

Hierarchical Bayes Estimators of the Error Variance in Balanced Fixed-Effects Two-Way ANOVA Models¹⁾

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Abstract

We propose a class of hierarchical Bayes estimators of the error variance under the relative squared error loss in balanced fixed-effects two-way analysis of variance models. Also, we provide analytic expressions for the risk improvement of the hierarchical Bayes estimators over multiples of the error sum of squares. Using these expressions we identify a subclass of the hierarchical Bayes estimators, each member of which dominates the best multiple of the error sum of squares which is known to be minimax. Numerical values of the percentage risk improvement are given in some special cases.

1. Introduction

For estimation of the error variance in a general balanced mixed linear model, Mathew, Sinha, and Sutradhar(1992) showed that the usual analysis of variance(ANOVA) estimator of the error variance, or its best multiple, can be uniformly improved by non-quadratic estimators and constructed two such classes of non-quadratic estimators. For the "within" component of variance, that is, the error variance in the one-way random-effects model, Kubokawa, Saleh, and Makita(1993) provided a class of improved estimators over the usual ANOVA estimator which included the general Bayes rule derived by Portnoy(1971). Ghosh(1994) and Datta and Ghosh(1995) proposed classes of hierarchical Bayes estimators of the error variance in balanced fixed-effects one-way analysis of variance models and identified such classes, each member of which dominates the best multiple of the error sum of squares which is known to be minimax under the relative squared error loss. Recently, Vounatsou and Smith(1997) presented a Bayesian analysis of variance component models via simulation. They showed the 2-component hierarchical design model under balanced and unbalanced experiments. Also, they considered 2-factor additive random effect models and mixed models in a cross-classified design. They assessed the sensitivity of inference to the choice of prior by a sampling/resampling technique.

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A closely related problem is that of estimating the variance of a normal distribution with an unknown mean. A comprehensive review and survey on this problem were given in Maatta and Casella(1990) which included results of Stein(1964), Brewester and Zidek(1974), Strawderman(1974), Rukhin(1987), and so on. Using a simple definite integral Kubokawa(1994) provided a class of improved estimators of powers of the variance. Recently, Arnold and Villasenor(1997) derived estimators of the variance by standard and variant Bayesian techniques.

In this paper we consider the hierarchical Bayesian estimation for the error variance σ_ϵ^2 under the relative squared error loss

$$L(d, \sigma_\epsilon^2) = (d\sigma_\epsilon^{-2} - 1)^2 = \sigma_\epsilon^{-4}(d - \sigma_\epsilon^2)^2 \quad (1.1)$$

in balanced fixed-effects two-way analysis of variance(ANOVA) models, where " d " denotes a decision which is an element of the decision space D .

In Section 2, we develop a class of hierarchical Bayes estimators of σ_ϵ^2 . In Section 3, we provide analytic expressions for the risk improvement of the hierarchical Bayes estimators over multiples of the error sum of squares. In Section 4, we identify a subclass, each member of which dominates the best multiple of the error sum of squares which is known to be minimax. Also, we provide a subclass of non-minimax hierarchical Bayes estimators. All the results of this paper can be regarded as a two-way extension of results by Ghosh(1994) and Datta and Ghosh(1995). In Section 5, we give numerical values of the percentage risk improvement of the hierarchical Bayes estimators over the best multiple estimator through computer simulation for some special cases. These calculations indicate the risk improvement over the best multiple estimator can often be quite substantial.

2. Hierarchical Bayes Estimators

Consider the following balanced fixed-effects two-way analysis of variance(ANOVA) models:

$$y_{ij} = \theta_{ij} + \epsilon_{ij}, \quad i = 1, 2, \dots, p(>1); j = 1, 2, \dots, q(>1),$$

where the ϵ_{ij} 's are independently and identically distributed(i.i.d.) as $N(0, \sigma_\epsilon^2)$. As a special case, if we let $\theta_{ij} = \theta_i$ for all $j = 1, 2, \dots, q$, then we have balanced fixed-effects one-way ANOVA models in which Ghosh(1994) and Datta and Ghosh(1995) considered the hierarchical Bayesian estimation of σ_ϵ^2 under the loss (1.1). The UMVUE $\hat{\sigma}_U^2 = \frac{S}{(p-1)(q-1)}$ of σ_ϵ^2 is inadmissible since it is dominated by the best multiple estimator of σ_ϵ^2 ,

$\hat{\sigma}_M^2 = \frac{1}{(p-1)(q-1)+2} S$, which is best invariant constant risk minimax estimator under the loss (1.1), where $S = \sum_{i=1}^p \sum_{j=1}^q (y_{ij} - \bar{y}_{..} - \bar{y}_{.j} + \bar{y}_{i.})^2$, $\bar{y}_{..} = \frac{1}{q} \sum_{j=1}^q y_{ij}$, $\bar{y}_{.j} = \frac{1}{p} \sum_{i=1}^p y_{ij}$, and

$$\bar{y}_{..} = \frac{1}{pq} \sum_{i=1}^p \sum_{j=1}^q y_{ij}.$$

Here, we use the notations

$$\mathbf{y} = (y_{11}, \dots, y_{pq})^T \text{ and } \boldsymbol{\theta} = (\theta_{11}, \dots, \theta_{pq})^T.$$

The hierarchical Bayesian model is given in (a)–(c) below:

- (a) Conditionally on $\boldsymbol{\theta}$, μ , $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_p)^T$ with $\sum_{i=1}^p \alpha_i = 0$, σ_β^2 , and σ_ϵ^2 , $\mathbf{y} \sim N_{pq}(\boldsymbol{\theta}, \sigma_\epsilon^2 I_{pq})$;
- (b) Conditionally on μ , $\boldsymbol{\alpha}$, σ_β^2 , and σ_ϵ^2 , $\boldsymbol{\theta} \sim N_{pq}(\mu \mathbf{1}_p + \boldsymbol{\alpha} \otimes \mathbf{1}_q, \sigma_\beta^2 J_p \otimes I_q)$, where $\mathbf{1}_k$ is a $k \times 1$ vector of ones, I_k is the $k \times k$ identity matrix, J_k is the $k \times k$ matrix of ones, and \otimes denotes the usual kronecker's product;
- (c) $(\mu, \boldsymbol{\alpha}, \sigma_\beta^2, \sigma_\epsilon^2) \sim (\sigma_\epsilon^2)^{\frac{a-b}{2}-1} (\sigma_\epsilon^2 + p\sigma_\beta^2)^{-\frac{4-b}{2}}$, $0 \leq a \leq p(q-1)+2$, $0 \leq b \leq q+1$.

