell\ell}(K)$ are independent mean corrected sum of squares of $X_{\ell j}(i)'$ s and $v_{\ell\ell}=\sum_{i=1}^{K} v_{\ell\ell}(i)$.

Lemma 2.1 notes that the likelihood criterion for the hypothesis H_B is the product of the likelihood criteria for H_1 through H_p,

$$\lambda_{H_B}=\prod_{\ell=1}^{p}\lambda_\ell=\left(\frac{N^{N/2}}{\prod_{i=1}^{K} N_i^{N_i/2}}\right)^p\prod_{\ell=1}^{p}\left(\frac{\prod_{i=1}^{K} v_{\ell\ell}(i)^{N_i/2}}{v_{\ell\ell}^{N/2}}\right). \tag{2.4}$$

Similarly, the likelihood criterion λ_H for H is the product of the criterion λ_{H_A} for H_A and λ_{H_B} for H_B. Thus the criterion for H is

$$\lambda_H=\lambda_{H_A}\lambda_{H_B}=\left(\frac{N^{N/2}}{\prod_{i=1}^{K} N_i^{N_i/2}}\right)^p\left(\frac{\prod_{i=1}^{K}|V(i)|^{N_i/2}}{\prod_{\ell=1}^{p} v_{\ell\ell}^{N/2}}\right). \tag{2.5}$$

It can be easily seen that λ_{H_A}, λ_{H_B} and λ_H are invariant with respect to change of location within populations and a common linear transformation

$$Y_j(i)=CX_j(i)+\psi(i),\quad i=1,\ldots,K,\quad j=1,\ldots,N_i. \tag{2.6}$$

where C is a singular.

3. Distribution of the Criterion

In this section we shall consider distribution of the test criterion. The criterion considered is the likelihood ratio criterion λ_H modified for unbiased estimates. To distinguish it from λ_H, it is asterisked:

$$\lambda^*_H=\lambda^*_{H_A}\ \lambda^*_{H_B}, \tag{3.1}$$

where

$$\lambda^*_{H_A}=\prod_i^K\frac{|V(i)|^{\nu_i/2}}{\prod_{j=1}^p v_{\ell\ell}(i)^{\nu_i/2}},\quad \lambda^*_{H_B}=\prod_{\ell=1}^p\lambda_\ell=\left(\frac{\nu^{\nu/2}}{\prod_{i=1}^K\nu_i^{\nu_i/2}}\right)^p\prod_{\ell=1}^p\left(\frac{\prod_{i=1}^K v_{\ell\ell}(i)^{\nu_i/2}}{v_{\ell\ell}^{\nu/2}}\right),$$

$\nu_i=N_i-1$ and $\nu=\sum_{i=1}^K\nu_i$.

Bartlett(1937) gave an intuitive argument for the use of the modified criterion in place of the likelihood ratio criterion. Perlman(1980) has shown that the tests based on the modified criteria are unbiased.

Lemma 3.1 When the hypothesis H_A is true, hth moment of $\lambda^*_{H_A}$ is

$$E\lambda^{*h}_{H_A}=K_1\frac{\prod_{i=1}^K\prod_{\ell=1}^p\Gamma\{\nu_i(1+h)/2+(1-\ell)/2\}}{\prod_{i=1}^K\Gamma^p\{\nu_i(1+h)/2\}},\tag{3.2}$$

where $K_1=\prod_{i=1}^K\Gamma\{\nu_i/2\}^2/\prod_{i=1}^K\prod_{\ell=1}^p\Gamma\{(\nu_i+1-\ell)/2\}$.

Proof. Under the hypothesis, $Z_i=|V(i)|/\prod_{\ell=1}^p v_{\ell\ell}(i)$ is distributed as $\prod_{\ell=2}^p X_\ell$ where the X_ℓ's are independent and X_ℓ has beta distribution with parameters $(\nu_i-\ell+1)/2$ and $(\ell-1)/2$, i.e. $Z_i\sim\beta((\nu_i-\ell+1)/2,(\ell-1)/2)$ (cf. Anderson 1984, p.383), where $\prod_{i=1}^K Z_i^{\nu_i/2}=\lambda^*_{H_A}$. Moments of Z_i are

$$E[Z_i^{\delta_i}]=\prod_{\ell=2}^p\left(\frac{\Gamma\{(\nu_i-\ell+1)/2+\delta_i\}}{\Gamma\{(\nu_i-\ell+1)/2\}}\frac{\Gamma\{\nu_i/2\}}{\Gamma\{\nu_i/2+\delta_i\}}\right)$$
$$=\frac{\Gamma^p\{\nu_i/2\}}{\Gamma^p\{\nu_i/2+\delta_i\}}\prod_{\ell=1}^p\frac{\Gamma\{(\nu_i+1-\ell)/2+\delta_i\}}{\Gamma^2\{(\nu_i+1-\ell)/2\}},\quad \delta=0,1,\cdots.$$

