Block Replacement Policy by Multiple Choice with Used Items

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Abstract

A block replacement policy where at failure the item is either replaced by a new or used item or remains inactive until the next planned replacement is considered. In this paper, our interests are focused on reusing all the used items created by the policy. Numerical results for the case where the underlying life distribution is gamma are obtained.

1. Introduction

Under the standard block replacement policy(BRP), items are replaced preventively by new ones at time kT, $k = 1, 2, \cdots$ and at failure (Barlow and Proschan (1965)). This policy is rather wasteful, since sometimes almost new items are replaced at time kT, $k = 1, 2, \cdots$. To overcome this undesirable feature, standard BRP has been improved in several ways. Cox (1962) developed and Blanning(1965) corrected the policy to remain inactive if a failure occurs in $[kT - \delta, kT)$ for any k and for some δ. Bhat (1969) proposed replacing the failed item with a used one, which has been removed earlier after attaining the age T. Mixed modification have also been suggested. Tango (1978) suggested the policy in which if items fail in $[(k-1)T, kT - \delta)$, they are replaced by new items, and if in $[kT - \delta, kT)$, they are replaced by used items of age T. Murthy and Nguyen (1982) modified the Tango's policy so as to make use of all used items created by the policy. Kadi and Cléroux (1988) subdivided the interval $[(k-1)T, kT)$ into three parts $[(k-1)T, kT - \delta_1)$, $[kT - \delta_1, kT - \delta_2)$ and $[kT - \delta_2, kT)$, and suggested replacing failed items by new ones if the failure occurs in the first subinterval and by used ones of age T if it occurs in the second one and remaining inactive in the last one. Modified BRP (in which a item is replaced on failure and preventively at time kT, $k = 1, 2, \cdots$, if its age exceeds b for some b) is proposed by Berg and Epstein (1976). Lately, Archibald and Dekker (1996) extends the modified BRP of Berg and Epstein (1976) to the case of multi-component systems with discrete life time distribution.

Kadi and Cléroux's policy creates used items of age varying from δ_1 to T. However it uses only used items of age T and discards used items of age less than T. This is not a

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rational policy. In this paper we extend the Kadi and Cléroux's policy to the case where failed items in $[kT - \delta_1, kT - \delta_2)$ are replaced by used ones with age varying from δ_1 to T, as opposed to replacement by used ones of age T only.

2. Replacement Policy

The policy is defined in the following way:

(1) Items are exchanged for new items at kT, $k = 1, 2, \cdots$.

(2) If items fail in $[(k-1)T, kT - \delta_1)$, $0 \leq \delta_1 \leq T < \infty$, they are replaced by new ones.

(3) If items fail in $[kT - \delta_1, kT - \delta_2)$, $0 \leq \delta_2 \leq \delta_1 \leq T < \infty$, they are replaced by used ones created by this policy.

(4) If items fail in $[kT - \delta_2, kT)$, they remains inactive until the next planned replacement at kT.

The parameter T, δ_1 and δ_2 are unknown parameters which determine the replacement policy. We use the same assumptions as in Kadi and Cléroux's policy, i.e. (i) the lifetime distribution of new items is continuous and has an increasing failure rate, (ii) failures are instantly detected, (iii) preventive replacement and replacement at failure are made instantly, (iv) the number of new and used items available for replacement is sufficient to avoid shortages, (v) no item is used more than twice, (vi) used items will cost less than new items, (vii) the preventive replacement cost is less than the cost of replacement at failure.