Remark 2.1. Note that (a) and (b) represent a balanced two-way additive mixed model if we let $\theta_{ij} = \mu + \alpha_i + \beta_j$, $i = 1 \dots p$, $j = 1 \dots q$, where μ and α_i 's are fixed with $\sum_{i=1}^p \alpha_i = 0$, and β_j 's are i.i.d. as $N(0, \sigma_\beta^2)$ which are independent of ϵ_{ij} 's. A detailed Bayesian treatment of this model by using the noninformative reference prior for $(\mu, \boldsymbol{\alpha}, \sigma_\beta^2, \sigma_\epsilon^2)$ with $a = b = 2$ in (c), can be found in Box and Tiao(1973, Section 6.3).

Remark 2.2. (c) is equivalent to say that $(\mu, \boldsymbol{\alpha}) \propto 1$, $(\sigma_\beta^2, \sigma_\epsilon^2) \sim (\sigma_\epsilon^2)^{\frac{a-b}{2}-1} (\sigma_\epsilon^2 + p\sigma_\beta^2)^{-\frac{4-b}{2}}$, and $(\mu, \boldsymbol{\alpha})$ and $(\sigma_\beta^2, \sigma_\epsilon^2)$ are independent. This prior for $(\sigma_\beta^2, \sigma_\epsilon^2)$ contains Jeffreys' noninformative prior with $a = b = 2$ as a special case and was first used in Portnoy(1971) with slightly different notations who treated the problem of estimating σ_β^2 under the scale invariant loss.

From (a) and (b), the conditional density of \mathbf{y} , and $\boldsymbol{\theta}$ given μ , $\boldsymbol{\alpha}$, σ_β^2 and σ_ϵ^2 is

$$\begin{aligned} f(\mathbf{y}, \boldsymbol{\theta} | \mu, \boldsymbol{\alpha}, \sigma_\beta^2, \sigma_\epsilon^2) &= f(\mathbf{y} | \boldsymbol{\theta}, \mu, \boldsymbol{\alpha}, \sigma_\beta^2, \sigma_\epsilon^2) \cdot \pi(\boldsymbol{\theta} | \mu, \boldsymbol{\alpha}, \sigma_\beta^2, \sigma_\epsilon^2) \\ &\propto (\sigma_\epsilon^2)^{-\frac{pq}{2}} (\sigma_\beta^2)^{-\frac{pq}{2}} e^{-\frac{1}{2\sigma_\epsilon^2} \sum_{i=1}^p \sum_{j=1}^q (y_{ij} - \theta_{ij})^2 - \frac{1}{2\sigma_\beta^2} (\boldsymbol{\theta} - (\mu \mathbf{1}_p + \boldsymbol{\alpha} \otimes \mathbf{1}_q))^T (J_p \otimes I_q)^{-1} (\boldsymbol{\theta} - (\mu \mathbf{1}_p + \boldsymbol{\alpha} \otimes \mathbf{1}_q))}. \end{aligned} \quad (2.1)$$

Integrating out $\boldsymbol{\theta}$ in (2.1),

$$\begin{aligned} f(\mathbf{y} | \mu, \boldsymbol{\alpha}, \sigma_\beta^2, \sigma_\epsilon^2) &\propto \left| \left(\frac{1}{\sigma_\epsilon^2} I_{pq} + \frac{1}{\sigma_\beta^2} (J_p \otimes I_q)^{-1} \right)^{-1} \right|^{\frac{1}{2}} (\sigma_\epsilon^2)^{-\frac{pq}{2}} (\sigma_\beta^2)^{-\frac{pq}{2}} \end{aligned}$$

$$\begin{aligned}
& \cdot e^{-\frac{1}{2} \left[\frac{1}{\sigma_\epsilon^2} \mathbf{y}^\top \mathbf{y} - \left(\frac{1}{\sigma_\epsilon^2} \mathbf{y} + \frac{1}{\sigma_\beta^2} (J_p \otimes I_q)^{-1} (\mu \mathbf{1}_m + \boldsymbol{\alpha} \otimes \mathbf{1}_q) \right)^\top \left(\frac{1}{\sigma_\epsilon^2} I_m + \frac{1}{\sigma_\beta^2} (J_p \otimes I_q)^{-1} \right)^{-1} \left(\frac{1}{\sigma_\epsilon^2} \mathbf{y} + \frac{1}{\sigma_\beta^2} (J_p \otimes I_q)^{-1} (\mu \mathbf{1}_m + \boldsymbol{\alpha} \otimes \mathbf{1}_q) \right) \right]} \\
& \cdot e^{-\frac{1}{2} \left[\frac{1}{\sigma_\beta^2} (\mu \mathbf{1}_m + \boldsymbol{\alpha} \otimes \mathbf{1}_q)^\top (J_p \otimes I_q)^{-1} (\mu \mathbf{1}_m + \boldsymbol{\alpha} \otimes \mathbf{1}_q) \right]} \\
\propto & \left| \left(\frac{1}{\sigma_\epsilon^2} I_m + \frac{1}{\sigma_\beta^2} (J_p \otimes I_q)^{-1} \right)^{-1} \right|^{\frac{1}{2}} (\sigma_\epsilon^2)^{-\frac{p}{2}} (\sigma_\beta^2)^{-\frac{q}{2}} \\
& \cdot e^{-\frac{1}{2\sigma_\epsilon^2} [\mathbf{y} - (\mu \mathbf{1}_m + \boldsymbol{\alpha} \otimes \mathbf{1}_q)]^\top \left(\frac{\sigma_\beta^2}{\sigma_\epsilon^2} (J_p \otimes I_q) + I_m \right)^{-1} [\mathbf{y} - (\mu \mathbf{1}_m + \boldsymbol{\alpha} \otimes \mathbf{1}_q)]} \\
\propto & |\sigma_\beta^2 (J_p \otimes I_q) + \sigma_\epsilon^2 I_m|^{-\frac{1}{2}} e^{-\frac{1}{2} [\mathbf{y} - (\mu \mathbf{1}_m + \boldsymbol{\alpha} \otimes \mathbf{1}_q)]^\top (\sigma_\beta^2 (J_p \otimes I_q) + \sigma_\epsilon^2 I_m)^{-1} [\mathbf{y} - (\mu \mathbf{1}_m + \boldsymbol{\alpha} \otimes \mathbf{1}_q)]}. \tag{2.2}
\end{aligned}$$

Now, it can be easily shown that

$$(\sigma_\beta^2 (J_p \otimes I_q) + \sigma_\epsilon^2 I_m)^{-1} = -\frac{\sigma_\beta^2}{\sigma_\epsilon^2 (p\sigma_\beta^2 + \sigma_\epsilon^2)} (J_p \otimes I_q) + \frac{1}{\sigma_\epsilon^2} I_m$$

and

$$|\sigma_\beta^2 (J_p \otimes I_q) + \sigma_\epsilon^2 I_m|^{-1} = (p\sigma_\beta^2 + \sigma_\epsilon^2)^q (\sigma_\epsilon^2)^{q(p-1)}. \tag{2.3}$$

