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Mixed Replacement Designs for Life Testing with Interval Censoring[1]

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Abstract

The estimation of mean lifetimes in presence of interval censoring with mixed replacement procedure are examined when the distributions of lifetimes are exponential. It is assumed that, due to physical restrictions and/or economic constraints, the number of failures is investigated only at several inspection times during the lifetime test; thus there is interval censoring. Comparisons of mixed replacement designs are made with those with and without replacement. The maximum likelihood estimator is found in an implicit form. The Cramer-Rao lower bound, which is the asymptotic variance of the estimator, is derived. The test conditions for minimizing the Cramer-Rao lower bound and minimizing the test costs within a desired width of the Cramer-Rao lower bound have been studied.

1. Introduction

With the development of reliability, a lot of time and cost are involved in the estimation of lifetimes of products. In case of higher reliability products, especially, it may be impossible to observe the lifetime in the usual manner. Thus various methods such as censorings (Boardman, 1973) and accelerated testings (Nelson, 1990) are introduced in lifetime tests. Life testing procedures with interval censoring procedures can be divided into two classes: With replacement - in which the failures are replaced at each inspection time and Without replacement (Wei and Bau, 1987) - in which the failures are not replaced. In general, it is well known that with replacement procedure has better accuracy than without replacement one. In the with replacement procedure, however, one needs to prepare enough test items for replacement of failures at each inspection time. It is difficult to adopt the with replacement procedure in practice because one does not know exactly the number of items that will be required during the total time of test, which could be excessively large.

The practical difficulties in the with replacement procedure have the following two sides. First, the test may become too costly due to the overestimation of the number of failures. On the other hand, test items may run out before predetermined test time if the failures are

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underestimated. In the latter case, the test itself is terminated before predetermined test time.

In this paper an interval censoring model with mixed replacement procedure is examined which includes without replacement and with replacement procedure as its special cases. That is, we adopt with replacement procedure at the beginning of the test, and adopt without replacement one starting from the arbitrary but nonrandom inspection time. At first, we try to find the estimator for mean lifetime with the mixed replacement procedure. Optimal test designs for minimizing the asymptotic variance of the estimator and minimizing test cost with the mixed replacement procedure are studied, subsequently.

2. Parameter Estimation

2.1 Mixed Replacement Model

An interval censoring model with mixed replacement procedure in this paper may be described as follows. At first, n test items are placed on lifetime test and failures are observed at several arbitrary inspection times τ_j, $j=1,\cdots,I$. At each inspection time, failures are replaced by new ones till the arbitrary inspection time τ_K which runs out test items for further replacement. The lifetime test is continued for the prespecified time T, so that

$$0 = \tau_0 < \tau_1 < \cdots < \tau_K < \cdots < \tau_I = T .$$

Thus the mixed replacement procedure which is introduced in this paper can be the without replacement procedure if $K=0$ and can be the with replacement one if $K=I-1$.

Consider I test intervals Δ_j's which have magnitude

$$\Delta_j = \tau_j - \tau_{j-1}, \quad j=1,\cdots,I, \tag{2.1}$$

respectively.

Let the observed number of failures during the *jth* interval (2.1) be r_j and the number of test items unfailed till the last inspection time τ_I be r_{I+1}, then we have

$$r_{K+1} + r_{K+2} + \cdots\cdots + r_I + r_{I+1} = n .$$

Test items n_j which are placed at the beginning of the *jth* interval are all n in replaced intervals and are reduced with failures after $(K+1)$*th* inspection time, that is,

$$n_j = \begin{cases} n , & 1 \le j \le K+1 \\ n - \sum_{i=K+1}^{j-1} r_i , & K+2 \le j \le I . \end{cases}$$

2.2 Maximum Likelihood Estimator

The lifetimes of test items are assumed to be exponentially and independently distributed with density function

$$f(x) = \frac{1}{\theta} \exp\left(-\frac{x}{\theta} \right), \quad x > 0 , \tag{2.2}$$

where θ is an unknown positive parameter.

The likelihood function for θ in interval censoring with mixed replacement procedure can be divided into two parts

$$L = L_1 \times L_2 , \tag{2.3}$$

where L_1 is likelihood for replaced intervals and L_2 is likelihood for intervals not replaced.