3. Formulations and Analysis

For a fixed time t, let $N_1(t)$ denote the number of failure replacements by new items during $[0, t]$ and $N_2(t)$ denote the number of preventive replacements during $[0, t]$. Then $N_1(t) + N_2(t)$ represents the number of replacements by new items during $[0, t]$. Furthermore let $N_3(t)$ denote the number of replacements by used items during $[0, t]$ and D denote the length of inactivity period during $[0, t]$. By using arguments similar to Kadi and Cléroux (1988), the expected cost per unit time for an infinite time span is given by

$$C(T, \delta_1, \delta_2) = \lim_{t \to \infty} \left\{ c_1 \frac{E[N_1(t)]}{t} + c_2 \frac{E[N_2(t)]}{t} + c_3 \frac{E[N_3(t)]}{t} + c_4 \frac{E[D]}{t} \right\}$$

where

c_1 = cost suffered for each failure replacement by a new item,

c_2 = cost suffered for each preventive replacement,

c_3 = cost suffered for each failure replacement by a used item,

c_4 = cost suffered for each time unit of inactivity.

From classical results of renewal theory (see, for example Barlow and Proschan (1965)), we have

$$\lim_{t \to \infty} \frac{E[N_1(t)]}{t} = \frac{M(T - \delta_1)}{T}, \qquad \lim_{t \to \infty} \frac{E[N_2(t)]}{t} = \frac{1}{T},$$

where $M(t) = \sum_{n=1}^{\infty} F^{(n)}(t)$ is the renewal function corresponding to the renewal process of new items and $F^{(n)}(t)$ is the n-fold convolution of $F(t)$, failure time distribution of new items. Let $m(t) = dM(t)/dt$ be the renewal density of this process.

3.1 Computation of $\lim_{t \to \infty} E[N_3(t)]/t$

The failure time distribution of used items is a function of $F(t)$, T and δ_1. Let $F_U(t \mid T, \delta_1)$ be the failure time distribution of used items. $F_U(t \mid T, \delta_1)$ can be derived as follows. A used item of age ($\delta_1 + x$) is created in a cycle if it starts operating at time $T - (\delta_1 + x)$ where $x \in (0, T - \delta_1)$ and survive until time T. Thus the probability that a used item of age ($\delta_1 + x$) is created in a cycle is given by

$$dG(x \mid T, \delta_1) = \begin{cases} [1 - F(\delta_1 + x)]dM(T - \delta_1 - x) & \text{for } 0 < x < T - \delta_1, \\ 1 - F(T) & \text{for } x = T - \delta_1. \end{cases}$$

The probability that a used item is created is easily seen to be $\int_0^{T - \delta_1} dG(x \mid T, \delta_1)$.

The failure time distribution of a used item of age ($\delta_1 + x$) is easily seen to be $F_{\delta_1 + x}(t) = \{F(\delta_1 + x + t) - F(\delta_1 + x)\}/\{1 - F(\delta_1 + x)\}$. Therefore, $F_U(t \mid T, \delta_1)$ is obtained as

$$F_U(t \mid T, \delta_1) = \frac{\int_0^{T - \delta_1} \left\{ \dfrac{F(\delta_1 + x + t) - F(\delta_1 + x)}{1 - F(\delta_1 + x)} \right\} dG(x \mid T, \delta_1)}{\int_0^{T - \delta_1} dG(x \mid T, \delta_1)}$$

$$= \frac{F(T+t) - F(T) + \int_0^{T-\delta_1} \{ F(\delta_1 + x + t) - F(\delta_1 + x) \} \, dM(T-\delta_1 - x)}{1 - F(T) + \int_0^{T-\delta_1} \{ 1 - F(\delta_1 + x) \} dM(T-\delta_1 - x)}.$$

To evaluate the expected number of replacement by used items in the $[\ T-\delta_1,\ T-\delta_2\ )$, we note that replacement in this interval can be viewed as a modified renewal process with the first item in this interval having a failure distribution function $\Psi(x)$, $x \in$ $[\ T-\delta_1,\ T-\delta_2\ )$ and subsequent items having a failure distribution $F_U(t \mid T, \delta_1)$. $d\Psi(x)$ (the probability of failure between x and $x + dx$ where $x \in [\ T-\delta_1,\ T-\delta_2))$ can be derived as follows.

