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Almost Sure Convergence of Randomly Weighted Sums with Application to the Efron Bootstrap[1]

Tae-Sung Kim[2], Hyuk Joo Kim[3], Eun-Yang Seok[4]

Abstract

Let $\{X_{nj}, 1 \le j \le n, j \ge 1\}$ be a triangular array of random variables which are neither independent nor identically distributed. The almost sure convergences of randomly weighted partial sums of the form $\sum_{j=1}^{n} W_{nj} X_{nj}$ are studied, where $\{W_{nj}, 1 \le j \le n, n \ge 1\}$ is a triangular array of random weights. Application regarding the Efron bootstrap is also introduced.

1. Introduction

Let $\{X_n : n \ge 1\}$ be a sequence of independent random variables and $\{w_{nj}, 1 \le j \le n, n \ge 1\}$ a triangular array of numbers. The almost sure convergence of the partial sums $\sum_{j=1}^{n} w_{nj} X_j$ were extensively investigated by Chow(1966), Stout(1968), Rohatgi(1971), Chow and Lai(1973), Adler and Rosalsky(1987), Yu(1990) and Cuzick(1995) among others. For a triangular array of random weights $\{W_{nj}, 1 \le j \le n, n \ge 1\}$ the almost sure convergence of the randomly weighted partial sums $\sum_{j=1}^{n} W_{nj} X_j$ was studied by Buldygin and Solntev(1986), Cuzick(1995), and Arenal-Gutiérrez, Matrán and Cuesta-Albertors(1996).

In this paper, for a triangular array $\{W_{nj}, 1 \le j \le n, n \ge 1\}$ of random weights and a triangular array $\{X_{nj}, 1 \le j \le n, n \ge 1\}$ of random variables which are neither independent nor identically distributed, the almost sure convergences of the randomly weighted partial sums of

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2) Professor Department of Statistics WonKwang University Iksan, Chonbuk 570-749 Korea

3) Associate Professor Department of Statistics WonKwang University Iksan, Chonbuk 570-749 Korea

4) Lecturer Department of Statistics WonKwang University Iksan, Chonbuk 570-749 Korea

the form $\sum_{j=1}^{n} W_{nj}X_{nj}$ are derived.

In section 2, conditions are imposed on the triangular array of random weights $\{W_{nj}, 1\le j\le n, n\ge 1\}$ and on the triangular array of random variables $\{X_{nj}, 1\le j\le n, n\ge 1\}$ which ensure the almost sure convergence of $\sum_{j=1}^{n} W_{nj}X_{nj}$. Applications regarding the Efron bootstrap are also discussed in section 3.

2. Results

Let $\{X_{nj}, 1\le j\le n, n\ge 1\}$ be a triangular array of random variables which are neither independent nor identically distributed. In this section, we study the almost sure convergence of the form of $\sum_{j=1}^{n} W_{nj}X_{nj}$.

Theorem 2.1. Let $\{X_{nj}, 1\le j\le n, n\ge 1\}$ be a triangular array of positive random variables such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} X_{nj} = B \quad a.s. \tag{2.1}$$

where B is a constant or an almost sure finite random variable, and let $\{W_{nj}, 1\le j\le n, n\ge 1\}$ be a triangular array of random variables such that for all $\varepsilon > 0$

$$P\left\{ \lim_{n\to\infty} \sup \bigcup_{j=1}^{n} \left[\left| W_{nj} - \frac{1}{n} \right| \ge \frac{\varepsilon}{n} \right] \right\} = 0. \tag{2.2}$$

Then $\sum_{j=1}^{n} W_{nj}X_{nj}$ converges almost surely to the almost sure limit of $\frac{1}{n}\sum_{j=1}^{n} X_{nj}$.

Proof. From (2.2) we obtain that for all $\varepsilon > 0$

$$P\left(\left\{ \bigcup_{j=1}^{n} \left| W_{nj} - \frac{1}{n} \right| \ge \frac{\varepsilon}{n} \right\} \ i.o. \ (n) \right) = 0 \tag{2.3}$$

and thus in view of (2.3), for arbitrary $\varepsilon > 0$, in a set of probability one there exists a positive integer n_0 such that for all $n \ge n_0$

$$\left| \sum_{j=1}^{n} W_{nj}X_{nj} - \frac{1}{n} \sum_{j=1}^{n} X_{nj} \right| \le \sum_{j=1}^{n} \left| W_{nj} - \frac{1}{n} \right| X_{nj} \le \frac{\varepsilon}{n} \sum_{j=1}^{n} X_{nj} \tag{2.4}$$

which converges almost surely recalling (2.1). Since $\varepsilon > 0$ is arbitrary it follows that

$$\sum_{j=1}^{n} W_{nj}X_{nj} - \frac{1}{n} \sum_{j=1}^{n} X_{nj} \to 0 \ a.s. \tag{2.5}$$

Thus the desired result follows.