Since $Z_1,\ldots,Z_K$ are independent, we have $E\lambda^{*h}_{H_1}$ by $\prod_{i=1}^K EZ_i^{\delta_i}$, where $\delta=h\nu_i/2$.
We now find the moments of $\lambda^*_{H_B}$. Let

$$U_\ell=\left(\frac{\prod_{i=1}^K v_{\ell\ell}(i)^{\nu_i/2}}{v_{\ell\ell}^{\nu/2}}\right),\quad \ell=1,\ldots,p$$

and let $W=\prod_{\ell=1}^p U_\ell$, then the moments of W determine the distributions of W, and hence $\lambda^*_{H_B}$ uniquely, for $0\le W\le 1$.

Lemma 3.2 For U_ℓ's, $\ell = 1,\dots,p$, their moments are equal and

$$EU_\ell = \left(\frac{\Gamma\{\nu/2\} \prod_{i=1}^{K} \Gamma\{\nu_i(1+h)/2\}}{\Gamma\{\nu(1+h)/2\} \prod_{i=1}^{K} \Gamma\{\nu_i/2\}} \right), \quad h=0,1,\dots. \tag{3.3}$$

Proof. Given H_A is true, $U_1,\dots,U_p$ are independent. To derive the distributions of U_ℓ's, we make use of Theorem 10.4.1 and Theorem 10.4.2 in Anderson(1984). First let us consider U_1. If we set

$$U_{1j} = \frac{(v_{11}(1)+\cdots+v_{11}(j-1))^{(\nu_1+\cdots+\nu_{j-1})/2}\, v_{11}(j)^{\nu_j/2}}{(v_{11}(1)+\cdots+v_{11}(j))^{(\nu_1+\cdots+\nu_j/2)}}, \quad j=2,\dots,K,$$

then $U_1 = \prod_{j=2}^{K} U_{1j}$ and U_{1j}'s are independently distributed as

$$U_{1j} = X_{1j}^{(\nu_1+\cdots+\nu_{j-1})/2}(1-X_{1j})^{\nu_j/2},$$

where X_{1j} has the beta distribution with parameters $(\nu_1+\cdots+\nu_{j-1})/2$ and $\nu_j/2$, i.e. $\beta((\nu_1+\cdots+\nu_{j-1})/2,\ \nu_j/2)$, $j=2,\dots,K$. This gives

$$\begin{aligned} EU_1^h &= E\left[\prod_{j=2}^{K} X_{1j}^{h(\nu_1+\cdots+\nu_{j-1})/2}(1-X_{1j})^{h\nu_j/2} \right] \\ &= \prod_{j=2}^{K} \left(\frac{\Gamma\{(1+h)(\nu_1+\cdots+\nu_{j-1})/2\}}{\Gamma\{(\nu_1+\cdots+\nu_{j-1})/2\}} \frac{\Gamma\{(\nu_1+\cdots+\nu_j)/2\}}{\Gamma\{(1+h)(\nu_1+\cdots+\nu_j)/2\}} \right. \\ &\quad \times \left. \frac{\Gamma\{(1+h)\nu_j/2\}}{\Gamma\{\nu_j\}} \right). \end{aligned}$$

Simplifying the last term gives the result. Exactly the same derivation in the above applies for obtaining EU_2^h through EU_p^h.

Theorem 3.1 The hth moment of the criterion λ^*_H for H is

$$E\lambda_H^{*h} = \Delta \left(\frac{\nu^{\nu/2}}{\prod_{i=1}^{K} \nu_i^{\nu_i/2}} \right)^{ph} \frac{\prod_{i=1}^{K} \prod_{\ell=1}^{p} \Gamma\{\nu_i(1+h)/2 + (1-\ell)/2\}}{\Gamma^p\{\nu(1+h)/2\}}, \tag{3.4}$$

where

$$\Delta = \frac{\Gamma^p\{\nu/2\}}{\prod_{\ell=1}^{p} \Gamma\{(\nu_i+1-\ell)/2\}}.$$

Proof. The distribution of the likelihood criterion under the null hypothesis can be characterized by the independence of $\lambda^*_{H_A}$ and $\lambda^*_{H_B}$ and their respective moments. It is easy to show that if H_A is true the sample correlation coefficients $\{r_{\ell,\ell'}(i) = v_{\ell\ell'}(i)/(v_{\ell\ell}(i)v_{\ell'\ell'}(i))^{1/2},\ \ell \neq \ell'\}$, are distributed independently of the sample variances $\{v_{\ell\ell}(i)/\nu_i\}$ (cf. Anderson 1984, p.267). Since $\lambda^*_{H_A}$ and $\lambda^*_{H_B}$ in (3.1) depend only on $\{r_{\ell,\ell'}(i)\}$ and $\{v_{\ell\ell}(i)\}$ respectively, we see that they are independently distributed when H is true. This gives $E\lambda^{*h}_H = E\lambda^{*h}_{H_A}\ E\lambda^{*h}_{H_B}$, where

$$E\lambda^{*h}_{H_B} = \left(\frac{\nu^{\nu/2}}{\prod_{i=1}^{K} \nu_i^{\nu_i/2}} \right)^{hp} \prod_{\ell=1}^{p} EU^h_\ell .$$

Thus the result is obtained by Lemma 3.1 and Lemma 3.2.

