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Robustness in the Hierarchical Bayes Estimation of Normal Means

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Abstract

The paper considers the problem of robustness in hierarchical Bayesian models. In specific, we address Bayesian robustness in the estimation of normal means. We provide the ranges of the posterior means under ε-contamination class as well as the density ratio class of priors. For the class of priors that are uniform over a specified interval, we investigate the sensitivity as to the choice of the intervals. The methods are illustrated using the famous baseball data of Efron and Morris (1975).

1. Introduction

To perform a Bayesian analysis, specification of a prior distribution is needed. Much of the Bayesian literature deals with a single completely specified prior distribution. The most frequent criticism against such an approach is that it supposedly assumes an ability to quantify available prior information completely and accurately in terms of a single prior distribution. Given the common and unavoidable practical limitations on factors such as available prior elicitation techniques and time, it is rather unrealistic to be able to quantify prior information in terms of one distribution with complete accuracy. In view of this difficulty in prior elicitation, there has long been a robust Bayesian viewpoint that assumes only that subjective information can be quantified only in terms of a class Γ of prior distributions. A procedure is then said to be robust, if its inferences are relatively insensitive to the variation of the prior distribution over Γ. These general ideas can be found in Good (1965), and more recently in Berger (1984, 1985, 1990).

In this paper, we consider the robust Bayes (RB) idea in the context of the estimation of normal means. Specifically, the hierarchical Bayes (HB) procedure models the uncertainty in the prior information by assigning a single distribution (often noninformative or improper) to the prior parameters (usually called hyperparameters). Instead, the RB procedure attempts to quantify the subjective information in terms of a class Γ of prior distributions.

In order to study Bayesian robustness in the estimation of normal means, we consider the following HB model.

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I. Conditional on $\theta_1, \cdots, \theta_p$, μ and τ^2, let $Y_1, \cdots, Y_p$ be independently distributed with $Y_i \sim N(\theta_i, \sigma^2)$, $i=1, \cdots, p$, where the σ^2 is known positive constant;

II. Conditional on μ and τ^2, $\theta_1, \cdots, \theta_p$ are independently distributed with $\theta_i \sim N(\mu, \tau^2)$, $i=1, \cdots, p$;

III. Marginally μ and τ^2 are independent with $\mu \sim$ uniform(R) and τ^2 having a distribution $h(\tau^2)$ which belongs to a certain class of distribution Γ.

We shall use the notations $\boldsymbol{y} = (y_1, \cdots, y_p)^T$ and $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_p)^T$.

The outline of the remaining sections is as follows. In Section 2, we choose Γ to be ε-contamination class of priors where the contamination class includes all unimodal distributions. We provide the range where the posterior means belong under ε-contamination class.

In Section 3, we choose Γ to be the density ratio class of priors. As suggested by Wasserman and Kadane (1992), we use Gibbs sampling (Gelfand and Smith (1990), Gelfand et al. (1990)) to compute bounds on posterior expectations over the density ratio class.

In Section 4, we choose Γ to be the class of uniform priors on τ^2 with $\tau_1^2 \le \tau^2 \le \tau_2^2$. We are interested in the sensitivity analysis of the posterior quantity over Γ.

Finally, Section 5 contains the analysis of the real data to illustrate the results of the proceeding sections.

2. ε-contamination Class

In this section, we consider the class Γ of priors of the form

$$\Gamma = \{ h \ : \ h = (1-\varepsilon) h_0 + \varepsilon q, \quad q \in Q \}, \tag{2.1}$$

where $0 \le \varepsilon \le 1$ is given, h_o is the inverse gamma distribution with pdf

$$h_0(\tau^2) = \frac{\alpha_o^{\beta_0}}{\Gamma(\beta_0)} \frac{1}{(\tau^2)^{\beta_0+1}} \exp(-\alpha_0/\tau^2) I_{(0,\infty)}(\tau^2), \tag{2.2}$$

denoted by IG (α_0, β_0), and Q is the class of all unimodal distributions with the same mode τ_0^2 as that of h_0.