Corollary 2.2. Let $\{X_{nj}, 1 \le j \le n, n \ge 1\}$ be a triangular array of positive random variables satisfying (2.1). Let $\{W_{nj}, 1 \le j \le n, n \ge 1\}$ be a triangular array of positive random variables such that for all $\varepsilon > 0$

$$\sum_{n=1}^{\infty} \sum_{j=1}^{n} P\{| W_{nj} - \frac{1}{n} | \ge \frac{\varepsilon}{n} \} < \infty \quad . \tag{2.6}$$

Then $\sum_{j=1}^{n} W_{nj} X_{nj}$ converges almost surely to the almost sure limit of $\frac{1}{n} \sum_{j=1}^{n} X_{nj}$.

Proof. By the Borel-Cantelli lemma (2.6) implies (2.2). Thus the desired result follows by Theorem 2.1.

Lemma 2.3. Let $\{X_{nj}, 1 \le j \le n, n \ge 1\}$ be a triangular array of positive random variables and let $\{W_{nj}, 1 \le j \le n, n \ge 1\}$ be a triangular array of positive random variables such that for all $\varepsilon > 0$

$$P\Big(\lim_{n \to \infty} sup \bigcup_{j=1}^{n} \{| W_{nj} - E W_{nj}| \ge \varepsilon\, E W_{nj}\} \Big) = 0. . \tag{2.7}$$

If

$$\lim_{n \to \infty} \sum_{j=1}^{n} E(W_{nj}) X_{nj} = B \quad \text{a.s.} \tag{2.8}$$

where B is a constant or an almost sure finite random variable then $\sum_{j=1}^{n} W_{nj} X_{nj}$ converges almost surely to the almost sure limit of $\sum_{j=1}^{n} E(W_{nj}) X_{nj}$.

Proof. In the proof of Theorem 2.1 by putting $E W_{nj} = \frac{1}{n}$ we obtain the desired result.

Theorem 2.4. Let $\{X_{nj}, 1 \le j \le n, n \ge 1\}$ be a triangular array of nonnegative random variables satisfying (2.1) and let $\{W_{nj}, 1 \le j \le n, n \ge 1\}$ be a triangular array of positive random variables satisfying (2.7). Assume

$$\lim_{n \to \infty} n(\min_{1 \le j \le n} E W_{nj}) = 1 \quad \text{and} \quad \lim_{n \to \infty} \{n(\max_{1 \le j \le n} E W_{nj})\} = 1. \tag{2.9}$$

Then $\sum_{j=1}^{n} W_{nj} X_{nj}$ converges almost surely to the almost sure limit of $\frac{1}{n} \sum_{j=1}^{n} X_{nj}$.

Proof. It is enough to show that (2.8) holds : To see this, we observe the following inequality

$$
\begin{aligned}
n(\min_{1 \le j \le n} E W_{nj}) \frac{1}{n} \sum_{j=1}^{n} X_{nj} &= \min_{1 \le j \le n} E(W_{nj}) \sum_{j=1}^{n} X_{nj} \\
&\le \sum_{j=1}^{n} E(W_{nj}) X_{nj} \\
&\le \max_{1 \le j \le n} E(W_{nj}) \sum_{j=1}^{n} X_{nj} \\
&= n(\max_{1 \le j \le n} E(W_{nj})) \frac{1}{n} \sum_{j=1}^{n} X_{nj}.
\end{aligned}
\tag{2.10}
$$

It follows from (2.1) and (2.9) that both sides of (2.10) converge almost surely to B, that is, (2.8) is obtained and the proof is complete by Lemma 2.3.