The probabilities that failures happen in replaced intervals, that is, the probability that r_j failures are observed at the *jth* interval has the form of binomial distribution with parameters n and $1 - \exp(-\Delta_j/\theta)$ given as follows;

$$\frac{n!}{r_j!(n - r_j)!} \left\{ 1 - \exp\left(-\frac{\Delta_j}{\theta} \right) \right\}^{r_j} \left\{ \exp\left(-\frac{\Delta_j}{\theta} \right) \right\}^{n-r_j} .$$

So the likelihood function for replaced intervals is described

$$L_1 = \prod_{j=1}^{K} \frac{n!}{r_j!\,(n-r_j)!} \left\{ 1 - \exp\left(-\frac{\Delta_j}{\theta} \right) \right\}^{r_j} \left\{ \exp\left(-\frac{\Delta_j}{\theta} \right) \right\}^{n-r_j} . \tag{2.4}$$

The likelihood function for intervals not replaced has the form of multinomial distribution given as follows;

$$L_2 = \frac{n!}{r_{K+1}! \cdots r_{I+1}!} \prod_{j=K+1}^{I+1} P_j^{r_j} , \tag{2.5}$$

where

$$P_j = \exp\left(- \frac{\tau_{j-1} - \tau_K}{\theta} \right)\left\{1 - \exp\left(- \frac{\Delta_j}{\theta} \right)\right\}, \quad j = K+1, \cdots, I$$

and $$P_{I+1} = \exp\left(- \frac{\tau_I - \tau_K}{\theta} \right) .$$

According to (2.4) and (2.5), the likelihood function for mixed replacement procedure (2.3) becomes

$$L = \prod_{j=1}^{K} \frac{n!}{r_j!\,(n - r_j)!} \left\{1 - \exp\left(- \frac{\Delta_j}{\theta} \right)\right\}^{r_j} \left\{\exp\left(- \frac{\Delta_j}{\theta} \right)\right\}^{n-r_j} \frac{n!}{r_{K+1}! \cdots r_{I+1}!} \prod_{j=K+1}^{I+1} P_j^{r_j} . \tag{2.6}$$

Differentiating the log likelihood function with respect to θ, we get the first derivative

$$\frac{\partial \ln L}{\partial \theta} = -\frac{1}{\theta^2} \sum_{j=1}^{I} \frac{r_j \Delta_j \exp\left(- \frac{\Delta_j}{\theta} \right)}{1 - \exp\left(- \frac{\Delta_j}{\theta} \right)} + \frac{1}{\theta^2} \sum_{j=1}^{K} (n - r_j)\,\Delta_j + \frac{1}{\theta^2} \sum_{j=K+1}^{I} r_j(\tau_{j-1} - \tau_K) + \frac{r_{I+1}(\tau_I - \tau_K)}{\theta^2} . \tag{2.7}$$

Then the likelihood equation for estimating θ can be derived from (2.7) as

$$\sum_{j=1}^{I} \frac{r_j \Delta_j}{1 - \exp\left(- \frac{\Delta_j}{\hat{\theta}} \right)} - n\tau_K - \sum_{j=K+1}^{I} r_j(\tau_j - \tau_K) - r_{I+1}(\tau_I - \tau_K) = 0 . \tag{2.8}$$

Though the above maximum likelihood estimator has implicit form, it can be solved by using iterative methods such as Newton-Raphson's algorithm or EM algorithm suggested by Dempster et al.(1977) and it can be shown that the solution is unique.

2.3 Asymptotic Variance of the Estimator

As a prerequisite for the discussions provided in section 3 and 4, let us find the asymptotic variance of estimator. The Cramer-Rao lower bound can be substituted for asymptotic variance, considering the form of the estimator derived earlier.

From the first derivative, we get the second derivative

$$\frac{\partial^2 \ln L}{\partial \theta^2} = \frac{2}{\theta^3}\sum_{j=1}^{I}\frac{r_j \Delta_j}{1-\exp\left(-\frac{\Delta_j}{\theta}\right)} - \frac{1}{\theta^4}\sum_{j=1}^{I}\frac{r_j \Delta_j^2 \exp\left(-\frac{\Delta_j}{\theta}\right)}{\left\{1-\exp\left(-\frac{\Delta_j}{\theta}\right)\right\}^2} - \frac{2n\tau_K}{\theta^3}$$
$$- \frac{2}{\theta^3}\sum_{j=K+1}^{I} r_j(\tau_j - \tau_K) - \frac{2r_{I+1}(\tau_I - \tau_K)}{\theta^3} . \tag{2.9}$$