Then the joint (improper) pdf of $\boldsymbol{y}$, $\boldsymbol{\theta}$, μ and τ^2 is given by

$$f(\boldsymbol{y}, \boldsymbol{\theta}, \mu, \tau^2) \propto (\tau^2)^{-\frac{p}{2}} \exp\left[-\frac{1}{2} \left\{ \frac{1}{\sigma^2} (\boldsymbol{y}-\boldsymbol{\theta})^T (\boldsymbol{y}-\boldsymbol{\theta}) - \frac{1}{\tau^2} \| \boldsymbol{\theta} - \mu 1_p \|^2 \right\} \right]$$
$$\times \{ (1-\varepsilon) h_0(\tau^2) + \varepsilon q(\tau^2) \}. \tag{2.3}$$

Now, integrating with respect to μ in (2.3), one finds the joint (improper) pdf of $\boldsymbol{y}, \boldsymbol{\theta}$ and τ^2 given by

$$f(\mathbf{y}, \boldsymbol{\theta}, \tau^2) \propto (\tau^2)^{-(p-1)/2} \exp\left[-\frac{1}{2\sigma^2}\left\{(\boldsymbol{\theta}-D^{-1}\mathbf{y})^T D(\boldsymbol{\theta}-D^{-1}\mathbf{y}) - V\sum_{i=1}^{p}(y_i-\bar{y})^2\right\}\right]$$
$$\times\{(1-\varepsilon)h_0(\tau^2)+\varepsilon q(\tau^2)\}, \tag{2.4}$$

where $D = I_p + \frac{\sigma^2}{\tau^2}\left(I_p - \frac{J_p}{p}\right)$, $D^{-1} = (1-V)I_p + p^{-1}VJ_p$, and $V = \sigma^2/(\sigma^2+\tau^2)$. From (2.4), the posterior distribution of $\boldsymbol{\theta}$ given $\mathbf{y}$ and τ^2 is

$$N_p[(1-V)\mathbf{y} + V\bar{y}\mathbf{1}_p,\ \sigma^2(1-V)I_p + \sigma^2 p^{-1}VJ_p]. \tag{2.5}$$

Also, integrating with respect to $\boldsymbol{\theta}$ in (2.4), one gets the joint (improper) pdf of $\mathbf{y}$ and τ^2 is given by

$$f(\mathbf{y}, \tau^2) \propto |D^{-1}|^{\frac{p}{2}} \exp\left\{-\frac{1}{2(\tau^2+\sigma^2)}\sum_{i=1}^{p}(y_i-\bar{y})^2\right\}\{(1-\varepsilon)h_o(\tau^2)+\varepsilon q(\tau^2)\}. \tag{2.6}$$

Note that

$$f(\mathbf{y} \mid \tau^2) \propto (\tau^2+\sigma^2)^{-\frac{p-1}{2}} \exp\left(-\frac{1}{2(\tau^2+\sigma^2)}\sum_{i=1}^{p}(y_i-\bar{y})^2\right). \tag{2.7}$$

Hence, we have

$$h(\tau^2 \mid \mathbf{y}) \propto (\tau^2+\sigma^2)^{-\frac{p-1}{2}} \exp\left(-\frac{1}{2(\tau^2+\sigma^2)}\sum_{i=1}^{p}(y_i-\bar{y})^2\right)$$
$$\times\{(1-\varepsilon)h_o(\tau^2)+\varepsilon q(\tau^2)\}. \tag{2.8}$$

Using (2.5) and the iterated formulas for conditional expectations and variances, one gets

$$E[\theta_i \mid \mathbf{y}] = E[E(\theta_i \mid \mathbf{y}, \tau^2) \mid \mathbf{y}] = E[(1-V)y_i + V\bar{y} \mid \mathbf{y}] \tag{2.9}$$

and

$$\begin{aligned} Var[\theta_i \mid \mathbf{y}] &= Var[E(\theta_i \mid \mathbf{y}, \tau^2) \mid \mathbf{y}] + E[Var(\theta_i \mid \mathbf{y}, \tau^2) \mid \mathbf{y}] \\ &= Var[V(y_i-\bar{y}) \mid \mathbf{y}] + E[\sigma^2(1-V) + \sigma^2 p^{-1}V \mid \mathbf{y}] \end{aligned} \tag{2.10}$$

Thus, the posterior distribution of θ under the ε-contamination prior is obtained using (2.5) and (2.8). In addition, one uses (2.9) and (2.10) to find the posterior means and variances of θ_i under the ε-contamination prior. Similarly, by using the iterated formulas, posterior covariances may be obtained as well.