Substituting (2.3) into (2.2), the conditional density of \mathbf{y} given μ , $\boldsymbol{\alpha}$, σ_β^2 , and σ_ϵ^2 is

$$\begin{aligned}
& f(\mathbf{y} | \mu, \boldsymbol{\alpha}, \sigma_\beta^2, \sigma_\epsilon^2) \\
\propto & (p\sigma_\beta^2 + \sigma_\epsilon^2)^{-\frac{q}{2}} (\sigma_\epsilon^2)^{-\frac{q(p-1)}{2}} e^{-\frac{1}{2\sigma_\epsilon^2} \sum_{i=1}^n \sum_{j=1}^m (y_{ij} - \mu_j)^2 + \frac{p^2 \sigma_\beta^2}{2\sigma_\epsilon^2 (p\sigma_\beta^2 + \sigma_\epsilon^2)} \sum_{i=1}^n (\bar{y}_{i\cdot} - \bar{\mu}_{\cdot})^2} \\
\propto & (p\sigma_\beta^2 + \sigma_\epsilon^2)^{-\frac{q}{2}} (\sigma_\epsilon^2)^{-\frac{q(p-1)}{2}} e^{-\frac{1}{2} [\mathbf{y} - \mu \mathbf{1}_m - (\boldsymbol{\alpha} \otimes \mathbf{1}_q)]^\top \left(\frac{1}{\sigma_\epsilon^2} I_m - \frac{\sigma_\beta^2}{\sigma_\epsilon^2 (p\sigma_\beta^2 + \sigma_\epsilon^2)} (J_p \otimes I_q) \right) [\mathbf{y} - \mu \mathbf{1}_m - (\boldsymbol{\alpha} \otimes \mathbf{1}_q)]}. \tag{2.4}
\end{aligned}$$

Then the posterior density of μ , $\boldsymbol{\alpha}$, σ_β^2 and σ_ϵ^2 is, from (2.4) and (c),

$$\begin{aligned}
& p(\mu, \boldsymbol{\alpha}, \sigma_\beta^2, \sigma_\epsilon^2 | \mathbf{y}) \\
\propto & (\sigma_\epsilon^2)^{-\frac{a(p-1)-a+b-1}{2}} (p\sigma_\beta^2 + \sigma_\epsilon^2)^{-\frac{a-b+4}{2}} \\
& \cdot e^{-\frac{1}{2} [\mathbf{y} - \mu \mathbf{1}_m - (\boldsymbol{\alpha} \otimes \mathbf{1}_q)]^\top \left(\frac{1}{\sigma_\epsilon^2} I_m - \frac{\sigma_\beta^2}{\sigma_\epsilon^2 (p\sigma_\beta^2 + \sigma_\epsilon^2)} (J_p \otimes I_q) \right) [\mathbf{y} - \mu \mathbf{1}_m - (\boldsymbol{\alpha} \otimes \mathbf{1}_q)]} \\
= & (\sigma_\epsilon^2)^{-\frac{a(p-1)-a+b-1}{2}} (p\sigma_\beta^2 + \sigma_\epsilon^2)^{-\frac{a-b+4}{2}} \\
& \cdot e^{-\frac{1}{2} \left[\frac{p\sigma_\beta^2}{p\sigma_\beta^2 + \sigma_\epsilon^2} \left(\mu - \frac{1}{p\sigma_\beta^2} (\mathbf{y} - (\boldsymbol{\alpha} \otimes \mathbf{1}_q))^\top \mathbf{1}_m \right)^2 - \frac{1}{p\sigma_\beta^2 + \sigma_\epsilon^2} ((\mathbf{y} - (\boldsymbol{\alpha} \otimes \mathbf{1}_q))^\top \mathbf{1}_m)^2 \right]} \\
& \cdot e^{-\frac{1}{2} \left[(\mathbf{y} - (\boldsymbol{\alpha} \otimes \mathbf{1}_q))^\top \left(\frac{1}{\sigma_\epsilon^2} I_m - \frac{\sigma_\beta^2}{\sigma_\epsilon^2 (p\sigma_\beta^2 + \sigma_\epsilon^2)} (J_p \otimes I_q) \right) (\mathbf{y} - (\boldsymbol{\alpha} \otimes \mathbf{1}_q)) \right]}. \tag{2.5}
\end{aligned}$$

Integrating with respect to μ in (2.5), it follows that the posterior density of $\boldsymbol{\alpha}$, σ_β^2 and σ_ϵ^2 is

$$\begin{aligned}
& p(\boldsymbol{\alpha}, \sigma_\beta^2, \sigma_\epsilon^2 | \mathbf{y}) \\
& \propto (\sigma_\epsilon^2)^{-\frac{q(p-1)-a+b}{2}-1} (p\sigma_\beta^2 + \sigma_\epsilon^2)^{-\frac{q-b+3}{2}} \\
& \quad \cdot e^{-\frac{1}{2} \left[(\mathbf{y} - (\boldsymbol{\alpha} \otimes \mathbf{1}_p))^T \left(\frac{1}{\sigma_\epsilon^2} I_p - \frac{\sigma_\beta^2}{\sigma_\epsilon^2(p\sigma_\beta^2 + \sigma_\epsilon^2)} (J_p \otimes I_p) - \frac{1}{p\sigma_\beta^2 + \sigma_\epsilon^2} J_p \right) (\mathbf{y} - (\boldsymbol{\alpha} \otimes \mathbf{1}_p)) \right]} \\
& \propto (\sigma_\epsilon^2)^{-\frac{q(p-1)-a+b}{2}-1} (p\sigma_\beta^2 + \sigma_\epsilon^2)^{-\frac{q-b+3}{2}} e^{-\frac{1}{2} \left[\frac{1}{\sigma_\epsilon^2} \sum_{i=1}^p \sum_{j=1}^q (y_{ij} - \alpha_i)^2 - \frac{p^2 \sigma_\beta^2}{\sigma_\epsilon^2(p\sigma_\beta^2 + \sigma_\epsilon^2)} \sum_{i=1}^p \overline{y}_{i.}^2 - \frac{pq}{p\sigma_\beta^2 + \sigma_\epsilon^2} \overline{y}_{..}^2 \right]} \\
& \propto (\sigma_\epsilon^2)^{-\frac{q(p-1)-a+b}{2}-1} (p\sigma_\beta^2 + \sigma_\epsilon^2)^{-\frac{q-b+3}{2}} e^{-\frac{1}{2\sigma_\epsilon^2} (S + SS_\alpha) - \frac{1}{2(p\sigma_\beta^2 + \sigma_\epsilon^2)} SS_\beta}, \tag{2.6}
\end{aligned}$$