The expected value of r_j which is the observed number of failures during the jth interval and of r_{I+1} which is the number of unfailed items till τ_I, is given as follows;

$$E(r_j) = \begin{cases} n\left\{1-\exp\left(-\frac{\Delta_j}{\theta}\right)\right\}, & j=1,\cdots,K \\ n\exp\left(-\frac{\tau_{j-1}-\tau_K}{\theta}\right)\left\{1-\exp\left(-\frac{\Delta_j}{\theta}\right)\right\}, & j=K+1,\cdots,I \end{cases}$$

$$E(r_{I+1}) = n\exp\left(-\frac{\tau_I-\tau_K}{\theta}\right) .$$

Then, the Cramer-Rao lower bound of the estimator becomes

$$CRLB(\hat{\theta}) = \frac{\theta^4}{n}\left[\sum_{j=1}^{K}\frac{\Delta_j^2}{\exp\left(\frac{\Delta_j}{\theta}\right)-1} + \sum_{j=K+1}^{I}\frac{\Delta_j^2\exp\left(-\frac{\tau_{j-1}-\tau_K}{\theta}\right)}{\exp\left(\frac{\Delta_j}{\theta}\right)-1}\right]^{-1} . \tag{2.10}$$

3. Test Designs for Minimum Variance

We now derive the test conditions for minimizing the Cramer-Rao lower bound of the maximum likelihood estimator and for minimizing the test costs in interval censoring with mixed replacement procedure.

3.1 Optimal Interval-Size

From the Cramer-Rao lower bound derived in section 2, it is evident that it is a function of test items n, lifetime parameter θ, jth interval-size Δ_j, the number of total intervals I, and the number of replaced intervals K. An optimization of the test design, thus, means the determination of Δ_j with respect to the various values of K given n, θ and I for minimizing the Cramer-Rao lower bound. Let us begin by assuming that all of intervals have equal size Δ.

The Cramer-Rao lower bound for the case which has all the interval-size equal can be deduced from (2.10). It is given as follows;

$$CRLB(\hat{\theta}) = \frac{\theta^4}{n\Delta^2} \cdot \frac{\exp\left(\frac{\Delta}{\theta}\right) - 2 + \exp\left(-\frac{\Delta}{\theta}\right)}{K\left\{1 - \exp\left(-\frac{\Delta}{\theta}\right)\right\} + 1 - \exp\left(-\frac{(I-K)\Delta}{\theta}\right)} . \tag{3.1}$$

Differentiating (3.1) with respect to Δ, we get

$$\frac{\partial CRLB(\hat{\theta})}{\partial \Delta} = \frac{\theta^4}{n\Delta^3\left[K\left\{1 - \exp\left(-\frac{\Delta}{\theta}\right)\right\} + 1 - \exp\left(-\frac{(I-K)\Delta}{\theta}\right)\right]^2} \times$$

$$\left[\frac{\Delta}{\theta}\left\{\exp\left(\frac{\Delta}{\theta}\right) - \exp\left(-\frac{\Delta}{\theta}\right)\right\}\left[K\left\{1 - \exp\left(-\frac{\Delta}{\theta}\right)\right\} + \left\{1 - \exp\left(-\frac{(I-K)\Delta}{\theta}\right)\right\}\right]\right.$$

$$- \left\{\exp\left(\frac{\Delta}{\theta}\right) - 2 + \exp\left(-\frac{\Delta}{\theta}\right)\right\}\left[2K\left\{1 - \exp\left(-\frac{\Delta}{\theta}\right)\right\} + 2\left\{1 - \exp\left(-\frac{(I-K)\Delta}{\theta}\right)\right\}\right.$$

$$\left.\left. + K\frac{\Delta}{\theta}\exp\left(-\frac{\Delta}{\theta}\right) + \frac{(I-K)\Delta}{\theta}\exp\left(-\frac{(I-K)\Delta}{\theta}\right)\right]\right] . \tag{3.2}$$

We can prove from (3.2) the optimal interval-size is unique since the second derivative can be shown to be always positive.