Now we consider the problem of finding the range of the posterior mean of θ_i over Γ in (2.1). Using the expression $h(\tau^2 \mid \mathbf{y}) \propto f(\mathbf{y} \mid \tau^2)h(\tau^2)$, we have

$$\begin{aligned} E[\theta_i \mid \mathbf{y}] &= y_i - E[V(y_i-\bar{y}) \mid \mathbf{y}] \\ &= y_i - \frac{\int_0^\infty V(y_i-\bar{y})h(\tau^2 \mid \mathbf{y})d\tau^2}{\int_0^\infty h(\tau^2 \mid \mathbf{y})d\tau^2}. \end{aligned} \tag{2.11}$$

Simple modifications of the arguments of Sivaganesan and Berger (1989) lead to the following result

$$\sup(\inf)_{h\in\Gamma} E[\theta_i \mid \mathbf{y}]$$

$$= \sup(\inf)_t \frac{A + \frac{1}{t-\tau_0^2}\int_{\tau_0^2}^{t}(y_i - V(y_i-\bar{y}))f(\mathbf{y} \mid \tau^2)d\tau^2}{B + \frac{1}{t-\tau_0^2}\int_{\tau_0^2}^{t} f(\mathbf{y} \mid \tau^2)d\tau^2} \tag{2.12}$$

where

$$B = y_i B - \frac{1-\varepsilon}{\varepsilon}\int_0^\infty f(\mathbf{y} \mid \tau^2)h_0(\tau^2)d\tau^2 \tag{2.13}$$

and

$$A = y_i B - \frac{1-\varepsilon}{\varepsilon}\int_0^\infty V(y_i-\bar{y})f(\mathbf{y} \mid \tau^2)h_0(\tau^2)d\tau^2. \tag{2.14}$$

The above sup(inf) is obtained by numerical optimization.

3. Density Ratio Class

In this section we consider a class of prior, introduced by DeRobertis and Hartigan (1981), and called a density ratio class by Berger (1990),

$$\Gamma_{l,u}^{R} = \left\{ h : \frac{h(\tau^2)}{h(\tau'^2)} \le \frac{u(\tau^2)}{l(\tau'^2)} \text{ for all } \tau^2 \text{ and } \tau'^2 \right\}, \tag{3.1}$$

where l and u are two bounded nonnegative functions such that $l(\tau^2) \le u(\tau^2)$ for all τ^2. This class can be viewed as specifying ranges for the ratios of the prior density between any two points. By taking $u = kh_0$ and $l = h_0$, we have interesting subclass,

$$\Gamma_{k}^{R}(h_0) = \left\{ h : \frac{h(\tau^2)}{h(\tau'^2)} \le k\frac{h_0(\tau^2)}{h_0(\tau'^2)} \text{ for all } \tau^2 \text{ and } \tau'^2 \right\}, \tag{3.2}$$

where $k \ge 1$ is a constant. This class may be thought of as a neighborhood around the target prior h_0. The interpretation is that the odds of any pair of points are not misspecificated by more than a factor of k. This prior is especially useful when h_o is a default prior chosen mainly for convenience.

Because of the expression $h(\tau^2 \mid \mathbf{y}) \propto f(\mathbf{y} \mid \tau^2)h(\tau^2)$ we can view our problem as having just the parameter τ^2, h being the prior and $f(\mathbf{y} \mid \tau^2)$ the likelihood. Since

$$E[\theta_i \mid \mathbf{y}] = y_i - E[V(y_i - \bar{y}) \mid \mathbf{y}], \tag{3.3}$$

our problem reduces to finding

$$\sup(\inf)_{h\in\Gamma_k^R(h_0)} E[b(\tau^2) \mid \mathbf{y}] \tag{3.4}$$

where $b(\tau^2) = V(y_i - \bar{y}) = \frac{\sigma^2}{\sigma^2+\tau^2}(y_i - \bar{y})$.

