A Simple Geometric Approach to Evaluating a Bivariate Normal Orthant Probability

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Abstract

We present a simple geometric approach, which uses polar transformation and elementary trigonometry, to evaluating an orthant probability in a bivariate normal distribution. Figures are provided to illustrate the situation for varying correlation coefficient. We derive the distribution of the sample correlation coefficient from a bivariate normal distribution when the sample size is 2 as an application.

Keywords: Polar Coordinates, Correlation Coefficient

1. Main Results

Consider a bivariate normal random variable with correlation coefficent ρ. We want to compute the probability that both coordinates are greater than or equal to their corresponding means, which is called orthant probability.

Kepner, Harpner, and Keith (1989) presents an approach based on integral calculus combined with polar transformation. In reply to their approach Stigler (1989) gives an alternative non-integral approach based on distributional properties of normal random variables which lacks geometrical interpretation. Farebrother (1989) also gives another non-integral approach which uses reverse Cholesky Decomposition which requires somewhat higher mathematical background. Further references including the early work by Sheppard can be found in D. B. Owen's article in the *Encyclopedia of Statistical Sciences*.

Figure 1: Case of positive correlation. An orthant probability in a bivariate normal distribution expressed through polar coordinates. A solution to the equation $\tan\theta = \eta(\rho)$, where $\eta(\rho) = -\rho(1-\rho^2)^{1/2}$, is indicated by $s = -\sin^{-1}\rho$. Note the $\sin(-\theta)$ curve. The desired probability is $l/(2\pi)$.
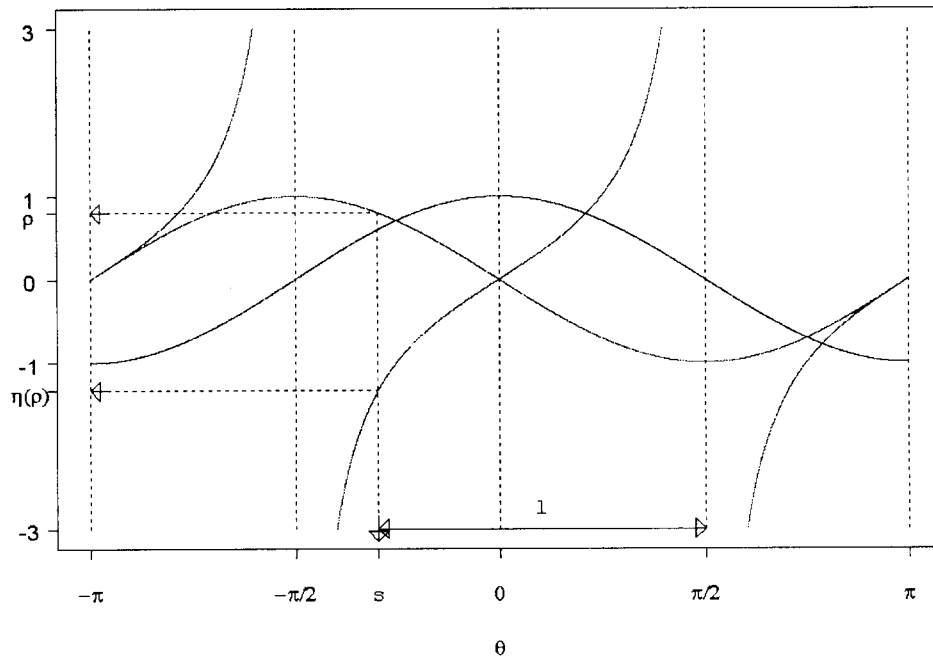
The aforementioned approaches in general require either higher mathematical background or lack geometrical interpretations. In this article, we present a simpler geometric approach based on polar coordinates and elementary trigonometry.

Without loss of generality, the problem reduces to evaluating

$$\Pr(X \geq 0, \ Y \geq 0),$$

where X and Y have zero means and unit variances with the same correlation coefficient ρ. Y may be written as $\rho X + (1 - \rho^2)^{1/2} Z$, where Z is another standard normal random variable independent of X. We can represent X and Z in polar coordinates as