Let t, the index of interval-size, be Δ/θ. Then (3.2) can be described as an expression involving t:

$$[t\{\exp(t) - \exp(-t)\} - 2\{\exp(t) - 2 + \exp(-t)\}]\,[K\{1 - \exp(-t)\} + 1 - \exp(-(I-K)t)]$$

$$- [\exp(t) - 2 + \exp(-t)]\,[K\,t\exp(-t) + (I-K)\,t\exp(-(I-K)\,t)] = 0 , \tag{3.3}$$

where $t = \Delta/\theta$.

The optimal indices can be solved by using the Newton-Raphson's iterative method for each pair of values I and K from 0 to 10 respectively. The results are shown in Table 1.

Table 1. Optimal Indices of Interval-Size

	K									
I	0	1	2	3	4	5	6	7	8	9
1	1.594									
2	1.207	1.594								
3	0.990	1.383	1.594							
4	0.847	1.267	1.451	1.594						
5	0.746	1.205	1.378	1.486	1.594					
6	0.669	1.174	1.346	1.434	1.507	1.594				
7	0.608	1.161	1.333	1.413	1.467	1.521	1.594			
8	0.559	1.155	1.329	1.405	1.451	1.488	1.532	1.594		
9	0.518	1.153	1.327	1.403	1.446	1.476	1.504	1.539	1.594	
10	0.483	1.152	1.327	1.402	1.445	1.473	1.494	1.515	1.545	1.594

In the case of $K=0$, the optimal indices of interval-size decrease from 1.594 to 0.483 as the total number of intervals I increases. It is equivalent to the results of Nelson(1977) which has studied the without replacement procedure. The optimal indices are all 1.594 in cases of with replacement procedures, i.e., $K=I-1$, regardless of the values of I. We can see that t increases as K increases under given I, and increases as I decreases under given K.

3.2 Relative Information

One can compare the relative accuracy of interval censored data with mixed replacement procedure to the complete data. From the Cramer-Rao lower bounds of interval censoring with mixed replacement and that for continuous inspection without replacement, we get the following ratio. It is defined as the relative information in Kulldorff(1961).

$$\frac{CRLB(Cont.\ without\ repl.)}{CRLB(Mixed)} = \frac{t^2\,[K\{1-\exp(-t)\}+1-\exp\{-(I-K)\,t\}]}{\{\exp(t)-2+\exp(-t)\}\,\{1-\exp(-It)\}}\ . \tag{3.4}$$

Applying the optimal indices of interval-size given in Table 1, the relative information of interval censoring with mixed replacement compared to the continuous model without replacement has been derived. It is presented in Table 2.

Table 2. Relative Information of Interval Censoring with Mixed Replacement Procedure to Continuous Model Without Replacement Procedure

	K									
I	0	1	2	3	4	5	6	7	8	9
1	0.813									
2	0.887	1.351								
3	0.922	1.465	1.959							
4	0.942	1.496	2.090	2.595						
5	0.955	1.505	2.124	2.731	3.239					
6	0.964	1.508	2.133	2.765	3.376	3.886				
7	0.970	1.509	2.136	2.773	3.409	4.022	4.533			
8	0.974	1.509	2.137	2.775	3.417	4.054	4.669	5.181		
9	0.978	1.510	2.137	2.776	3.419	4.062	4.700	5.316	5.829	
10	0.981	1.510	2.137	2.776	3.419	4.063	4.707	5.347	5.963	6.476

In case of without replacement, i.e., $K=0$, the relative informations of interval censoring are increased from 0.813 to 0.981 with the increase of I. With more than one replaced intervals, relative informations are larger than those of continuous inspection cases.

Similarly, we can derive

$$\frac{CRLB(Cont.\ with\ repl.)}{CRLB(Mixed)} = \frac{t\,[K\{1-\exp(-t)\}+1-\exp\{-(I-K)t\}]}{I\{\exp(t)-2+\exp(-t)\}} \quad . \tag{3.5}$$

Thus, we get the relative informations of interval censoring with mixed replacement compared to the continuous inspection with replacement model.