Wasserman and Kadane (1992) have developed a Monte Carlo technique which can be used to bound posterior expectations over the density ratio class. Wasserman (1992) has shown that the set of posteriors obtained by applying Bayes' theorem to the density ratio class $\Gamma^R_k(h_0)$ is the density ratio class $\Gamma^R_k(h^0_y)$, where $h^0_y(\tau^2) = h_0(\tau^2 \mid \boldsymbol{y})$ is the posterior corresponding to h_0. To see this all we need to do is to write

$$\Gamma^R_k(h^0_y) = \left\{ h_y : \frac{h_y(\tau^2)}{h_y(\tau'^2)} \le k \frac{h^0_y(\tau^2)}{h^0_y(\tau'^2)} \text{ for all } \tau^2 \text{ and } \tau'^2 \right\}, \tag{3.5}$$

and observe $\dfrac{h(\tau^2 \mid \boldsymbol{y})}{h(\tau'^2 \mid \boldsymbol{y})} = \dfrac{f(\boldsymbol{y} \mid \tau^2)h(\tau^2)}{f(\boldsymbol{y} \mid \tau'^2)h(\tau'^2)}$ and $\dfrac{h_0(\tau^2 \mid \boldsymbol{y})}{h_0(\tau'^2 \mid \boldsymbol{y})} = \dfrac{f(\boldsymbol{y} \mid \tau^2)h_0(\tau^2)}{f(\boldsymbol{y} \mid \tau'^2)h_0(\tau'^2)}$ so that $h \in \Gamma^R_k(h_0)$ is equivalent to $h_y \in \Gamma^R_k(h^0_y)$, where $h_y(\tau^2) = h(\tau^2 \mid \boldsymbol{y})$. Wasserman (1992) calls this as Bayes' invariance property. Hence, to bound the posterior expectation of $b(\tau^2)$, we only need bound the expectation of $b(\tau^2)$ on $\Gamma^R_k(h^0_y)$. To do so, we will need to draw a sample from the posterior h^0_y.

Following Wasserman and Kadane (1992), we can rely on recent sampling-based methods for Bayesian inference. Note in this case that

$$h_0(\tau^2 \mid \boldsymbol{y}) \propto (\tau^2 + \sigma^2)^{-\frac{p-1}{2}} \exp\left[-\frac{1}{2(\tau^2+\sigma^2)} \sum_{i=1}^{p} (y_i - \bar{y})^2 \right] h_0(\tau^2). \tag{3.6}$$

Let $\tau^2_1, \cdots, \tau^2_N$ be a random sample from $h_0(\tau^2 \mid y)$. Let $b_i = b(\tau^2_i)$, $i=1,\cdots,N$ and let $b_{(1)} \le b_{(2)} \le \cdots \le b_{(N)}$ be the corresponding order statistics. Also, let $c_i = -b(\tau^2_i)$, $i=1,\cdots,N$ and let $c_{(1)} \le c_{(2)} \le \cdots \le c_{(N)}$ be the corresponding order statistics. Let $\bar{b} = \sum_{i=1}^{N} b_i / N$ and $\bar{c} = \sum_{i=1}^{N} c_i / N$. Then following Wasserman and Kadane (1992), we get

$$\begin{aligned} \sup_{h \in \Gamma^R_k(h_0)} E[b(\tau^2) \mid \boldsymbol{y}] &= \sup_{h_y \in \Gamma^R_k(h^0_y)} E[b(\tau^2) \mid \boldsymbol{y}] \\ &\approx \max \left\{ \left(1 - \frac{i}{N}\right)\triangle + 1 \right\}^{-1} \left\{ \triangle \sum_{j=i}^{N} b_{(j)} / N + \bar{b} \right\} \end{aligned} \tag{3,7}$$