where $S = \sum_{i=1}^p \sum_{j=1}^q (y_{ij} - \overline{y}_{i.} - \overline{y}_{.j} + \overline{y}_{..})^2$ is the error sum of squares, $SS_\alpha = q \sum_{i=1}^p (\alpha_i - \overline{y}_{i.} + \overline{y}_{..})^2$ is sum of squares due to fixed effects, and $SS_\beta = p \sum_{j=1}^q (\overline{y}_{.j} - \overline{y}_{..})^2$ is sum of squares due to random effects. Now, since $\sum_{i=1}^p \alpha_i = 0$,

$$\begin{aligned}
SS_\alpha &= q \left\{ \left[\boldsymbol{\alpha} - \begin{pmatrix} \overline{y}_{1.} & \overline{y}_{..} \\ \vdots & \vdots \\ \overline{y}_{p.} & \overline{y}_{..} \end{pmatrix} \right]^T \left[\boldsymbol{\alpha} - \begin{pmatrix} \overline{y}_{1.} & \overline{y}_{..} \\ \vdots & \vdots \\ \overline{y}_{p.} & \overline{y}_{..} \end{pmatrix} \right] \right\} \\
&= q \left\{ \left[\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -1 & -1 & \cdots & -1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_{p-1} \end{pmatrix} - \begin{pmatrix} \overline{y}_{1.} & \overline{y}_{..} \\ \vdots & \vdots \\ \overline{y}_{p.} & \overline{y}_{..} \end{pmatrix} \right]^T \left[\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -1 & -1 & \cdots & -1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_{p-1} \end{pmatrix} - \begin{pmatrix} \overline{y}_{1.} & \overline{y}_{..} \\ \vdots & \vdots \\ \overline{y}_{p.} & \overline{y}_{..} \end{pmatrix} \right] \right\} \\
&= q \left\{ \left[\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_{p-1} \end{pmatrix} - \begin{pmatrix} \overline{y}_{1.} & \overline{y}_{..} \\ \vdots & \vdots \\ \overline{y}_{p-1.} & \overline{y}_{..} \end{pmatrix} \right]^T \left(I_{p-1} - \frac{1}{p} J_{p-1} \right)^{-1} \left[\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_{p-1} \end{pmatrix} - \begin{pmatrix} \overline{y}_{1.} & \overline{y}_{..} \\ \vdots & \vdots \\ \overline{y}_{p-1.} & \overline{y}_{..} \end{pmatrix} \right] \right. \\
&\quad \left. + \left(\begin{pmatrix} \overline{y}_{1.} & \overline{y}_{..} \\ \vdots & \vdots \\ \overline{y}_{p.} & \overline{y}_{..} \end{pmatrix} \right)^T \left[I_p - \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -1 & -1 & \cdots & -1 \end{pmatrix} \begin{pmatrix} 2 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 1 \\ \vdots & \ddots & \ddots & \vdots \\ 1 & 1 & \cdots & 2 \end{pmatrix}^{-1} \begin{pmatrix} 1 & \cdots & 0 & -1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & -1 \\ 0 & \cdots & 1 & -1 \end{pmatrix} \begin{pmatrix} \overline{y}_{1.} & \overline{y}_{..} \\ \vdots & \vdots \\ \overline{y}_{p.} & \overline{y}_{..} \end{pmatrix} \right] \right\}.
\end{aligned}$$

In the above, we used the following matrix inverse:

$$\begin{pmatrix} 2 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 1 \\ \vdots & \ddots & \ddots & \vdots \\ 1 & 1 & \cdots & 2 \end{pmatrix}^{-1} = (I_{p-1} + J_{p-1})^{-1} = I_{p-1} - \frac{1}{p} J_{p-1}.$$

Next, Integrating with respect to $\boldsymbol{\alpha}$ in (2.6), it follows that the posterior density of σ_β^2 and σ_ϵ^2 is

$$\begin{aligned}
& p(\sigma_\beta^2, \sigma_\epsilon^2 | \mathbf{y}) \\
& \propto (\sigma_\epsilon^2)^{-\frac{q(p-1)-a+b}{2}-1} (p\sigma_\beta^2 + \sigma_\epsilon^2)^{-\frac{q-b+3}{2}} e^{-\frac{1}{2\sigma_\epsilon^2} S - \frac{1}{2(p\sigma_\beta^2 + \sigma_\epsilon^2)} SS_\beta} \int \dots \int e^{-\frac{1}{2\sigma_\epsilon^2} SS_\alpha} d\boldsymbol{\alpha} \\
& \propto (\sigma_\epsilon^2)^{-\frac{(p-1)(q-1)-a+b}{2}-1} (p\sigma_\beta^2 + \sigma_\epsilon^2)^{-\frac{q-b+3}{2}} e^{-\frac{1}{2\sigma_\epsilon^2} S - \frac{1}{2(p\sigma_\beta^2 + \sigma_\epsilon^2)} SS_\beta}. \tag{2.7}
\end{aligned}$$

Now we use the transformation $r = \frac{1}{\sigma_\epsilon^2}$ and $u = \frac{\sigma_\epsilon^2}{p\sigma_\beta^2 + \sigma_\epsilon^2}$. Then, from (2.7), we get the posterior density of R and U given by

$$p(r, u | \mathbf{y}) \propto r^{\frac{p(q-1)-a}{2}} u^{\frac{q-b-1}{2}} e^{-\frac{r}{2}(S+uSS_\beta)}. \quad (2.8)$$

From (2.8) we have

$$p(r|u, \mathbf{y}) = \frac{1}{\Gamma\left(\frac{p(q-1)-a}{2} + 1\right) 2^{\frac{p(q-1)-a}{2} + 1}} (S+uSS_\beta)^{\frac{p(q-1)-a}{2} + 1} r^{\frac{p(q-1)-a}{2}} e^{-\frac{r}{2}(S+uSS_\beta)}$$

and

$$\begin{aligned} p(u|\mathbf{y}) &\propto u^{\frac{q-b-1}{2}} \int_0^\infty r^{\frac{p(q-1)-a}{2}} e^{-\frac{r}{2}(S+uSS_\beta)} dr \\ &= u^{\frac{q-b-1}{2}} (S+uSS_\beta)^{-\frac{p(q-1)-a}{2}-1} \int_0^\infty y^{\frac{p(q-1)-a}{2}} e^{-\frac{y}{2}} dy \\ &= \Gamma\left(\frac{p(q-1)-a}{2} + 1\right) 2^{\frac{p(q-1)-a}{2} + 1} u^{\frac{q-b-1}{2}} (S+uSS_\beta)^{-\frac{p(q-1)-a}{2}-1}. \end{aligned} \quad (2.9)$$