$$\begin{cases} X = R\cos\Theta \\ Y = R\sin\Theta, \end{cases}$$

Where R^2 has a chi-square distribution with two degrees of freedom and Θ has a

Figure 2: Case of negative correlation. The solution to the equation $\tan\theta = -\eta(\rho)$ remains $s = -\sin^{-1}\rho$, and the desired probability is now $l'/(2\pi)$. We used the same correlation coefficient as in Figure 1 but with the opposite sign. Note that $l + l' = \pi$.
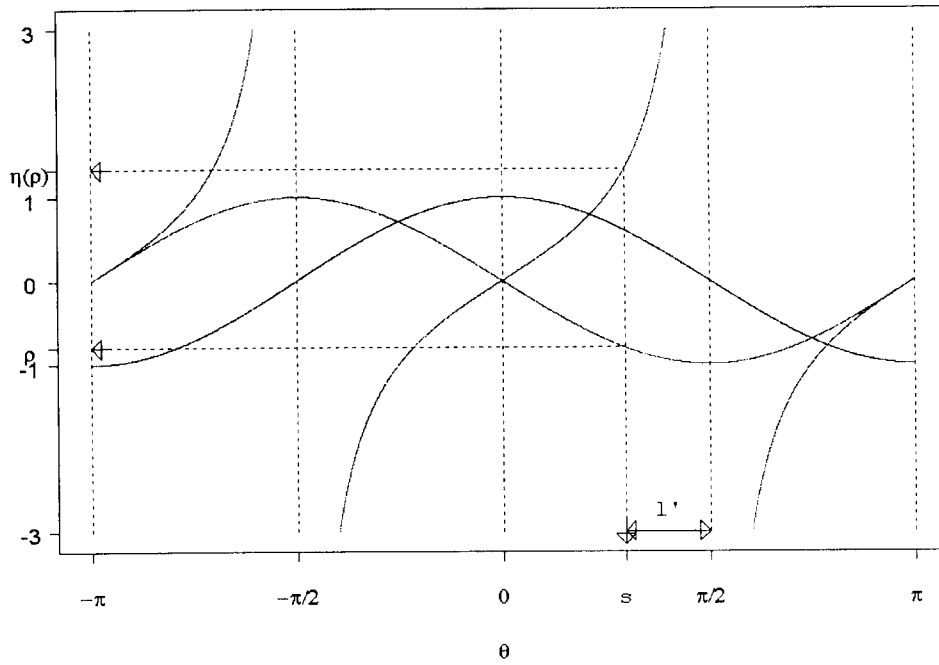

Uniform distribution over the interval $-\pi$ to π, with R^2 and Θ independent. Note that the negative correlation case can be obtained by substituting $-Y$ for Y in the positive correlation case. Then, we have

$$\Pr(X \geq 0, Y \leq 0) = 1/2 - \Pr(X \geq 0, Y \geq 0).$$

Let us first assume that ρ is positive. The desired probability is th

$$\Pr\{X \geq 0, \rho X + (1 - \rho^2)^{1/2}, Z \geq 0\}$$

$$= \Pr\{\cos\Theta \geq 0, \rho\cos\Theta + (1 - \rho^2)^{1/2}\sin\Theta \geq 0\}$$