and

$$\begin{aligned} \inf_{h \in \Gamma^R_k(h_0)} E[b(\tau^2) \mid \boldsymbol{y}] &= \inf_{h_y \in \Gamma^R_k(h^0_y)} E[b(\tau^2) \mid \boldsymbol{y}] \\ &\approx \max \left\{ \left(1 - \frac{i}{N}\right)\triangle + 1 \right\}^{-1} \left\{ \triangle \sum_{j=i}^{N} c_{(j)} / N + \bar{c} \right\} \end{aligned} \tag{3.8}$$

where $\triangle = k - 1$. This gives the posterior bound for a given k.

To generate the sample from h^0_y, we use Gibbs sampling. The Gibbs sampling analysis is based on the following posterior distributions:

(i) $\mu \mid \boldsymbol{y}, \theta, z \sim N(\theta, z^{-1})$;

(*ii*) $\theta_1, \cdots, \theta_m \mid \boldsymbol{y},\ \mu,\ z \overset{ind}{\sim} N(\,(1-B)y_i + B\mu,\ z^{-1}B\,)$;

(*iii*) $\tau^2 \mid \boldsymbol{y},\ \mu,\ z \sim IG(\,\alpha_0 + \frac{1}{2}\sum_{i=1}^{m}(\theta_i-\mu)^2, \frac{1}{2}p+\beta_0\,)$;

where $z=(\tau^2)^{-1}$ and $B=z/(z+\zeta)$, where $\zeta=1/\sigma^2$, $i=1,\cdots,p$.

Note that our target is to draw samples from $h_0(\tau^2 \mid \boldsymbol{y})$. However, as is well known in the Gibbs sampling literatures, the histogram of the $(\tau^2)^t, t=1,2,\cdots,q$, samples drawn from the conditionals $[\tau^2 \mid \boldsymbol{y}, \mu, \theta]$ converges to the conditional distribution $[\tau^2 \mid \boldsymbol{y}]$ as $q \to \infty$. This along with (3.7) and (3.8) facilitates the computations of upper and lower bounds of $E[b(\tau^2) \mid \boldsymbol{y}]$.

4. Class of Uniform Priors

In usual HB model in our setting, one might use the diffuse prior on τ^2. Instead, we consider the class of uniform prior on τ^2 with constraints of the form

$$\Gamma = \{\tau^2 : \tau_1^2 \le \tau^2 \le \tau_2^2\}, \tag{4.1}$$

where τ_1^2 and τ_2^2 are arbitrary nonnegative numbers such that $\tau_1^2 < \tau_2^2$.

This class of prior is attractive because of its mathematical convenience, and indeed give a good enough class of priors for an honest robustness check. Classes of conjugate priors having parameters in certain ranges have been studied by Leamer (1982) and Polasek (1985).

From (2.5), we have

$$\sup(\inf)_{\tau_1^2 \le \tau^2 \le \tau_2^2} E[\theta_i \mid \boldsymbol{y}, \tau^2] = y_i - \inf(\sup)_{\tau_1^2 \le \tau^2 \le \tau_2^2} \frac{\sigma^2}{\sigma^2+\tau^2}(y_i - \bar{y}). \tag{4.2}$$

Hence, we can find the range of the posterior quantity $E[\theta_i \mid \boldsymbol{y}, \tau^2]$ over Γ in (4.1), and investigate the sensitivity with different choices of τ_1^2 and τ_2^2.

5. Data Analysis

In this section we illustrate the methods suggested in preceding sections with an analysis of real data set. We consider the famous baseball data of Efron and Morris (1975). The data consist of the batting averages of 18 major league players through their first 45 official at bats of the 1970 season.

To begin, the binomial model is fitted to the 45 at-bats for each player i, but, to simplify the computations, the binomial likelihood is approximated by a normal likelihood under the arcsin transformation. The arcsin transformation is used to create normal random variables of roughly unit variance for all values of the parameter θ_i that are not close to 0 or 1.