If we let E_1 and E_2 be the independent and mutually exclusive events defined by $E_1 =$ {the new item installed at time $t = 0$ fails for the first time between x and $x + dx$ where $x \in [\ T-\delta_1,\ T-\delta_2\ )\}$, $E_2 =$ {the last renewal occurred between y and $y + dy$ where $y \in [\ 0,\ T-\delta_1)$ and the new item installed at that time fails between x and $x + dx$, where $x \in [\ T-\delta_1,\ T-\delta_2\ )\}$, then we have $d\Psi(x) = P(E_1) + P(E_2)$, where $P(E_1) = f(x) \, dx$ and $P(E_2) = \int_0^{T-\delta_1} m(y) f(x-y) \, dy \, dx$. Thus, the expected number of used items needed per cycle is given by $M_D(T, \delta_1, \delta_2)$, i.e.

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\begin{aligned}
M_D(T, \delta_1, \delta_2) &= \int_{T-\delta_1}^{T-\delta_2} \{ 1 + M_U(T-\delta_2 - x \mid T, \delta_1) \} d\Psi(x) \\
&= \int_{T-\delta_1}^{T-\delta_2} \{ 1 + M_U(T-\delta_2 - x \mid T, \delta_1) \} f(x) dx \\
&\quad + \int_0^{T-\delta_1} \int_{T-\delta_1}^{T-\delta_2} \{ 1 + M_U(T-\delta_2 - x \mid T, \delta_1) \} m(y) f(x-y) dx dy,
\end{aligned}
$$

where $M_U(t \mid T, \delta_1)$ is the renewal function for a renewal process with failure time distribution $F_U(t \mid T, \delta_1)$.
Hence we have

$$\lim_{t \to \infty} \frac{E[N_3(t)]}{t} = \frac{M_D(T, \delta_1, \delta_2)}{T}.$$

3.2 Computation of $\lim\limits_{t\to\infty} E[D]/t$

We have

$$\lim_{t\to\infty} \frac{E[D]}{t} = \frac{L(T, \delta_1, \delta_2)}{T},$$

where $L(T, \delta_1, \delta_2)$ is the average duration of the inactivity period in $[T - \delta_2, T)$. If x is the instant of failure in $[T - \delta_2, T)$ then the number of time units of inactivity is equal to $T - x$. Therefore

$$L(T, \delta_1, \delta_2) = \int_{T-\delta_2}^{T} (T - x)\phi(x)\,dx,$$

where $\phi(x)dx$ is the probability of a failure between x and $x + dx$ where $x \in [T - \delta_2, T)$. In order to obtain $\phi(x)dx$ we consider the following independent events: For $x \in [T - \delta_2, T)$, $y \in (0, T - \delta_2)$, $v \in [T - \delta_1, T - \delta_2)$ and $u \in [0, T - \delta_2 - v]$,

E_3 = {the new item installed at time $t = 0$ fails for the first time between x and $x + dx$}

E_4 = {a renewal occurs between y and $y + dy$}

E_5 = {the new item installed at time y fails between x and $x + dx$}

E_6 = {the new item installed at time $t = 0$ fails between v and $v + dv$}

E_7 = {the new item installed at time v fails between x and $x + dx$}

E_8 = {the new item installed at time y fails between v and $v + dv$}

E_9 = {a renewal with a used item occurs between $v + u$ and $v + u + du$}

E_{10}= {the used item installed at time $v + u$ fails between x and $x + dx$}

The event {a failure occurs between x and $x + dx$ where $x \in [T - \delta_2, T)$} is the union of the following mutually exclusive scenarios:

$S_1 : E_3$

$S_2 : E_4$ followed by E_5

$S_3 : E_6$ followed by E_7

$S_4 : E_4$ followed by E_8 followed by E_7

$S_5 : E_6$ followed by E_9 followed by E_{10}

$S_6 : E_4$ followed by E_8 followed by E_9 followed by E_{10}

and therefore $\phi(x)dx = \sum_{i=1}^{6} P(S_i)$, where

$P(S_1) = f(x)dx$

$P(S_2) = \int_0^{T-\delta_1} m(y)f(x-y)dy\,dx$

$P(S_3) = \int_{T-\delta_1}^{T-\delta_2} f(v)f_U(x-v)dv\,dx$

$P(S_4) = \int_0^{T-\delta_1}\int_{T-\delta_1}^{T-\delta_2} m(y)f(v-y)f_U(x-v)dvdy\,dx$