Table 3. Relative Information of Interval Censoring with Mixed Replacement Procedure to Continuous Model With Replacement Procedure

	K									
I	0	1	2	3	4	5	6	7	8	9
1	.406									
2	.335	.406								
3	.295	.348	.406							
4	.269	.293	.359	.406						
5	.250	.249	.308	.367	.406					
6	.236	.214	.264	.321	.373	.406				
7	.225	.186	.229	.280	.332	.378	.406			
8	.216	.163	.201	.247	.294	.341	.381	.406		
9	.208	.146	.179	.220	.263	.306	.347	.384	.406	
10	.201	.131	.161	.198	.237	.276	.315	.353	.386	.406

It is reasonable that relative informations are decreasing with the increase of I. The relative informations are increasing with the increase of K for the cases of $I \leq 4$. When $I \geq 5$, but, relative informations decrease first and then increase with the increase of K. It can be in part interpreted that the increasing effects of T are larger than those of K. When $I=7$, for example, relative informations decrease from 0.225 to 0.186 with the change of K from 0 to 1, since total test time T is increased from 4.256 to 8.127 which amounts to change in optimal indices of interval-size from 0.608 to 1.161. On the other hand, the relative informations converge to 0.406 with increase of K regardless of I.

3.3 Sensitivity Analysis for Relative Information

Unknown parameter θ is involved in the index of interval-size t as follows.

$$t = \Delta / \theta .$$

Table 4. Effects of Preestimates θ^* to Relative Information of Interval Censoring with Mixed Replacement Procedure

	$I=3$		$I=6$			$I=9$			
θ^* K	0	2	0	3	5	0	3	6	8
$\frac{1}{2}\theta$	0.980	1.721	0.991	2.350	3.153	0.994	2.402	3.885	4.693
	(106.3)	(87.9)	(102.8)	(85.0)	(81.1)	(101.6)	(86.5)	(82.7)	(80.5)
$\frac{2}{3}\theta$	0.965	1.866	0.984	2.595	3.583	0.990	2.623	4.371	5.366
	(104.7)	(95.3)	(102.1)	(93.9)	(92.2)	(101.2)	(94.5)	(93.0)	(92.1)
$\frac{3}{4}\theta$	0.955	1.913	0.979	2.674	3.723	0.988	2.693	4.524	5.580
	(103.6)	(97.7)	(101.6)	(96.7)	(95.8)	(101.0)	(97.0)	(96.3)	(95.7)
θ	0.922	1.959	0.964	2.765	3.886	0.978	2.776	4.700	5.829
$\frac{4}{3}\theta$	0.867	1.841	0.936	2.644	3.675	0.961	2.664	4.475	5.512
	(94.0)	(94.0)	(97.1)	(95.6)	(94.6)	(98.3)	(96.0)	(95.2)	(94.6)
$\frac{3}{2}\theta$	0.835	1.730	0.920	2.517	3.457	0.951	2.545	4.239	5.185
	(90.6)	(88.3)	(95.4)	(91.0)	(89.0)	(97.2)	(91.7)	(90.2)	(89.0)
2θ	0.729	1.313	0.863	2.012	2.625	0.915	2.059	3.316	3.937
	(79.1)	(67.0)	(89.5)	(72.8)	(67.6)	(93.6)	(74.2)	(70.6)	(67.5)

NOTE: Values within ( ) are percentages of relative informations with true value and preestimate of θ

We have to preestimate the parameter θ for its application in determining interval-size Δ. For this reason, sensitivity analysis for relative information which takes into account preestimation of θ is needed. Results of sensitivity analysis for the case of interval censoring with mixed replacement procedure versus continuous model without replacement are shown in Table 4.

We can see that the loss rates of relative information are increased with increase in the ratio of replacement, even though absolute values of relative information are increased. The preestimation-error within 33% results in maximum 7.9% loss of relative information regardless of I and K. Therefore, it is concluded that the error in preestimation in the determination of optimal interval-size does not have crucial effect on the interval censoring with mixed replacement.

4. Test Designs for Minimum Cost

In this section test conditions for minimizing the test costs within a desired width of Cramer-Rao lower bound are examined.

4.1 Cost Function and Optimal Solution

The following four components of test costs are assumed,

C_i = the cost of placing an item on test,
C_t = the cost of the test running for a unit time,
C_s = the cost of test facility for a unit time,
C_u = the cost of inspection.