We consider the HB model as given in Section 1. We choose Γ to be ε-contamination class includes all unimodal distributions. We find the ranges of the posterior means under ε -contamination class. Table 5.1 provides the sensitivity index (= (sup-inf)/inf) for posterior means under ε-contamination class when $(\alpha_0, \beta_0)=(1,10)$ and $(\alpha_0, \beta_0)=(7,3)$.

For $(\alpha_0, \beta_0)=(1,10)$, the sensitivity indices are fairly small and robustness seems to be achieved using this class for all ε values. But for $(\alpha_0, \beta_0)=(7,3)$, the sensitivity indices are relativity larger in comparison with $(\alpha_0, \beta_0)=(1,10)$. As one might expect, the choice of h_0, that is, the elicited inverse gamma prior, seems to have some effect on the ranges of the posterior means for ε-contamination class. Note that the inverse gamma prior of τ^2 with $(\alpha_0, \beta_0)=(1,10)$ has coefficient of variation 1/8 compared to 1 with $(\alpha_0, \beta_0)=(7,3)$. Although the two priors have very similar tails, the former is much flatter than the latter even for small values of τ^2. This suggests that the bigger the coefficient of variation of the assumed inverse gamma prior, the wider is the range of the posterior means of the θ_i's.

We also find the ranges of the posterior means under density ratio classes. In computing the bounds, we have considered Gibbs sampler with 5 independent sequences, each with a sample of size 1000 with a burn-in sample of another 1000. We adopt the basic approach of Gelman and Rubin (1992) to monitor the convergence of the Gibbs sampler. We have obtained $\hat{R}$ values (the potential scale reduction factors) corresponding to the estimand θ_i based on 5 $\times$1000=5000 simulated values. The fact that all the point estimates $\hat{R}$ are equal to 1 as well as the near equality of these point estimates and the corresponding 97.5 quantiles sugests that convergence is achieved in the Gibbs sampler.

Table 5.2 provides the sensitivity index for the posterior means under density ratio classes when $(\alpha_0, \beta_0)=(1,10)$ and $(\alpha_0, \beta_0)=(7,3)$. Note however that here the indices given for IG(7,3) are much smaller than the ones for IG(1,10). The intuitive explanation for this phenomenon is that while the ratio $\dfrac{h_0(\tau^2)}{h_0(\tau'^2)}$ can be extremely large for certain choices (τ^2, τ'^2) corresponding to the IG(1,10) prior, the ratio $\dfrac{h_0(\tau^2)}{h_0(\tau'^2)}$ is more under control for the IG(7,3) prior. This density ratio classes are very convenient to represent vague prior knowledge and robustness seems to be achieved using the IG(7,3) prior.

Finally, we consider the ranges of the posterior quantitiles $E[\theta_i \mid \mathbf{y}, \tau^2]$ over the class of uniform priors. From Table 5.3, we can see that the ranges are not sensitive to the choice of the upper bound of τ^2. Also, for most of the players, the ranges of the posterior means do not seem to be too wide.

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Table 5.1 : Sensitivity Index ((sup-inf)/inf) for Posterior Mean of θ_i under ε-contamination Class of Priors

i	θ_i	$(\alpha_0, \beta_0)=(1,10)$				$(\alpha_0, \beta_0)=(7,3)$			
		$\varepsilon=.1$	$\varepsilon=.2$	$\varepsilon=.5$	$\varepsilon=1$	$\varepsilon=.1$	$\varepsilon=.2$	$\varepsilon=.5$	$\varepsilon=1$
1	.346	.01708	.02001	.02602	.03291	.14166	.15546	.17457	.18055
2	.298	.01660	.01988	.02597	.02981	.07957	.09588	.10487	.13563
3	.276	.01350	.01642	.01910	.02444	.06767	.08347	.10037	.12910
4	.222	.00661	.00771	.00822	.01724	.05034	.05580	.09432	.11330
5	.273	.00561	.00622	.00803	.01601	.02369	.05437	.06982	.09388
6	.270	.00561	.00622	.00803	.01601	.02369	.05437	.06982	.09388
7	.263	.00305	.00603	.00710	.01458	.01805	.03964	.05414	.08116
8	.210	.00028	.00057	.00129	.00230	.00824	.00861	.00882	.01071
9	.269	.00272	.00543	.00688	.00859	.00652	.00735	.01497	.04640
10	.230	.00272	.00543	.00688	.00859	.00652	.00735	.01497	.04640
11	.264	.00583	.00703	.00795	.01620	.01765	.02926	.03329	.08242
12	.256	.00583	.00703	.00795	.01620	.01765	.02926	.03329	.09377
13	.303	.00583	.00703	.00795	.01620	.01765	.02926	.03329	.09377
14	.264	.00583	.00703	.00795	.01620	.01765	.02926	.03329	.09377
15	.226	.00583	.00703	.00795	.01620	.01765	.02926	.03329	.09377
16	.285	.00707	.00924	.01030	.02139	.05385	.07927	.09385	.11500
17	.316	.01571	.01993	.02355	.02840	.06132	.08461	.11649	.13821
18	.200	.01609	.02015	.02796	.03174	.11441	.12571	.13074	.17153