$P(S_5) = \int_{T-\delta_1}^{T-\delta_2}\int_v^{T-\delta_2} f(v)\,m_U(u-v)\,f_U(x-u)dudv\,dx$

$P(S_6) = \int_{T-\delta_1}^{T-\delta_2}\int_v^{T-\delta_2}\int_0^{T-\delta_1} m(y)f(v-y)\,m_U(u-v)\,f_U(x-u)dydudv\,dx$.

Combining these results we finally obtain

$$\phi(x) = f(x) + \int_0^{T-\delta_1} m(y)f(x-y)dy + \int_{T-\delta_1}^{T-\delta_2} f(v)f_U(x-v)dv$$

$$+ \int_0^{T-\delta_1}\int_{T-\delta_1}^{T-\delta_2} m(y)f(v-y)f_U(x-v)dvdy$$

$$+ \int_{T-\delta_1}^{T-\delta_2}\int_v^{T-\delta_2} f(v)\,m_U(u-v)\,f_U(x-u)dudv$$

$$+ \int_{T-\delta_1}^{T-\delta_2}\int_v^{T-\delta_2}\int_0^{T-\delta_1} m(y)f(v-y)\,m_U(u-v)\,f_U(x-u)dydudv$$.

The problem is thus to minimize

$$C(T, \delta_1, \delta_2) = \frac{1}{T}[c_1M(T-\delta_1) + c_2 + c_3M_D(T,\delta_1,\delta_2) + c_4L(T,\delta_1,\delta_2)]$$

with respect to T, δ_1, δ_2 under the constraints $\delta_1 \geq 0$, $\delta_2 \geq 0$, $\delta_1 \geq \delta_2$, $0 \leq T < \infty$, $T - \delta_1 \geq 0$.

3.3 Special Cases
This policy can be regarded as the generalization for the special cases summarized in the Table 1.

Table 1 Special cases of our policy

Case	Cost Function	Description
$\delta_1 = \delta_2 = 0$	$C(T, 0, 0) = \frac{1}{T}[c_1 M(T) + c_2]$	standard BRP. Barlow and Proschan(1965)
$\delta_1 = \delta_2 = T$	$C(T, T, T) = \frac{1}{T}[c_2 + c_4 L(T, T, T)]$	a item is replaced preventively at time kT, $k = 1, 2, \cdots$ and remains inactive at failure.
$0 < \delta_1 = \delta_2 < T$	$C(T, \delta_2, \delta_2) = \frac{1}{T}[\ c_1 M(T - \delta_2) + c_2 + c_4 L(T, \delta_2, \delta_2)]$	a item is replaced preventively at time kT, $k = 1, 2, \cdots$, if fails in $[\ (k-1)T, kT - \delta_2)$, it is replaced by a new item, and if in $[kT - \delta_2, kT)$, it remains inactive. Cox (1962), Blanning(1965)
$\delta_1 = T, \delta_2 = 0$	$C(T, T, 0) = \frac{1}{T}[c_2 + c_3 M_D(T, T, 0)]$	a item is replaced preventively at time kT, $k = 1, 2, \cdots$ and replaced by a used item at failure.
$0 < \delta_1 < T, \delta_2 = 0$	$C(T, \delta_1, 0) = \frac{1}{T}[\ c_1 M(T - \delta_1) + c_2 + c_3 M_D(T, \delta_1, 0)]$	a item is replaced preventively at time kT, $k = 1, 2, \cdots$, if fails in $[\ (k-1)T, kT - \delta_1)$, it is replaced by a new item, and if in $[kT - \delta_1, kT)$, it is replaced by a used item. Murthy and Nguyen (1982)