Theorem 2.1 Under the loss (1.1) the hierarchical Bayes estimator of $\sigma_\epsilon^2 = \frac{1}{r}$ is given by

$$d_{a,b}^{HB}(\mathbf{y}) = \frac{S}{(p-1)(q-1)-a+b+2} [1 - \phi_{a,b}(V)], \quad (2.10)$$

$$\text{where } \phi_{a,b}(V) = \frac{2}{p(q-1)-a+4} \frac{V^{\frac{q-b-1}{2}} (1-V)^{\frac{(p-1)(q-1)-a+b+2}{2}}}{\int_0^V z^{\frac{q-b-1}{2}} (1-z)^{\frac{(p-1)(q-1)-a+b+2}{2}} dz} \quad \text{with } V = \frac{SS_\beta}{S+SS_\beta}.$$

Proof. Under the loss (1.1) the posterior risk of an estimator $d(\mathbf{y})$ of $\sigma_\epsilon^2 = \frac{1}{R}$ is given by

$$\begin{aligned} E[\sigma_\epsilon^{-4}(d(\mathbf{y}) - \sigma_\epsilon^2)^2 | \mathbf{y}] &= E\left[R^2\left(d(\mathbf{y}) - \frac{1}{R}\right)^2 | \mathbf{y}\right] \\ &= E(R^2 | \mathbf{y}) \left[d(\mathbf{y}) - \frac{E(R | \mathbf{y})}{E(R^2 | \mathbf{y})}\right]^2 + 1 - \frac{[E(R | \mathbf{y})]^2}{E(R^2 | \mathbf{y})}. \end{aligned}$$

Hence the posterior risk is minimized when

$$d(\mathbf{y}) = d_{a,b}^{HB}(\mathbf{y}) = \frac{E(R | \mathbf{y})}{E(R^2 | \mathbf{y})} = \frac{E\{E(R | u, \mathbf{y}) | \mathbf{y}\}}{E\{E(R^2 | u, \mathbf{y}) | \mathbf{y}\}}. \quad (2.11)$$

Now,

$$\begin{aligned} E(R | u, \mathbf{y}) &= \int_0^\infty \frac{1}{\Gamma\left(\frac{p(q-1)-a}{2} + 1\right) 2^{\frac{p(q-1)-a}{2} + 1}} (S+uSS_\beta)^{\frac{p(q-1)-a}{2} + 1} r^{\frac{p(q-1)-a}{2} + 1} e^{-\frac{r}{2}(S+uSS_\beta)} dr \\ &= \frac{1}{\Gamma\left(\frac{p(q-1)-a}{2} + 1\right) 2^{\frac{p(q-1)-a}{2} + 1}} (S+uSS_\beta)^{-1} \Gamma\left(\frac{p(q-1)-a}{2} + 2\right) 2^{\frac{p(q-1)-a}{2} + 2} \\ &= (p(q-1)-a+2)(S+uSS_\beta)^{-1}. \end{aligned} \quad (2.12)$$

Similarly

$$E(R^2 | u, \mathbf{y}) = (p(q-1) - a + 4)(p(q-1) - a + 2)(S + u SS_\beta)^{-2}. \quad (2.13)$$

From (2.11), (2.12), and (2.13), we have, with $w = \frac{SS_\beta}{S}$ and $\frac{uw}{1+uw} = z$,

$$\begin{aligned} \frac{E(R | \mathbf{y})}{E(R^2 | \mathbf{y})} &= \frac{1}{p(q-1) - a + 4} \frac{E[(S + u SS_\beta)^{-1} | \mathbf{y}]}{E[(S + u SS_\beta)^{-2} | \mathbf{y}]} \\ &= \frac{1}{p(q-1) - a + 4} \frac{\int_0^1 u^{\frac{q-b-1}{2}} (S + u SS_\beta)^{-\frac{p(q-1)-a}{2}-2} du}{\int_0^1 u^{\frac{q-b-1}{2}} (S + u SS_\beta)^{-\frac{p(q-1)-a}{2}-3} du} \\ &= \frac{S}{p(q-1) - a + 4} \left[1 + \frac{\int_0^v z^{\frac{q-b-1}{2}+1} (1-z)^{\frac{(p-1)(q-1)-a+b}{2}} dz}{\int_0^v z^{\frac{q-b-1}{2}} (1-z)^{\frac{(p-1)(q-1)-a+b+2}{2}} dz} \right]. \end{aligned} \quad (2.14)$$

Now, using integration by parts, we get

$$\begin{aligned} &\int_0^v z^{\frac{q-b-1}{2}+1} (1-z)^{\frac{(p-1)(q-1)-a+b}{2}} dz \\ &= -\frac{2}{(p-1)(q-1)-a+b+2} v^{\frac{q-b-1}{2}+1} (1-v)^{\frac{(p-1)(q-1)-a+b+2}{2}} \\ &\quad + \frac{q-b+1}{(p-1)(q-1)-a+b+2} \int_0^v z^{\frac{q-b-1}{2}} (1-z)^{\frac{(p-1)(q-1)-a+b+2}{2}} dz. \end{aligned} \quad (2.15)$$

Combining (2.14) and (2.15) we have

$$\begin{aligned} d_{a,b}^{HB}(\mathbf{y}) &= \frac{S}{p(q-1) - a + 4} \left[1 + \frac{q-b+1}{(p-1)(q-1) - a + b + 2} \right. \\ &\quad \left. - \frac{2}{(p-1)(q-1) - a + b + 2} \frac{v^{\frac{q-b+1}{2}} (1-v)^{\frac{(p-1)(q-1)-a+b+2}{2}}}{\int_0^v z^{\frac{q-b-1}{2}} (1-z)^{\frac{(p-1)(q-1)-a+b+2}{2}} dz} \right] \\ &= \frac{S}{(p-1)(q-1) - a + b + 2} [1 - \phi_{a,b}(v)] \end{aligned}$$

$$\text{where } \phi_{a,b}(v) = \frac{2}{p(q-1) - a + 4} \frac{v^{\frac{q-b+1}{2}} (1-v)^{\frac{(p-1)(q-1)-a+b+2}{2}}}{\int_0^v z^{\frac{q-b-1}{2}} (1-z)^{\frac{(p-1)(q-1)-a+b+2}{2}} dz}.$$

Hence the proof is completed.

3. Exact Expressions for the Risk Difference of Estimators

We first provide the exact expressions for the risk difference of $d_{a,b}^{HB}(\mathbf{y})$ and $d_{a,b}(S) = \frac{1}{(p-1)(q-1) - a + b + 2} S$ under the loss (1.1).