Table 5.2 : Sensitivity Index ((sup-inf)/inf) for Posterior Mean of θ_i under the Density Ratio Class of Priors

i	θ_i	$(\alpha_0, \beta_0)=(1,10)$			$(\alpha_0, \beta_0)=(7,3)$		
		$K=2$	$K=6$	$K=10$	$K=2$	$K=6$	$K=10$
1	.346	.15336	.21881	.38081	.04594	.10140	.19197
2	.298	.10067	.17066	.24178	.04171	.09349	.17375
3	.276	.09548	.15242	.21762	.03739	.07802	.09654
4	.222	.09117	.12698	.19333	.03262	.07327	.08938
5	.273	.08119	.11920	.15343	.02409	.05607	.06946
6	.270	.08071	.11720	.15101	.02414	.05597	.06911
7	.263	.07311	.09571	.12617	.01991	.03474	.04705
8	.210	.06132	.08318	.09368	.01370	.02384	.02918
9	.269	.07154	.10016	.10346	.01534	.03291	.04213
10	.230	.07150	.10012	.10348	.01539	.03312	.04169
11	.264	.08749	.14948	.16364	.02756	.05778	.08364
12	.256	.08767	.15053	.16459	.02753	.05724	.08324
13	.303	.08774	.15015	.16568	.02751	.05717	.08383
14	.264	.08779	.14875	.16613	.02758	.05785	.08339
15	.226	.08789	.15142	.16843	.02750	.05765	.08402
16	.285	.10847	.18627	.21908	.03501	.07653	.09238
17	.316	.15317	.19231	.23018	.04184	.08668	.14336
18	.200	.17827	.25211	.38536	.04793	.11992	.20767

Table 5.3 : Sensitivity Index ((sup-inf)/inf) for
$E[\theta_i \mid \mathbf{y}, \tau^2]$ under the Class of Uniform Priors

i	θ_i	$0 \le \tau^2 \le 1$	$0 \le \tau^2 \le 5$	$0 \le \tau^2 \le 10$
1	.346	0.2535795	0.4227581	0.4611906
2	.298	0.2121326	0.3534287	0.3858327
3	.276	0.1706887	0.2844762	0.3104747
4	.222	0.1273549	0.2121326	0.2317257
5	.273	0.0859080	0.1431801	0.1563677
6	.270	0.0859080	0.1431801	0.1563677
7	.263	0.0444611	0.0742275	0.0810097
8	.210	0.0030143	0.0048983	0.0056519
9	.269	0.0420102	0.0718901	0.0793005
10	.230	0.0420102	0.0718901	0.0793005
11	.264	0.0890439	0.1579406	0.1748561
12	.256	0.0890439	0.1579406	0.1748561
13	.303	0.0890439	0.1579406	0.1748561
14	.264	0.0890439	0.1579406	0.1748561
15	.226	0.0890439	0.1579406	0.1748561
16	.285	0.1405242	0.2584163	0.2889752
17	.316	0.1971132	0.3779854	0.4276492
18	.200	0.2596108	0.5235361	0.5997588