4. Comparison and Numerical Solution of the Model

To compare this policy with Kadi and Cléroux's policy, we note that the failure time distribution of used items in our policy is given by $F_U(t \mid T, \delta_1)$ while that in Kadi and Cléroux's policy is given by $F_T(t \mid T) = \{F(T + t) - F(T)\}/\{1 - F(T)\}$. Thus for Kadi and Cléroux's policy the expected number of used items per cycle is given by

$$M_K(T, \delta_1, \delta_2) = \int_{T - \delta_1}^{T - \delta_2} \{1 + M_T(T - \delta_2 - x \mid T)\} d\Psi(x),$$

where $M_T(t \mid T)$ is the renewal function for a renewal process with failure time distribution

$F_T(t \mid T)$. Since $F(t)$ has an increasing failure rate, it follows that $F_U(t \mid T, \delta_1) \leq F_T(t \mid T)$ and $M_U(t \mid T, \delta_1) \leq M_T(t \mid T)$. Therefore $M_D(T, \delta_1, \delta_2) \leq M_K(T, \delta_1, \delta_2)$, i.e. for the same T, δ_1 and δ_2, the expected number of used items per cycle for our policy is smaller than that of Kadi and Cléroux's policy. Thus for the same T, δ_1 and δ_2, the expected cost per unit of time for our policy is smaller than that of Kadi and Cléroux's policy. As a result, our policy is always better than Kadi and Cléroux's policy.

The optimal policy for our model is given by T^*, δ_1^* and δ_2^* which minimize $C(T, \delta_1, \delta_2)$. And T^*, δ_1^* and δ_2^* can be obtained by solving $\partial C(T, \delta_1, \delta_2)/\partial T = 0$, $\partial C(T, \delta_1, \delta_2)/\partial \delta_1 = 0$ and $\partial C(T, \delta_1, \delta_2)/\partial \delta_2 = 0$. In general one cannot obtain an explicit analytical form for these equations since the renewal functions $M(t)$ and $M_U(t)$ and the renewal densities $m(t)$ and $m_U(t)$ cannot be explicitly written. Thus a computational scheme is needed to obtain the solution. We shall now consider a situation in which $F(t)$ is given by a gamma distribution of order 2 with parameter λ, that is, $F(t) = 1 - (1 + \lambda t)e^{-\lambda t}$ for $t \geq 0$ and $\lambda > 0$. In this case $M(t) = \lambda t/2 - 1/4 + 1/4e^{-2\lambda t}$, $m(t) = (\lambda/2)(1 + e^{-2\lambda t})$. From this, $F_U(t \mid T, \delta_1)$ can be computed numerically and this provided a use of the algorithm developed by Cléroux and McConalogue(1976) to calculate $M_U(t)$, $m_U(t)$. Without loss of generality the computation can be made for only one value of λ. Here the problem has been solved for $\lambda = 0.001$ so that the mean and the standard deviation of $F(t)$ are respectively, $\mu = 2,000$ and $\sigma = 1414.21$.

As was done by Tango(1978), Murthy and Nguyen(1982), Kadi and Cléroux(1988) etc., the cost have been reparameterized in the following way: $P_2 = c_2/c_1$, $P_4 = c_4/c_1$ and $d = (c_1 - c_3)/c_2$. Then we have $0 < d \leq 1$, and if $d \to 0$, then $c_3 \to c_1$, while if $d = 1$, then $c_3 = c_1 - c_2$. Thus the value of d is a measure of the relative location of c_3 between $c_1 - c_2$ and c_1. The computation have been carried out for all combinations of $P_2 = 0.10$, 0.15, 0.20, $d = 0.05$, 0.50, 1.00 and $P_4 = 0.01$, 0.03, 0.05, 0.07. A FORTRAN program run on Pentium computer system was used for this computation. And the standard package IMSL "International Mathematical and Statistical Library" was used to calculate convolutions for $M_U(t)$ and $m_U(t)$. To find T^*, δ_1^* and δ_2^* which minimize $C(T, \delta_1, \delta_2)$, univariate minimization method(see Fox(1971) or Rao(1996)) was used. This numerical procedure requires an initial value of (T, δ_1, δ_2). Many trials were carried out with various initial values. In some cases, changing the initial value resulted in a change in the solution. However, the optimal cost does not change greatly. The minimum obtained will be the relative minimum nearest to the starting points. But in the engineering problem, we seldom need to use