Theorem 3.1 Under the loss (1.1)

$$\begin{aligned} & E_{\theta, \sigma_\epsilon^2} [L(d_{a,b}(S), \sigma_\epsilon^2)] - E_{\theta, \sigma_\epsilon^2} [L(d_{a,b}^{HB}(\mathbf{y}), \sigma_\epsilon^2)] \\ &= E[(p(q-1) + 2L)(p(q-1) + 2L + 2)C_{L,a,b}], \end{aligned} \quad (3.1)$$

where

$$\begin{aligned} C_{L,a,b} = & E \left[\frac{(1-V)\phi_{a,b}(V)}{((p-1)(q-1)-a+b+2)(p(q-1)-a+4)} \times \right. \\ & \left\{ (2L+a-2) \left(\frac{2}{p(q-1)+2L+2} - \frac{1-V}{p(q-1)-a+b+2} \right) \right. \\ & \left. \left. + \frac{a-b}{(p-1)(q-1)-a+b+2} \right\} \mid L \right], \end{aligned} \quad (3.2)$$

$$\text{where } L \sim \text{Poisson} \left(\frac{\sum_{i=1}^p \sum_{j=1}^q (\theta_{ij} - \bar{\theta}_{..})^2}{2\sigma_\epsilon^2} \right), \text{ and } V \mid L \sim \text{Beta} \left(\frac{q-1+2L}{2}, \frac{(p-1)(q-1)}{2} \right).$$

Proof. Let $G = T + S$ where $T = SS_a$ and $S = SS_\epsilon$ so that $V = \frac{T}{G}$. Write $d \equiv d(a, b) = (p-1)(q-1) - a + b + 2$. Then $d_{a,b}(S) = G\psi_{a,b}(V)$ and $d_{a,b}^{HB}(\mathbf{y}) = G\psi_{a,b}^*(V)$ where $\psi_{a,b}(V) = \frac{1-V}{d}$ and $\psi_{a,b}^*(V) = \psi_{a,b}(V)(1 - \phi_{a,b}(V))$.

Now,

$$E_{\theta, \sigma_\epsilon^2} [\sigma^{-4}(G\psi(V) - \sigma_\epsilon^2)^2] = E_{\frac{\theta}{\sigma_\epsilon}, 1} [(G\psi(V) - 1)^2]$$

Write $\lambda = \frac{1}{2\sigma_\epsilon^2} \sum_{i=1}^p \sum_{j=1}^q (\theta_{ij} - \bar{\theta}_{..})^2$. Under the reparametrization $\left(\frac{\theta}{\sigma_\epsilon}, 1 \right)$, S and T are independently distributed with $S \sim \chi^2_{(p-1)(q-1)}$ and $T \sim \chi^2_{q-1}(\lambda)$. Consider the dummy random variable $L \sim \text{Poisson}(\lambda)$. Then it can be shown that conditionally on L , V and G are independent with

$$V \mid L \sim \text{Beta} \left(\frac{q-1+2L}{2}, \frac{(p-1)(q-1)}{2} \right) \text{ and } G \mid L \sim \chi^2_{\lambda(q-1)+2L}.$$

Hence

$$\begin{aligned} & E_{\frac{\theta}{\sigma_\epsilon}, 1} [(G\psi(V) - 1)^2 \mid L] \\ &= E_{\frac{\theta}{\sigma_\epsilon}, 1} [G^2 \mid L] E_{\frac{\theta}{\sigma_\epsilon}, 1} [\psi^2(V) \mid L] - 2E_{\frac{\theta}{\sigma_\epsilon}, 1} [G \mid L] E_{\frac{\theta}{\sigma_\epsilon}, 1} [\psi(V) \mid L] + 1 \\ &= (p(q-1) + 2L)(p(q-1) + 2L + 2) E \left[\left(\psi(V) - \frac{1}{p(q-1) + 2L + 2} \right)^2 \mid L \right] \\ & \quad + 1 - \frac{p(q-1) + 2L}{p(q-1) + 2L + 2}. \end{aligned} \quad (3.3)$$

Hence we evaluate, using (3.3),

$$\begin{aligned} & E[(G\psi_{a,b}(V) - 1)^2 \mid L] - E[G\psi_{a,b}^*(V) - 1]^2 \mid L] \\ &= (p(q-1) + 2L)(p(q-1) + 2L + 2) E \left[\left(\frac{(1-V)^2}{d^2} (2\phi_{a,b}(V) - \phi_{a,b}^2(V)) \right. \right. \\ & \quad \left. \left. - \frac{2(1-V)}{p(q-1) + 2L + 2} \phi_{a,b}(V) \right) \mid L \right]. \end{aligned} \quad (3.4)$$

Now,

$$\begin{aligned} \dot{\phi}_{a,b}(v) \\ = & \left(\frac{(q-1)-b+2}{2v} - \frac{(p-1)(q-1)-a+b+2}{2(1-v)} \right) \dot{\phi}_{a,b}(v) - \frac{p(q-1)-a+4}{2v} \phi_{a,b}^2(v). \end{aligned} \quad (3.5)$$

Hence we have, from (3.5),

$$\begin{aligned} & \frac{2v}{p(q-1)-a+4} \dot{\phi}_{a,b}(v) \\ & = \frac{(q-1)-b+2-(p(q-1)-a+4)v}{(1-v)(p(q-1)-a+4)} \phi_{a,b}(v) - \phi_{a,b}^2(v) \end{aligned}$$

and therefore

$$\begin{aligned} & 2\phi_{a,b}(v) - \phi_{a,b}^2(v) \\ & = \frac{[2(p(q-1)-a+4)-(q-1)-b+2]-(p(q-1)-a+4)v}{(1-v)(p(q-1)-a+4)} \phi_{a,b}(v) \\ & \quad + \frac{2v\dot{\phi}_{a,b}(v)}{p(q-1)-a+4}. \end{aligned} \quad (3.6)$$

Hence substituting (3.6) into (3.4) gives

$$\begin{aligned} & E[(G\phi_{a,b}(V)-1)^2|L] - E[(G\phi_{a,b}^*(V)-1)^2|L] \\ & = (p(q-1)+2L)(p(q-1)+2L+2) \\ & \quad \cdot E\left[\frac{(1-V)^2}{d^2}\left\{\frac{[2(p(q-1)-a+4)-(q-1)-b+2]-(p(q-1)-a+4)V}{(1-V)(p(q-1)-a+4)} \phi_{a,b}(V)\right.\right. \\ & \quad \left.\left. + \frac{2V\dot{\phi}_{a,b}(V)}{p(q-1)-a+4}\right\} - \frac{2(1-V)}{(p(q-1)+2L+2)d} \phi_{a,b}(V)\right|L]. \end{aligned} \quad (3.7)$$