Theorem 2.5. Let $\{X_{nj}, 1 \le j \le n, n \ge 1\}$ be a triangular array of nonnegative random variables. Let $\{W_{nj}, 1 \le j \le n, n \ge 1\}$ be a triangular array of random variables satisfying (2.2). Assume that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} (X_{nj} - EX_{nj}) = 0 . a. s. \tag{2.11}$$

and that there exists a positive finite constant A such that for all $n \ge 1$

$$\frac{1}{n} \sum_{j=1}^{n} EX_{nj} < A. \tag{2.12}$$

Then $\sum_{j=1}^{n} W_{nj} X_{nj}$ converges to the almost sure limit of $\frac{1}{n} \sum_{j=1}^{n} X_{nj}$. Moreover, $\sum_{j=1}^{n} W_{nj} X_{nj}$ converges to the almost sure limit of $\sum_{j=1}^{n} W_{nj} E(X_{nj})$.

Proof. It follows from (2.2) and (2.12) that for arbitrary $\varepsilon > 0$, in a set of probability one, there exists a positive integer n_0 such that for all $n \ge n_0$

$$
\begin{aligned}
\left| \sum_{j=1}^{n} W_{nj} X_{nj} - \frac{1}{n} \sum_{j=1}^{n} X_{nj} \right| &\le \sum_{j=1}^{n} \left| W_{nj} - \frac{1}{n} \right| X_{nj} \\
&\le \frac{\varepsilon}{n} \sum_{j=1}^{n} X_{nj} \\
&\le \frac{\varepsilon}{n} \sum_{j=1}^{n} (X_{nj} - EX_{nj}) + \varepsilon A.
\end{aligned}
\tag{2.13}
$$

which converges almost surely recalling (2.11). Since $\varepsilon > 0$ is arbitrary it follows that

$$\sum_{j=1}^{n} W_{nj} X_{nj} - \frac{1}{n} \sum_{j=1}^{n} X_{nj} \to 0 \quad a.\ s. \tag{2.14}$$

and the first result follows. It follows also from (2.2) and (2.12) that for arbitrary $\varepsilon > 0$, in a set of probability one, there exists a positive integer n_0 such that for all $n \geq n_0$

$$\begin{aligned} \left| \sum_{j=1}^{n} W_{nj} E(X_{nj}) - \frac{1}{n} \sum_{j=1}^{n} E(X_{nj}) \right| &\leq \sum_{j=1}^{n} \left| W_{nj} - \frac{1}{n} \right| E(X_{nj}) \\ &\leq \frac{\varepsilon}{n} \sum_{j=1}^{n} E(X_{nj}) < \varepsilon A. \end{aligned} \tag{2.15}$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$\sum_{j=1}^{n} W_{nj} E(X_{nj}) - \frac{1}{n} \sum_{j=1}^{n} E(X_{nj}) \to 0 \quad a.\ s. \tag{2.16}$$

Note that

$$\begin{aligned} &\left| \sum_{j=1}^{n} W_{nj} X_{nj} - \sum_{j=1}^{n} W_{nj} E(X_{nj}) \right| \\ &= \left| \sum_{j=1}^{n} W_{nj} X_{nj} - \frac{1}{n} \sum_{j=1}^{n} X_{nj} + \frac{1}{n} \sum_{j=1}^{n} X_{nj} - \frac{1}{n} \sum_{j=1}^{n} EX_{nj} + \frac{1}{n} \sum_{j=1}^{n} EX_{nj} - \sum_{j=1}^{n} W_{nj} E(X_{nj}) \right| \\ &\leq \left| \sum_{j=1}^{n} W_{nj} X_{nj} - \frac{1}{n} \sum_{j=1}^{n} X_{nj} \right| + \left| \frac{1}{n} \sum_{j=1}^{n} (X_{nj} - EX_{nj}) \right| + \left| \sum_{j=1}^{n} W_{nj} E(X_{nj}) - \frac{1}{n} \sum_{j=1}^{n} EX_{nj} \right|. \end{aligned} \tag{2.17}$$

By (2.14) the first term on the right-hand side of (2.17) converges to 0 a. s., the second term converges to 0 a. s. by (2.11), and the third term converges to 0 a. s. by (2.16) respectively. Thus the second desired result also follows.

Theorem 2.6. Let $\{X_{nj}, 1 \leq j \leq n, n \geq 1\}$ be a triangular array of nonnegative random variables such that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} | X_{nj} - E X_{nj} | = 0 \text{ a.s.} \tag{2.18}$$

and let $\{W_{nj}, 1 \leq j \leq n, n \geq 1\}$ be a triangular array of nonnegative integrable random variables, such that

$$P\left(\lim_{n \to \infty} sup\{ \max_{1 \leq j \leq n} | W_{nj} - E W_{nj} | \geq \frac{\varepsilon}{n} \} \right) = 0.. \tag{2.19}$$

Assume that (2.9) and (2.12) hold. Then $\sum_{j=1}^{n} W_{nj} X_{nj}$ converges to the almost sure limit of $\sum_{j=1}^{n} W_{nj} EX_{nj}$.

