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Higher Order Expansions of the Cumulants and the Modified Normalizing Process of Multi-dimensional Maximum Likelihood Estimator[1)]

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Abstract

In this paper we derive the higher order expansions of the first four cumulants of multi-dimensional Maximum Likelihood Estimator (MLE) under the general parametric model up to and including terms of order $O(n^{-1})$. Also, we obtain the explicit form of the expansion of the normalizing transformation of multi-dimensional MLE and show that the suggested normalizing process is much better than the normal approximation based on central limit theorem through example.

1. Introduction

A lot of studies related to the normalizing process are given by Cornish and Fisher (1937), Wallace (1958), and McCullagh (1987), etc. In paticular, Cornish and Fisher (1937) and McCullagh (1987) suggested the general form of the normalizing process of the univariate and multivariate random vectors of general type, respectively. But, the explicit forms of cumulants are needed to apply their method to a specified statistics such as maximum likelihood estimators (including multi-dimensional cases) and so far no explicit calculations of cumulants are known for multi-dimensional MLE.

In Section 2, we derive the explicit forms of the first four cumulants of multi-dimensional MLE under the general parametric model up to and including terms of order $O(n^{-1})$. By using these results, we show that the faster normalizing transformation due to McCullagh (1987) has a definite form in the case of multi-dimensional MLE. A practical example of univariate case is also given to show the efficiency of the suggested normalizing process. The detailed derivations of the theorems in Section 2 are also given in Appendix. To prove the theorems in this paper, the computational method of generalized cumulants and the basic lemmas concerned with tensor method are needed. The papers related to the calculations of

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generalized cumulants are also given by Speed (1983) and McCullagh (1984, 1987). In particular, we shall adopt the notation of *Einstein summation convention* (or index notation) and the expressions related to tensor methods used by McCullagh (1987). Also the studies concerned tensor methods are given by many authors including McCullagh (1987), Barndorff-Nielsen and Cox (1989), Stafford and Andrews (1993), Stafford (1994), and Stafford, et al. (1994).

2. Higher Order Expansions of the First Four Cumulants of Multi-dimensional MLE

Suppose that $Y=(Y_1,\dots,Y_n)$ has n independent and identically distributed components so that the log-likelihood for the full data may be written as the sum

$$l(\theta\,;Y)=\sum_i l(\theta\,;Y_i)\ .$$

The parameter vector θ with components $\theta^1,\dots,\theta^p$ is assumed to in some subset, Θ , of R^p . By convention, we use superscripts to represent the coordinates of an arbitrary point in Θ . Suppose that $U_r, U_{rs},\dots$ is the log likelihood derivatives at an arbitrary point θ , i.e.,

$$U_I=\partial_I l(\theta\,;Y)=(\frac{\partial}{\partial\theta})_I\ l(\theta\,;Y) \tag{2.1}$$

In (2.1), I stands for an index set $i_1\dots i_\nu$ and $|I|$ is the number of blocks in I, i.e., $|I|=\nu$.

Let

$$x_I=E(U_I\,;\theta)=E(\ (\frac{\partial}{\partial\theta})_I l(\theta\,;Y)\ ;\theta\ )$$

$$\bar{x}_I\ =\ E(\ (\frac{\partial}{\partial\theta})_I\ l(\theta\,;Y_1)\ ;\theta\ )\ .$$

The normalized log likelihood derivatives $Z_r, Z_{rs},\ \dots$ are defined by

$$Z_I=\frac{U_I-x_I}{\sqrt{n}}=\frac{U_I-n\,\bar{x}_I}{\sqrt{n}}\,.$$

More generally, let

$$\bar{x}_{I_1,\dots,I_v} = cum\left(\left(\frac{\partial}{\partial\theta}\right)_{I_1} l(\theta\,; Y_1), \dots, \left(\frac{\partial}{\partial\theta}\right)_{I_v} l(\theta\,; Y_1) \right)$$

and

$$\begin{aligned} x_{I_1,\dots,I_v} &= cum\left(\left(\frac{\partial}{\partial\theta}\right)_{I_1} L(\theta\,; Y), \dots, \left(\frac{\partial}{\partial\theta}\right)_{I_v} L(\theta\,; Y) \right) \\ &= n\, \bar{x}_{I_1,\dots,I_v} \,. \end{aligned}$$

Further, we adopt the *summation convention* according to which if an index occurs both as a superscript and as a subscript in a single expression then summation over that index is understood. Thus, for instance, $a^i b_i = a^1 b_1 + \cdots + a^m b_m$ and $a^{ijk} b_{il} = a^{1jk} b_{1l} + \cdots + a^{mjk} b_{ml}$.