Now, by integration by parts, we have

$$\begin{aligned} & E[V(1-V)^2\dot{\phi}_{a,b}(V)|L] \\ & = E\left[\left\{\frac{(p-1)(q-1)+2}{2} V(1-V) - \frac{(q-1)+4}{2} (1-V)^2\right\} \phi_{a,b}(V)\right|L]. \end{aligned} \quad (3.8)$$

Substituting (3.8) into (3.7) we get

$$\begin{aligned} & E[(G\phi_{a,b}(V)-1)^2|L] - E[(G\phi_{a,b}^*(V)-1)^2|L] \\ & = (p(q-1)+2L)(p(q-1)+2L+2) \\ & \quad \cdot E\left[\frac{(1-V)^2}{d^2}\left\{\frac{[2(p(q-1)-a+4)-(q-1)-b+2]-(p(q-1)-a+4)V}{(1-V)(p(q-1)-a+4)} \phi_{a,b}(V)\right.\right. \\ & \quad \left.\left. + \frac{2}{d^2} \frac{1}{p(q-1)-a+4} \left\{\frac{(p-1)(q-1)+2}{2} V(1-V) - \frac{(q-1)+2L}{2} (1-V)^2\right\} \phi_{a,b}(V)\right.\right. \\ & \quad \left.\left. - \frac{2(1-V)}{(p(q-1)+2L+2)d} \phi_{a,b}(V)\right\}\right|L] \end{aligned}$$

$$= (p(q-1) + 2L)(p(q-1) + 2L + 2) \\ E \left[\frac{1-V}{d} \phi_{a,b}(V) \frac{1}{p(q-1)-a+4} \left\{ (2L+a-2) \frac{2}{p(q-1)+2L+2} - \frac{1-V}{d} \right\} + \frac{a-b}{d} \right] | L] .$$

Hence we complete the proof.

4. Minimaxity and Non-minimaxity

In this section, we only consider the special case $a=b$ and find conditions on a under which $d_{a,a}^{HB}(\mathbf{y})$ dominates $d_{a,a}(S) = \frac{1}{(p-1)(q-1)+2} S$ which is known to be minimax.

Theorem 4.1 $d_{a,a}^{HB}(\mathbf{y})$ is a minimax estimator of σ_e^2 under the loss (1.1) if $2 \leq a < q+1$ and is nonminimax when $0 \leq a < 2$.

Proof. From (3.1) and (3.2),

$$E_{\theta, \sigma_e^2} [L(d_{a,a}(S), \sigma_e^2)] - E_{\theta, \sigma_e^2} [L(d_{a,a}^{HB}(\mathbf{y}), \sigma_e^2)] \\ = E[(p(q-1) + 2L)(p(q-1) + 2L + 2)C_{L,a,a}], \quad (4.1)$$

where for $0 \leq a < q+1$,

$$C_{L,a,a} = E \left[\frac{(1-V)\phi_{a,a}(V)}{((p-1)(q-1)+2)(p(q-1)-a+4)} \times \right. \\ \left. (2L+a-2) \left(\frac{2}{p(q-1)+2L+2} - \frac{1-V}{p(q-1)+2} \right) | L \right]. \quad (4.2)$$

From (4.2), $C_{0,2,2} = 0$.

We first claim that

$$C_{l,2,2} > 0 \quad \text{for all } l=1, 2, \dots, \\ C_{l,a,a} > 0 \quad \text{for all } a \in (2, q+1), \text{ and for all } l=0, 1, 2, \dots, \\ \text{and } C_{0,a,a} < 0 \quad \text{for all } 0 \leq a < 2.$$

To prove the Claim, it suffices to show that

$$E \left[(1-V)\phi_{a,a}(V) \left\{ \frac{2}{p(q-1)+2L+2} - \frac{1-V}{p(q-1)+2} \right\} | L = l \right] > 0 \quad (4.3)$$

for $2 \leq a < q+1, l=0, 1, 2, \dots$. Now, the proof of (4.3) is omitted since it is similar to those in Ghosh(1994) and Datta and Ghosh(1995). This leads to the fact that $d_{a,a}^{HB}(\mathbf{y})$ dominates (in frequentist risk) the constant risk minimax estimator $\frac{1}{(p-1)(q-1)+2} S$ of σ^2 when $2 \leq a < q+1$. On the other hand, minimaxity fails when $0 \leq a < 2$, since if $\theta_{11} = \dots = \theta_{pq}$ so that $P(L=0)=1$, then $P(C_{L,a,a} < 0) = P(C_{0,a,a} < 0) = 1$ for $0 < a < 2$, and hence the risk difference in (4.1) will be negative due to $C_{0,a,a} < 0$ with probability 1. This completes the proof.

5. Numerical Computations

We now find numerically percentage risk improvement(PRI) of $d_{a,a}^{HB}(\mathbf{y})$ over $\widehat{\sigma}_M^2 = d_{a,a}(S)$ which is defined by $PRI(d_{a,a}^{HB}(\mathbf{y}), d_{a,a}(S)) = \frac{[R(d_{a,a}, \sigma_\varepsilon^2) - R(d_{a,a}^{HB}, \sigma_\varepsilon^2)]}{R(d_{a,a}, \sigma_\varepsilon^2)} \times 100$, where $R(d, \sigma_\varepsilon^2)$ denotes the risk of an estimator d of σ_ε^2 . Note that $R(d_{a,a}, \sigma_\varepsilon^2) = R(\widehat{\sigma}_M, \sigma_\varepsilon^2) = \frac{2}{(p-1)(q-1)+2}$. As found in Theorem 3.1, the percentage risk improvement depends on θ and σ_ε^2 only through $\lambda = \frac{1}{2\sigma_\varepsilon^2} \sum_{i=1}^p \sum_{j=1}^q (\theta_{ij} - \bar{\theta}_{..j})^2$.

Figure 5.1 considers graphs of PRI for $p=4$ and $q=7$ when $a=0,1,2,\dots,7$, and values of PRI are given in Table 5.1. As proved in Theorem 4.1, $d_{0,0}^{HB}(\mathbf{y})$ and $d_{1,1}^{HB}(\mathbf{y})$ is non-minimax, that is, it is possible to have negative risk improvement over $\widehat{\sigma}_M$ (since $a<2$). However, in spite of the non-minimaxity of $d_{0,0}^{HB}(\mathbf{y})$ and $d_{1,1}^{HB}(\mathbf{y})$, they can perform much better than minimax estimators for a wide range of values of λ . Finally, when $\lambda \rightarrow \infty$, the PRI's of all the hierarchical Bayes estimators seem to be very close.