Proof. First note that (2.18) implies (2.11). It follows from (2.12) and (2.19) that

$$\left| \sum_{j=1}^{n} W_{nj} X_{nj} - E(W_{nj}) X_{nj} \right| \le \frac{\varepsilon}{n} \sum_{j=1}^{n} X_{nj}$$

$$\le \frac{\varepsilon}{n} \sum_{j=1}^{n} (X_{nj} - EX_{nj}) + \varepsilon A$$

which converges almost surely by (2.11). Since $\varepsilon > 0$ is arbitrary

$$\sum_{j=1}^{n} W_{nj} X_{nj} - E(W_{nj}) X_{nj} \to 0 \quad \text{a.s.} \tag{2.20}$$

As in the proof of Theorem 2.5 we consider

$$\left| \sum_{j=1}^{n} W_{nj} X_{nj} - \sum_{j=1}^{n} W_{nj} EX_{nj} \right|$$

$$\le \left| \sum_{j=1}^{n} W_{nj} X_{nj} - \sum_{j=1}^{n} E\, W_{nj} X_{nj} \right| + \left| \sum_{j=1}^{n} E\, W_{nj} X_{nj} - \sum_{j=1}^{n} E\, W_{nj}\, EX_{nj} \right|$$

$$+ \left| \sum_{j=1}^{n} W_{nj} EX_{nj} - \sum_{j=1}^{n} E\, W_{nj}\, EX_{nj} \right| . \tag{2.21}$$

One observes

$$\left| \sum_{j=1}^{n} E\, W_{nj} X_{nj} - \sum_{j=1}^{n} E\, W_{nj} EX_{nj} \right| = \left| \sum_{j=1}^{n} E\, W_{nj} (X_{nj} - EX_{nj}) \right| \le \max{}_{1 \le j \le n} E\, W_{nj} \sum_{j=1}^{n} | X_{nj} - EX_{nj} | \tag{2.22}$$

$$= \{ n (\max{}_{1 \le j \le n} E\, W_{nj}) \} \frac{1}{n} \sum_{j=1}^{n} | X_{nj} - EX_{nj} | \to 0 \ a.s.$$

by (2.9) and (2.18) and that

$$\left| \sum_{j=1}^{n} W_{nj} EX_{nj} - \sum_{j=1}^{n} E\, W_{nj} EX_{nj} \right| \le \sum_{j=1}^{n} | W_{nj} - E\, W_{nj} | EX_{nj}$$

$$\le \frac{\varepsilon}{n} \sum_{j=1}^{n} EX_{nj} \qquad \text{(by 2.19)}$$

$$< \varepsilon A \to 0 \ a.s. \qquad \text{(by 2.12)} \tag{2.23}$$

Thus it follows from (2.20), (2.22) and (2.23) that the right hand-side of (2.21) converges to 0 a.s. and the desired result follows

Theorem 2.7. Let $\{X_{nj}, 1 \le j \le n,\ n \ge 1\}$ be a triangular array of positive random variables and $\{W_{nj} : 1 \le j \le n,\ n \ge 1\}$ be a triangular array of integrable random variables. Assume that there exists a triangular array of positive constants $\{a_{nj} : 1 \le j \le n,\ n \ge 1\}$ such that

$$\lim_{n \to \infty} \sum_{j=1}^{n} a_{nj} X_{nj} = B \ a.s. \tag{2.24}$$

where B is a constant or an almost sure finite random variable, and for all $\varepsilon > 0$

$$P\left\{ \limsup_{n\to\infty} \bigcup_{j=1}^{n} \left[\left| W_{nj} - \frac{1}{n} \right| \geq \varepsilon\, a_{nj} \right] \right\} = 0. \tag{2.25}$$

and there exists a positive constant C such that

$$\sum_{j=1}^{n} a_{nj} E X_{nj} < C \tag{2.26}$$

for all $n \geq 1$ and $1 \leq j \leq n$. If (2.11) and (2.12) hold then $\sum_{j=1}^{n} W_{nj} X_{nj}$ converges almost surely to the almost sure limit of $\sum_{j=1}^{n} W_{nj} E X_{nj}$.