Box (1949) suggested a general asymptotic expansion of the distribution of a random variable whose moments are certain functions of gamma functions. By use of the expansion, we derive the distribution of λ^*_H for testing H as follows.

Theorem 3.2. A close approximation to the distribution of λ^{*H} under H is given by

$$P\{-2\rho\log\lambda^*_H \le t\} = P\{\chi^2(f) \le t\} + \omega_2[P\{\chi^2(f+4)\le t\} - P\{\chi^2(f)\le t\}] + O(\nu^{-3}), \qquad (3.5)$$

where $f = p(Kp+K-2)/2$,

$$\rho = 1 - \frac{2}{3(pK+K-2)}\left(\frac{2p^2+3p-1}{4} \sum_{i=1}^{K} \frac{1}{\nu_i} - \frac{1}{\nu} \right), \text{ and}$$

$$\omega_2 = \frac{1}{6\rho^2}\left\{ \frac{3p(Kp+K-2)}{4}(1-\rho)^2 - (1-\rho)p\left(\frac{2p^2+3p-1}{4} \sum_{i=1}^{K} \frac{1}{\nu_i} - \frac{1}{\nu} \right) + \frac{(p-1)p(p+1)(p+2)}{8} \sum_{i=1}^{K} \frac{1}{\nu_i^2} \right\}.$$

Proof. Set

$b = p, y_j = \nu/2,\ \eta_j = 0\ ;\ j = 1,\ldots,p,$

$a = pK, x_k = \nu_i/2,\ k = p(i-1)+1,\ldots,p_i,\ i = 1,\ldots,K,$

$\delta_k = (1-\ell)/2,\ k = \ell, 2+\ell,\ldots,2(K-1)+\ell,\ \ell = 1,\ldots,p.$

Then the moment of λ^{*h}_H can be expressed as

$$E\lambda^{*h_H} = \Delta^* \left(\frac{\prod_{j=1}^{b}(y_j)^{y_j}}{\prod_{k=1}^{a}(x_k)^{x_k}} \right)^h \frac{\prod_{k=1}^{a}\Gamma\{x_k(1+h)+\delta_k\}}{\prod_{j=1}^{b}\Gamma\{y_j(1+h)+\eta_j\}}, \quad h = 0,1,2,\ldots,$$

where $\Delta^* = \prod_{j=1}^{b}\Gamma\{y_j+\eta_j\}/\prod_{k=1}^{a}\Gamma\{x_k+\delta_k\}$. Since $\sum_{k=1}^{a} x_k = \sum_{j=1}^{b} y_\ell = p\nu/2$ and $E\lambda^{*0}_H = 1$, the random variable λ_H^{*h}, whose moments are certain functions of gamma functions, satisfies the conditions for Box's(1949) theorem of a general asymptotic expansion of the random variable(see e.g. Anderson(1984, p.311)). Applying Box's theorem and taking a second order approximation to the distribution of $M=-2\rho\log\lambda_H^{*h}$, we have the result.

4. Concluding Remarks

In this paper, we have suggested a modified likelihood ratio criterion for testing H that covariance matrices of K multivariate normal populations are equal and diagonal. This is obtained by observing that H is a combination of the hypothesis H_A: Σ_i's are diagonal or the components of $X_j(i)'$ s are independent and H_B: The diagonal elements of Σ_i's are equal given Σ_i's are diagonal or the variances of a component of $X_j(i)$'s are equal across the K populations given the components are independent. The likelihood ratio criterion leads to a simple test as well as to accurate asymptotic distribution of the test statistic via the general result by Box(1949). When H is true, the analytical ease with which result can be obtained using the test criterion makes it attractive for use in analyzing the multivariate linear models. For example, in the one-way MANOVA problem, if the suggest test accepts H, it will be straightforward to analyze treatment effects simply under a set of one-way ANOVA models.

While the suggested test criterion is useful, it is recognized that criteria for other types of hypotheses, for example, a criterion for testing that K covariance matrices are equal and intraclass, are worthy to study. Further study pertaining to deriving the test criteria would be useful and is left for future research.

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