Then the cost of interval censoring with mixed replacement procedure is

$$C = C_i\left(n + \sum_{j=1}^{K} r_j\right) + C_t\, T + C_s\, n + C_u\, I \,. \tag{4.1}$$

The total test time T in (4.1) can be described as

$$T = \Delta I \,. \tag{4.2}$$

The expected number of failures in replaced intervals becomes

$$E\left(\sum_{j=1}^{K} r_j\right) = K\, n\left\{1 - \exp\left(-\frac{\Delta}{\theta}\right)\right\} \,. \tag{4.3}$$

From (4.2) and (4.3) the expected cost derived as a function of n and I is given as

$$C(n, I) = \left[C_i + C_s + K\left\{1 - \exp\left(-\frac{\Delta}{\theta}\right)\right\}\right] n + (C_t \Delta + C_u) I . \tag{4.4}$$

A constraint in this test can be assumed as

$$CRLB(\hat{\theta}) \leq \alpha^2 \theta^2 ,$$

where α^2 is a constant for determining the width of Cramer-Rao lower bound.

Applying the Cramer-Rao lower bound derived in (3.1), we get

$$\frac{\theta^4}{n\Delta^2} \cdot \frac{\exp\left(\frac{\Delta}{\theta}\right) - 2 + \exp\left(-\frac{\Delta}{\theta}\right)}{K\left\{1 - \exp\left(-\frac{\Delta}{\theta}\right)\right\} + \left\{1 - \exp\left(-\frac{(I-K)\Delta}{\theta}\right)\right\}} \leq \alpha^2 \theta^2 . \tag{4.5}$$

We can formulate the following nonlinear minimizing problem from (4.4) and (4.5).

$$\textit{Minimize} \quad C(n, I) = \left[C_i + C_s + K\left\{1 - \exp\left(-\frac{\Delta}{\theta}\right)\right\}\right] n + (C_t \Delta + C_u) I$$

$$s.t. \quad \frac{\theta^2}{\alpha^2\Delta^2} \cdot \frac{\exp\left(\frac{\Delta}{\theta}\right) - 2 + \exp\left(-\frac{\Delta}{\theta}\right)}{K\left\{1 - \exp\left(-\frac{\Delta}{\theta}\right)\right\} + \left\{1 - \exp\left(-\frac{(I-K)\Delta}{\theta}\right)\right\}} \leq n .$$

Then the cost function could be solved using the general method for the nonlinear problem.

4.2 Example

Let us consider an example for finding the solution. WLOG in this purpose, let

$$\theta = 200, \quad \alpha = 0.05 .$$

Assume K, the number of replaced intervals, be the half of I which is the number of total intervals, even though K is not determined before the test. And let the four components of test cost be all 10 and change each component to 100 by turns. The results of finding optimal solutions are shown in Table 5 for the cases of various t.

Table 5. Optimal n and I of Interval Censoring with Mixed Replacement Procedure

t	C_i	C_s	C_t	C_u	K	I	n	$C(n,I)$	$CRLB$
0.67	10	10	10	10	2	4	243	10,497.3	99.64
	100	10	10	10	3	5	189	27,816.9	99.72
	10	100	10	10	3	5	189	27,816.9	99.72
	10	10	100	10	0	1	851	30,430.0	99.92
	10	10	10	100	2	4	243	10,857.3	99.64
0.8	10	10	10	10	1	2	383	11,090.9	100.00
	100	10	10	10	3	5	173	27,365.0	99.51
	10	100	10	10	3	5	173	27,365.0	99.51
	10	10	100	10	0	1	766	31,330.0	100.00
	10	10	10	100	1	2	383	11,270.9	100.00
1.0	10	10	10	10	1	2	344	11,117.5	99.90
	100	10	10	10	3	5	158	27,729.6	99.59
	10	100	10	10	3	5	158	27,729.6	99.59
	10	10	100	10	0	1	688	33,770.0	99.90
	10	10	10	100	1	2	344	11,297.5	99.90
1.2	10	10	10	10	1	2	323	11,505.7	99.76
	100	10	10	10	3	5	150	28,864.5	99.89
	10	100	10	10	3	5	150	28,864.5	99.89
	10	10	100	10	0	1	645	36,910.0	99.92
	10	10	10	100	1	2	323	11,685.7	99.76
1.4	10	10	10	10	1	2	312	12,095.1	99.92
	100	10	10	10	3	5	147	30,552.3	99.88
	10	100	10	10	3	5	147	30,552.3	99.88
	10	10	100	10	0	1	624	40,490.0	99.92
	10	10	10	100	1	2	312	12,275.1	99.92
1.59	10	10	10	10	1	2	309	12,806.0	99.94
	100	10	10	10	3	5	147	32,471.1	99.95
	10	100	10	10	3	5	147	32,471.1	99.95
	10	10	100	10	0	1	618	44,170.0	99.94
	10	10	10	100	1	2	309	12,986.0	99.94

We can see that n is decreased and I is increased when C_i or C_s is large. Similarly, n is increased and I is decreased when C_t is large. On the other hand, the change of Δ has no crucial effect on optimal solution. It means that there are a lot of flexibilities for interval-size in this test procedure.