Proof. It follows from (2.25) that for all $\varepsilon > 0$

$$P\left\{ \limsup_{n\to\infty} \left[\max_{1\leq j\leq n} \left| W_{nj} - \frac{1}{n} \right| \geq \varepsilon\, a_{nj} \right] \right\} = 0 \tag{2.27}$$

and thus in view of (2.27) for arbitrary $\varepsilon > 0$, in a set of probability one there exists a positive integer n_0 such that for all $n \geq n_0$

$$\left| \sum_{j=1}^{n} W_{nj} X_{nj} - \frac{1}{n} \sum_{j=1}^{n} X_{nj} \right| \leq \varepsilon \sum_{j=1}^{n} a_{nj} X_{nj} \tag{2.28}$$

which converges almost surely recalling (2.24). Since $\varepsilon > 0$ is arbitrary, it follows that

$$\sum_{j=1}^{n} W_{nj} X_{nj} - \frac{1}{n} \sum_{j=1}^{n} X_{nj} \to 0 \quad a.s. \tag{2.29}$$

Finally, as in the proof of Theorem 2.6 we consider the following inequality.

$$\begin{aligned} &\left| \sum_{j=1}^{n} W_{nj} X_{nj} - \sum_{j=1}^{n} W_{nj} E X_{nj} \right| \\ &\leq \left| \sum_{j=1}^{n} W_{nj} X_{nj} - \frac{1}{n} \sum_{j=1}^{n} X_{nj} \right| + \left| \frac{1}{n} \sum_{j=1}^{n} X_{nj} - \frac{1}{n} \sum_{j=1}^{n} E X_{nj} \right| \\ &\quad + \left| \sum_{j=1}^{n} W_{nj} E X_{nj} - \frac{1}{n} \sum_{j=1}^{n} E X_{nj} \right| \end{aligned} \tag{2.30}$$

The first term on the right-hand sider of (2.30) converges to 0 a.s. by (2.29), the second term converges to 0 a.s. by (2.11). Now it remains to show that the third term converges almost surely to zero. Note that

$$\begin{aligned} \left| \sum_{j=1}^{n} W_{nj} E X_{nj} - \frac{1}{n} \sum_{j=1}^{n} E X_{nj} \right| &\leq \sum_{j=1}^{n} \left| W_{nj} - \frac{1}{n} \right| E X_{nj} \\ &\leq \varepsilon \sum_{j=1}^{n} a_{nj} E X_{nj} \to 0 \quad \text{a. s.} \quad (\text{by } 2.26) \end{aligned}$$

since $\varepsilon > 0$ is arbitrary. Thus the desired result follows.

Theorem 2.8. Let $\{X_{nj},\ n \geq 1\}$ be a triangular array of positive random variables and

$\{W_{nj},\ 1\le j\le n,\ n\ge 1\}$ be a triangular array of integrable random variables. There exists a triangular array of positive constants $\{a_{nj},\ 1\le j\le n,\ n\ge 1\}$ satisfying (2.24), (2.26) and for all $\varepsilon > 0$

$$P\left\{ \lim_{n\to\infty} sup \bigcup_{j=1}^{n} [\,|W_{nj} - EW_{nj}| \ge \varepsilon\, a_{nj}] \right\} = 0 \quad . \tag{2.31}$$

Assume that

$$\lim_{n\to\infty} \sum_{j=1}^{n} E\, W_{nj}(X_{nj} - EX_{nj}) = 0 \quad a.\ s. \tag{2.32}$$

Then $\sum_{j=1}^{n} W_{nj}X_{nj}$ converges almost surely to the almost sure limit of $\sum_{j=1}^{n} W_{nj}\, EX_{nj}$.

Proof. It follows frow (2.31) that

$$\left| \sum_{j=1}^{n} W_{nj}X_{nj} - \sum_{j=1}^{n} W_{nj}EX_{nj} \right|$$

$$\le \left| \sum_{j=1}^{n} W_{nj}X_{nj} - \sum_{j=1}^{n} E\,W_{nj}X_{nj} \right| + \left| \sum_{j=1}^{n} E\,W_{nj}X_{nj} - \sum_{j=1}^{n} E\,W_{nj}EX_{nj} \right| + \left| \sum_{j=1}^{n} W_{nj}EX_{nj} - \sum_{j=1}^{n} E\,W_{nj}EX_{nj} \right|$$

$$\le \varepsilon \sum_{j=1}^{n} a_{nj}X_{nj} + \left| \sum_{j=1}^{n} E\,W_{nj}(X_{nj} - EX_{nj}) \right| + \varepsilon \sum_{j=1}^{n} a_{nj}EX_{nj}. \tag{2.33}$$

The first term on the fight-hand side of (2.33) converges to 0 a.s. by (2.24), the second term converges to 0 a.s. by (2.32) and the third term converges to 0 a.s. by (2.26). Hence the desired result follows.