The likelihood equations $U_r(\hat{\theta}\,; r) = 0$ may be expanded in a Taylor series in $\hat{\delta} = \sqrt{n}\,(\hat{\theta} - \theta)$ to give

$$\begin{aligned} & n^{1/2} Z_r + (n x_{rs} + \sqrt{n} Z_{rs})\, \hat{\delta}^s / n^{1/2} + (n x_{rst} + n^{1/2} Z_{rst})\, \hat{\delta}^s\, \hat{\delta}^t / (2n) \\ & + n x_{rstu}\, \hat{\delta}^s\, \hat{\delta}^t\, \hat{\delta}^u / (6 n^{3/2}) + O_p(n^{-1}) = 0 \end{aligned}$$

Let $\bar{x}^{r,s}$ is the inverse of $\bar{x}_{r,s}$, where

$$\bar{x}_{r,s} = -\bar{x}_{rs} = -E\left(\frac{\partial^2}{\partial\theta^r \partial\theta^s} L(\theta\,; Y_1)\,; \theta \right) \tag{2.2}$$

Note the expression (2.2) represent the Fisher's information numbers per observation and $\bar{x}^{rs}$, $\bar{x}^{rst}$, $\bar{x}^{rstu}$, ... are defined as follows.

$$\begin{aligned} \bar{x}^{rs} &= \bar{x}^{r,i}\, \bar{x}^{s,j}\, \bar{x}_{ij} \,, \\ \bar{x}^{rst} &= \bar{x}^{r,i}\, \bar{x}^{s,j}\, \bar{x}^{t,k}\, \bar{x}_{ijk} \,, \\ \bar{x}^{rstu} &= \bar{x}^{r,i}\, \bar{x}^{s,j}\, \bar{x}^{t,k}\, \bar{x}^{u,l}\, \bar{x}_{ijkl} \,, \\ & \dots \end{aligned}$$

In this paper, we derive the explicit forms of the first four cumulants of the solutions of the likelihood equation up to and including terms of order $O(n^{-1})$.

Lemma 1. (McCullagh (1987)) Let $\hat{\delta}=\sqrt{n}(\hat{\theta}-\theta)$, where $\hat{\theta}$ satisfies the likelihood equation $U_r(\hat{\theta})=0$. Then

$$\hat{\delta}^r = a_1{}^r + n^{-1/2}\, a_2{}^r + n^{-1}\, a_3{}^r + O(n^{-3/2})$$

where

$$a_1{}^r = \bar{x}^{r,s} Z_s$$

$$a_2{}^r = \bar{x}^{r,s}\, \bar{x}^{t,u} Z_{st} Z_u + x^{rst} Z_s Z_t /2$$

$$a_3{}^r = \bar{x}^{r,s}\, \bar{x}^{t,u}\, \bar{x}^{v,w}\, Z_{st} Z_{uv} Z_w + \bar{x}^{rst}\, \bar{x}^{u,v}\, Z_s Z_{tu} Z_v$$

$$+ \bar{x}^{r,s}\, \bar{x}^{tuv}\, Z_{st} Z_u Z_v /2 + \bar{x}^{rst}\, \bar{x}^{uvw}\, \bar{x}_{t,v}\, Z_s Z_u Z_w /2$$

$$+ \bar{x}^{r,s}\, \bar{x}^{t,u}\, \bar{x}^{v,w}\, Z_{suw} Z_t Z_v /2 + \bar{x}^{rstu}\, Z_s Z_t Z_u /6$$

Lemma 2. In the notation introduced above, we have the following results.

(1) $E(Z_I)=0$

(2) $cum(Z_{I_1},\dots,Z_{I_\nu})=(1/\sqrt{n})^{\nu-2}\, x_{I_1,\dots,I_\nu}$, $\nu\geq 2$

(3) $\bar{x}_{[I]}=\sum_{I_1|\dots|I_\nu\in P(I)} \bar{x}_{I_1,\dots,I_\nu}=0$

Here, $I_1|\dots|I_\nu\in P(I)$ means all partitions of the index set I. For examples,

$$\bar{x}_a=0,\quad \bar{x}_{a,b}=-\bar{x}_{ab}$$
$$\bar{x}_{a,b,c}+\bar{x}_{a,bc}[3]+\bar{x}_{abc}=0$$
$$\bar{x}_{a,b,c,d}+\bar{x}_{a,bcd}[4]+\bar{x}_{ab,cd}[3]+\bar{x}_{a,b,cd}[6]+\bar{x}_{abcd}=0$$

...

where the bracket notation $[k]$ is simply a convenience to avoid the sum of k similar terms, i.e., the expression $\bar{x}_{a,bc}[3]$ is the abbreviation of $\bar{x}_{a,bc}+\bar{x}_{b,ac}+\bar{x}_{c,ab}$.