arbitrary starting point and to be anxious about relative minima, because the relative minima often have physical significance. The results are shown in Table 2.

It is seen, as expected, that

1) if P_2 increases, then the interval between two consecutive planned replacements T^* and optimal cost $C(T^*, \delta_1^*, \delta_2^*)$ increases.

2) if P_4 increases, with P_2 and d remaining fixed, then the length δ_2^* of the inactivity period decreases and optimal cost $C(T^*, \delta_1^*, \delta_2^*)$ increases.

As it is not included in the Table 2, the case of $\delta_1^* > \delta_2^*$ and $\delta_2^* \neq 0$ was found for some combinations of (P_2, d, P_4). For example, when $(P_2, d, P_4) = (0.20, 0.05, 0.065)$, optimal policy $(T^*, \delta_1^*, \delta_2^*, C(T^*, \delta_1^*, \delta_2^*))$ is $(1409.80, 15.20, 13.90, 47.31 \times 10^{-3})$.

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Table 2 Numerical results for the case where the underlying life distribution $F(t)$ is given by a gamma distribution of order 2 with parameter λ = 0.001.

Cost structure			Optimal solution			Optimal cost
P_2	d	P_4	T^*	δ_1^*	δ_2^*	$C(T^*, \delta_1^*, \delta_2^*) \times 10^3$
0.10	0.05	0.01	641.60	44.30	44.30	35.29
		0.03	629.40	44.30	44.30	35.94
		0.05	619.20	44.30	44.30	36.58
		0.07	636.40	31.90	31.90	36.84
	0.50	0.01	640.20	44.30	44.30	35.29
		0.03	629.40	44.30	44.30	35.94
		0.05	619.20	44.30	44.30	36.58
		0.07	641.60	30.40	30.30	36.82
	1.00	0.01	641.60	44.30	44.30	35.29
		0.03	629.40	44.30	44.30	35.94
		0.05	623.40	42.10	42.10	36.55
		0.07	643.40	28.60	28.50	36.80
0.15	0.05	0.01	923.40	44.30	44.30	41.81
		0.03	910.70	44.30	44.30	42.56
		0.05	930.60	32.90	32.90	42.99
		0.07	956.20	22.30	22.30	43.09
	0.50	0.01	923.40	44.30	44.30	41.81
		0.03	910.60	44.30	44.30	42.56
		0.05	937.50	30.40	30.40	42.95
		0.07	961.50	20.70	20.70	43.07
	1.00	0.01	923.20	44.30	44.30	41.81
		0.03	910.40	44.30	44.30	42.56
		0.05	945.80	27.90	27.80	42.92
		0.07	964.60	19.10	19.10	43.06
0.20	0.05	0.01	1309.40	44.30	44.30	46.40
		0.03	1345.40	34.70	34.70	47.07
		0.05	1404.40	20.70	20.70	47.25
		0.07	14107.20	252.30	00.00	49.32
	0.50	0.01	1309.40	44.30	44.30	46.40
		0.03	1345.80	37.80	37.80	47.10
		0.05	1409.50	20.70	20.70	47.25
		0.07	1435.30	14.30	14.30	47.32
	1.00	0.01	1309.40	44.30	44.30	46.40
		0.03	1358.40	33.20	33.20	47.06
		0.05	1422.30	19.10	19.10	47.24
		0.07	1436.80	13.40	13.40	47.32