3. Application

Finally, we introduce the strong law of large number for the bootstrap mean and show the bootstrap strong law of large numbers by applying Theorem 2.5 (see Theorem 3.2). The ordinary Efron bootstrap sample $Y_{n1}, Y_{n2}, \cdots, Y_{nm(n)}$ of size $m(n)$ picks randomly, with replacement, $m(n)$ elements from the set $\{X_1, \cdots, X_n\}$. Therefore, if $\{Z_{ni}\}\ n\ge 1$, $1\le i\le m(n)$ is a triangular array of random variables, with Z_{ni} uniformly distributed on $\{1, 2, \cdots, n\}$ and $Z_{n1}, Z_{n2}, \cdots, Z_{nm(n)}$ independent for every $n = 1, 2, \cdots$, the bootstrap variables can be represented as $Y_{ni} = X_{Z_{ni}}$. Now, introducing the weights

$$W_{nj} = \frac{1}{m(n)} \sum_{i=1}^{m(n)} I_{\{Z_{ni} = j\}}, \quad j = 1, 2, \cdots, n,$$

the bootstrap sample mean, $\overline{X_n^*}$, is

$$\begin{aligned}
\overline{X_n^*} &= \frac{1}{m(n)} \sum_{i=1}^{m(n)} Y_{ni} \\
&= \frac{1}{m(n)} \sum_{i=1}^{m(n)} X_{Z_{ni}} \\
&= \sum_{j=1}^{n} \frac{1}{m(n)} \sum_{i=1}^{m(n)} X_j I_{\{Z_{ni}=j\}} \\
&= \sum_{j=1}^{n} \left(\frac{1}{m(n)} \sum_{i=1}^{m(n)} I_{\{Z_{ni}=j\}} \right) X_j \\
&= \sum_{j=1}^{n} W_{nj} X_j
\end{aligned}$$

when using respectively Athreya's (1983) and Csörgö's (1992) notation.

Note that the random vectors $\{m(n)(W_{n1}, W_{n2}, \cdots, W_{nn})'\}_{n=1}^{\infty}$ have a multinomial distribution with parameters $(m(n), \frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n})$ and also that the whole sequence is independent of the sequence $\{X_n, n \geq 1\}$. In the sequal, we say that the bootstrap strong law of large numbers holds if

$$\lim_{n\to\infty} \overline{X_n^*} = EX_1 \quad a.s.$$

in the identically distributed case, or in general if

$$\begin{aligned}
\lim_{n\to\infty} \left(\overline{X_n^*} - \sum_{j=1}^{n} W_{nj} EX_j \right) &= \lim_{n\to\infty} \left(\sum_{j=1}^{n} W_{nj} X_j - \sum_{j=1}^{n} W_{nj} EX_j \right) \\
&= \lim_{n\to\infty} \left(\sum_{j=1}^{n} W_{nj} (X_j - EX_j) \right) = 0 \quad a.s. \quad (see\,[2])
\end{aligned} \tag{3.1}$$

The following result was proved by Arenal-Gutiérres, Matrán and Cuesta-Albertos(1996).

Lemma 3.1. (Arenal-Gutiérres, etal., 1996) If

$$\lim_{n\to\infty} n \log n / m(n) = 0 \tag{3.2}$$

then

$$P\left(\lim_{n\to\infty} \left\{ \max_{1\leq j\leq n} \left| W_{nj} - \frac{1}{n} \right| \geq \frac{\varepsilon}{n} \right\} \right) = 0.$$

From Theorem 2.5 and Lemma 3.1 we have :

Theorem 3.2. Let $\{X_n, n \geq 1\}$ be a sequence of nonnegative random variables and let $\{W_{nj}, 1 \leq j \leq n, n \geq 1\}$ be a triangular array of random variables. Assume

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} (X_j - EX_j) = 0 \quad a.s.$$

and there exists a positive finite constant A such that for all $n \geq 1$

$$\frac{1}{n} \sum_{j=1}^{n} X_j < A.$$

If (3.2) is satisfied then the bootstrap strong law of large number holds, that is, (3.1) holds.

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