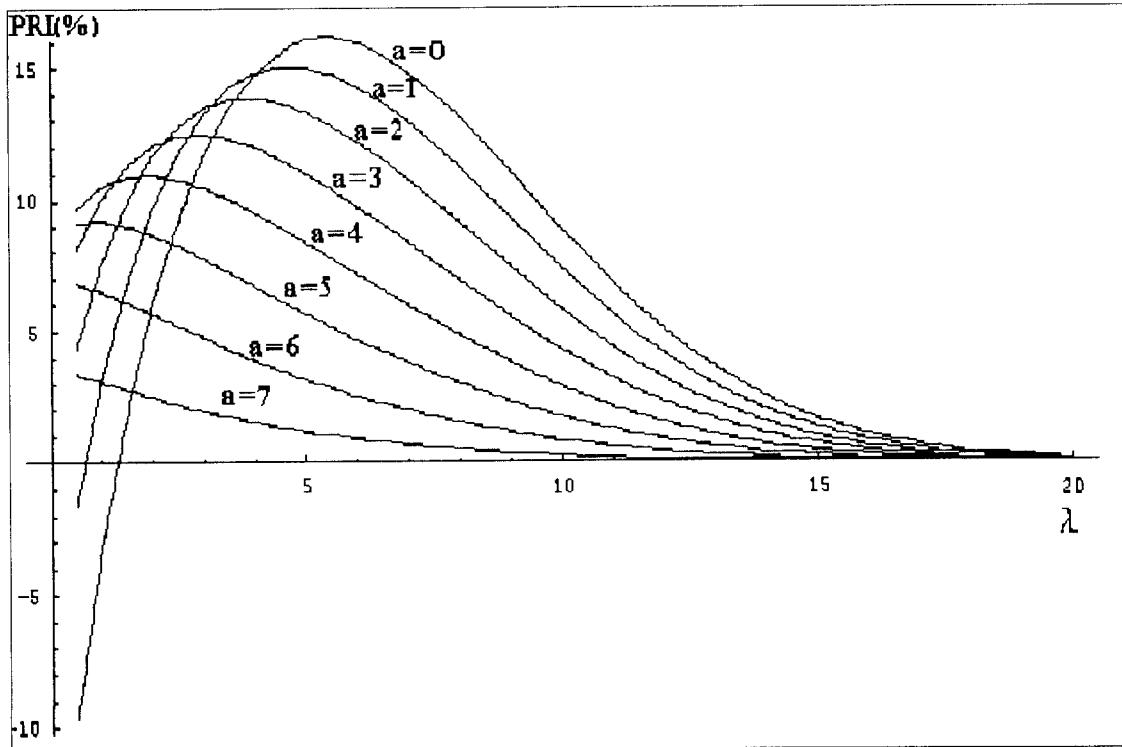


Figure 5.1. Graphs of PRI for $p=4$ and $q=7$.

Table 5.1. Values of PRI for p=4 and q=7.

λ	a=0	a=1	a=2	a=3	a=4	a=5	a=6	a=7
0.5	-9.66	-1.64	4.33	8.13	9.70	9.14	6.80	3.37
1.0	-3.18	3.17	7.60	10.03	10.50	9.18	6.48	3.07
1.5	2.01	6.90	9.99	11.28	10.87	8.99	6.07	2.77
2.0	6.12	9.72	11.68	12.04	10.94	8.66	5.63	2.48
2.5	9.32	11.80	12.80	12.40	10.78	8.23	5.18	2.21
3.0	11.74	13.28	13.48	12.47	10.45	7.73	4.73	1.97
3.5	13.53	14.25	13.80	12.32	10.01	7.21	4.30	1.74
4.0	14.78	14.82	13.84	11.99	9.50	6.68	3.89	1.54
4.5	15.58	15.06	13.66	11.54	8.94	6.15	3.50	1.35
5.0	15.99	15.02	13.30	11.00	8.35	5.63	3.15	1.19
5.5	16.08	14.76	12.80	10.40	7.75	5.14	2.82	1.04
6.0	15.89	14.31	12.20	9.74	7.15	4.66	2.51	0.91
6.5	15.46	13.70	11.50	9.06	6.55	4.21	2.23	0.80
7.0	14.84	12.96	10.75	8.35	5.96	3.78	1.98	0.70
7.5	14.05	12.13	9.94	7.64	5.39	3.37	1.74	0.60
8.0	13.14	11.23	9.11	6.93	4.84	3.00	1.53	0.52
8.5	12.13	10.28	8.26	6.23	4.31	2.64	1.33	0.45
9.0	11.07	9.31	7.43	5.55	3.81	2.32	1.16	0.39
9.5	9.99	8.34	6.61	4.91	3.34	2.01	1.00	0.33
10.0	8.91	7.39	5.82	4.30	2.91	1.74	0.85	0.28
10.5	7.86	6.49	5.08	3.73	2.51	1.49	0.73	0.24
11.0	6.86	5.64	4.40	3.21	2.15	1.27	0.62	0.20
11.5	5.93	4.85	3.77	2.74	1.82	1.07	0.52	0.17
12.0	5.07	4.13	3.20	2.32	1.54	0.90	0.43	0.14
12.5	4.30	3.49	2.69	1.94	1.28	0.75	0.36	0.11
13.0	3.61	2.92	2.25	1.62	1.06	0.62	0.29	0.09
13.5	3.00	2.43	1.86	1.34	0.88	0.51	0.24	0.08
14.0	2.48	2.00	1.53	1.09	0.72	0.41	0.19	0.06
14.5	2.03	1.63	1.25	0.89	0.58	0.33	0.16	0.05
15.0	1.65	1.32	1.01	0.72	0.47	0.27	0.13	0.04
15.5	1.33	1.07	0.81	0.58	0.37	0.21	0.10	0.03
16.0	1.06	0.85	0.65	0.46	0.30	0.17	0.08	0.02
16.5	0.85	0.68	0.51	0.36	0.24	0.13	0.06	0.02
17.0	0.67	0.53	0.40	0.29	0.18	0.11	0.05	0.01
17.5	0.52	0.42	0.32	0.22	0.14	0.08	0.04	0.01
18.0	0.41	0.33	0.25	0.17	0.11	0.06	0.03	0.01
18.5	0.32	0.25	0.19	0.13	0.09	0.05	0.02	0.01
19.0	0.24	0.19	0.15	0.10	0.07	0.04	0.02	0.01
19.5	0.19	0.15	0.11	0.08	0.05	0.03	0.01	0.00
20.0	0.14	0.11	0.09	0.06	0.04	0.02	0.01	0.00

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