For comparison purpose, we can find the optimal solutions in cases of $K=0$ and $K=I-1$ as shown in Table 6 and Table 7 respectively.

Table 6. Optimal Solutions of Without Replacement Procedure

C_i	C_s	C_t	C_u	t	K	I	n	$C(n,I)$	$CRLB$
10	10	10	10	1.21	0	2	496	14,780.0	99.86
100	10	10	10	0.67	0	6	423	54,630.0	99.95
10	100	10	10	0.67	0	6	423	54,630.0	99.95
10	10	100	10	1.59	0	1	618	44,170.0	99.94
10	10	10	100	1.21	0	2	496	14,960.0	99.86

Table 7. Optimal Solutions of With Replacement Procedure

C_i	C_s	C_t	C_u	t	K	I	n	$C(n,I)$	$CRLB$
10	10	10	10	1.59	1	2	309	12,806.0	99.94
100	10	10	10	1.59	4	5	124	29,984.9	99.62
10	100	10	10	1.59	4	5	124	29,984.9	99.62
10	10	100	10	1.59	0	1	618	44,170.0	99.94
10	10	10	100	1.59	1	2	309	12,986.0	99.94

From Table 5 and Table 6, we can know that the costs of the mixed replacement procedure are always lower than those of the without replacement procedure. And comparing Table 5 and Table 7, the costs of the mixed replacement procedure are always better than the with replacement procedure if $t \leq 1.2$, and partially better if $t \geq 1.4$.

In the case which has 10, 100, 10 and 10 as the four components of test costs respectively, for example, comparisons of the minimum costs in three different replacement procedures are described in Table 8.

Table 8. Comparisons of Optimal Solutions in Three Different Replacement Procedures

t	C_i	C_s	C_t	C_u	K	I	n	$C(n,I)$	$CRLB$
0.67	10	100	10	10	0	6	423	54,630.0	99.95
0.67	10	100	10	10	3	5	189	27,816.9	99.72
0.8	10	100	10	10	3	5	173	27,365.0	99.51
1.0	10	100	10	10	3	5	158	27,729.6	99.59
1.2	10	100	10	10	3	5	150	28,864.5	99.89
1.4	10	100	10	10	3	5	147	30,552.3	99.88
1.59	10	100	10	10	3	5	147	32,471.1	99.95
1.59	10	100	10	10	4	5	124	29,984.9	99.62

The minimum costs of the without replacement procedure and that with replacement one are 54,630.0 and 29,984.9, respectively. The minimum costs of the mixed replacement procedure are distributed from 27,365.0 to 32,471.1 according to the value of t. So, we can expect that the interval censoring with mixed replacement procedure has advantages in terms of test costs as well as test convenience.

5. Summary

The mixed replacement procedure studied in this paper can be useful in practical life testing because it is a very flexible test scheme. As a special case, it can be a without replacement procedure or a with replacement one. The maximum likelihood estimator is obtainable. The Cramer-Rao lower bound is derived instead of variance, considering the implicit form of estimator. The optimal indices of interval-size for minimizing the Cramer-Rao lower bound of estimator are derived in the cases for various pairs of values for I and K. The results of Nelson(1977) which has studied the without replacement procedure are included as special cases of this mixed replacement one. The relative informations of the interval censoring with mixed replacement to the continuous models with and without replacement are examined. Through the sensitivity analysis for relative information, it is shown that the preestimation-error in the determination of the optimal interval-size does not have crucial effect on the test. The optimal magnitudes of n and I for minimizing the test costs within a desired width of Cramer-Rao lower bound are analyzed. Through a general example, it is shown that the test costs of the mixed replacement procedure are always lower than those of the without replacement one and are partially better than the with replacement one. Finally, it is reemphasized that the interval censoring with mixed replacement procedure has advantages in terms of test costs as well as test flexibility and convenience.

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