$$= \Pr\{\cos\Theta \geq 0, \tan\Theta \geq -\rho(1 - \rho^2)^{1/2}\}.$$

Figure 3: Orthant probabilities for $\rho = \sin(\pi/3) = 0.8660$, $\sin(\pi/4) = 0.7071$, $\sin(\pi/6) = 0.5$, $-\sin(\pi/6) = -0.5$, $-\sin(\pi/4) = -0.7071$, $-\sin(\pi/3) = -0.8660$.
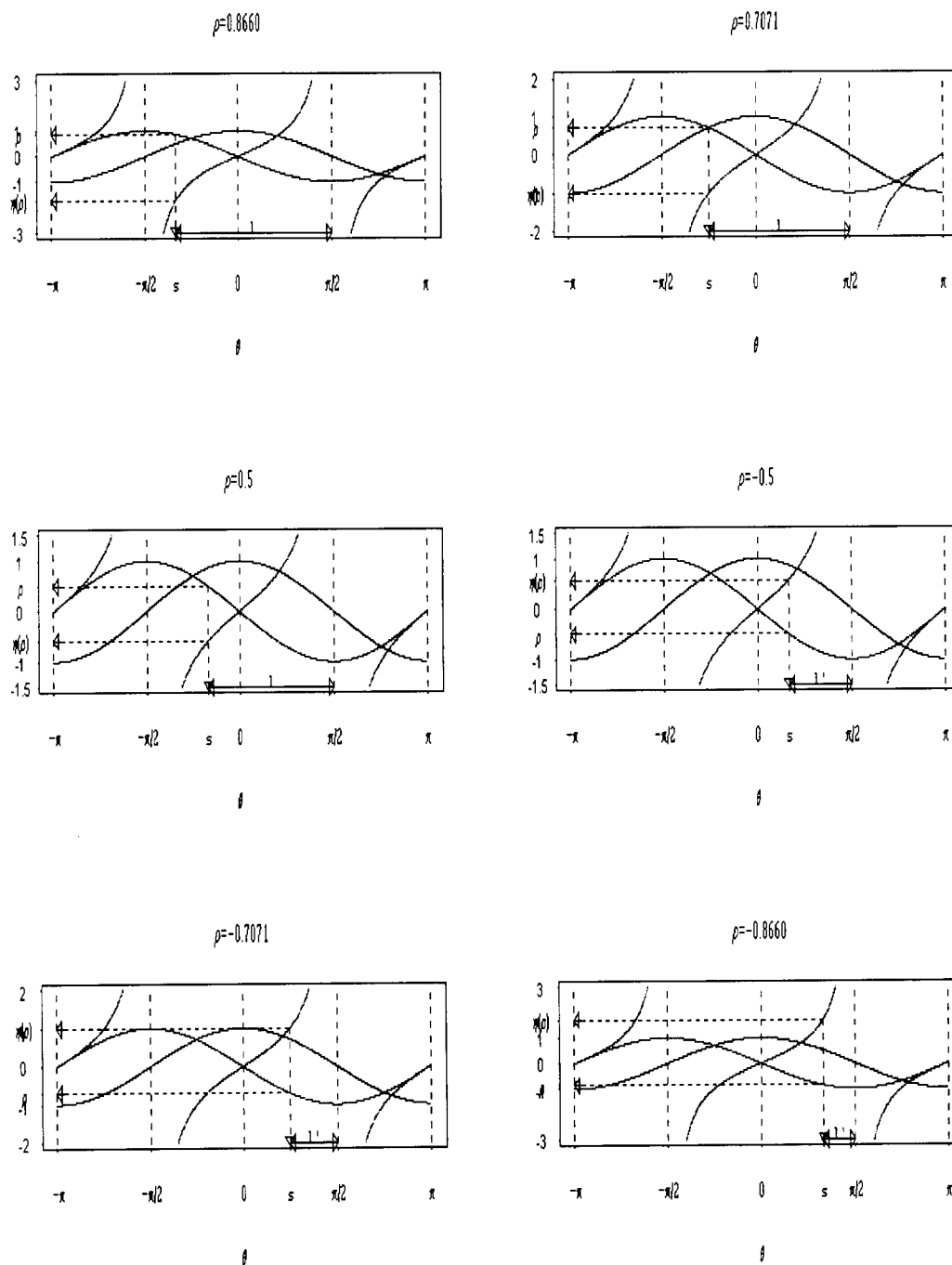
Note that $\cos\theta \geq 0$ holds on $-\pi/2 \leq \theta \leq \pi/2$. On this interval, a solution to the equation $\tan\theta = -\rho(1-\rho^2)^{1/2}$ is given by $s = -\sin^{-1}\rho$. From Figure 1, the desired probability reduces to

$$\Pr(-\sin^{-1}\rho \leq \Theta \leq \pi/2) = (2\pi)^{-1}(\pi/2 + \sin^{-1}\rho)$$

$$= 1/4 + (2\pi)^{-1}\sin^{-1}\rho.$$

Figure 2 depicts the case of negative correlation, and Figure 3 gives six diagrams for various values of ρ.

2. An Application

Let (X_1, Y_1) and (X_2, Y_2) be independent and identically distributed bivariate normal random vectors with correlation coefficient ρ. Then, the sample correlation coefficient r is given by

$$r = \frac{(X_1 - X_2)(Y_1 - Y_2)}{|(X_1 - X_2)(Y_1 - Y_2)|}$$

$$= \begin{cases} -1 & \text{if } (X_1 - X_2)(Y_1 - Y_2) < 0, \\ 1 & \text{if } (X_1 - X_2)(Y_1 - Y_2) > 0 \end{cases}$$

Note that the joint distribution of $(X_1 - X_2)/\sqrt{2}$ and $(Y_1 - Y_2)/\sqrt{2}$ is the same as that of X and Y in the previous section. Note also that the joint distribution of (X, Y) and that of $(-X, -Y)$ are the same. Therefore,

$$\Pr(r=1) = \Pr(X<0, Y<0) + \Pr(X>0, Y>0)$$

$$= 1/2 + \pi^{-1}\sin^{-1}\rho.$$

Hence, $\Pr(r=-1) = 1/2 - \pi^{-1}\sin^{-1}\rho$. The mean and variance of r are given by $2\pi^{-1}\sin^{-1}\rho$ and $1 - (2\pi^{-1}\sin^{-1}\rho)^2$, respectively. For early work on the distribution of the sample correlation coefficient, see Fisher (1915).

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