(proof) The results of (1) and (2) are obvious from the definition of Z_I and the property of multilinearity of cumulant tensor. For the proof of (3), see the exercise 7.1 (on p.222) of McCullagh (1987).

By using Lemma 1 and 2 above, we can derive the first four cumulants of multi-dimensional MLE.

Theorem 1. The mean and the covariance of multi-dimensional MLE are given by

(A) $E(\hat{\delta}^r) = -n^{-1/2}\bar{x}^{r,s}\bar{x}^{t,u}(\bar{x}_{s,t,u} + \bar{x}_{s,tu})/2 + O(n^{-3/2})$

(B)

$$Cov(\hat{\delta}^r, \hat{\delta}^s) = \bar{x}^{r,s} + n^{-1/2}(\bar{x}^{r,i}\bar{x}^{s,j}\bar{x}^{k,l}\bar{x}_{i,jk,l} + \bar{x}^{r,s,t}\bar{x}^{i,j}\bar{x}_{jst})$$

$$+ n^{-1}\bar{x}^{r,s}\bar{x}^{t,u}\bar{x}^{i,j}(\bar{x}_{ju,st} + \bar{x}^{k,l}\bar{x}_{l,st}\bar{x}_{u,jk} + 2\bar{x}^{b,l}\bar{x}_{bju}\bar{x}_{l,st}$$

$$+ \bar{x}^{a,b}\bar{x}_{ast}\bar{x}_{bju}/2) + O(n^{-3/2})$$

(Proof) See Appendix

Theorem 2. The third and the fourth cumulants of multi-dimensional MLE are given by

(C)

$$cum(\hat{\delta}^r, \hat{\delta}^s, \hat{\delta}^t) = n^{-1/2}\bar{x}^{r,i}\bar{x}^{s,j}\bar{x}^{t,k}(\bar{x}_{i,j,k} + 6\bar{x}_{i,jk} + 3\bar{x}_{ijk}) + O(n^{-3/2})$$

$$= n^{-1/2}\bar{x}^{r,i}\bar{x}^{s,j}\bar{x}^{t,k}(\bar{x}_{ijk} - \bar{x}_{i,j,k}) + O(n^{-3/2})$$

(D)

$$cum(\hat{\delta}^r, \hat{\delta}^s, \hat{\delta}^t, \hat{\delta}^u)$$

$$= n^{-1}\bar{x}^{r,i}\bar{x}^{s,j}\bar{x}^{t,k}\bar{x}^{u,l}\{\bar{x}_{i,j,k,l} + 24\bar{x}_{i,j,kl} + 12\bar{x}^{a,b}\bar{x}_{a,i,j}\bar{x}_{bku}$$

$$+ 24\bar{x}^{a,b}\bar{x}_{i,bu}\bar{x}_{j,lk} + 24\bar{x}^{a,b}\bar{x}_{bui}\bar{x}_{j,ak} + 12\bar{x}^{a,b}\bar{x}_{sku}\bar{x}_{i,bl}$$

$$+ 12\bar{x}_{iua}\bar{x}_{jkb} + 12\bar{x}_{i,jkl} + 4\bar{x}_{uijk} + 48\bar{x}_{ij,kl} + 48\bar{x}^{a,b}\bar{x}_{jkl}\bar{x}_{u,bi}$$

$$+ 12\bar{x}^{a,b}\bar{x}_{ali}\bar{x}_{bkj}\} + O(n^{-3/2})$$

(Proof) See Appendix

3. Modified Normalizing Process of MLE

The results in Section 2 are applicable to the normalizing transformation introduced by Cornish and Fisher (1937) and McCullagh (1987). They suggested the polynomial transformations of general statistics which converge to the normal distribution as a device for the faster normalizing transformation by using the knowledge of the first four cumulants of the statistics. We introduce the results of McCullagh (1987). Assume that

$X=(X^1, X^2, \ldots, X^p)$ is a standardized random variable with cumulants

$$0,\quad \delta^{ij},\quad n^{-1/2}\chi^{i,j,k},\quad n^{-1}\chi^{i,j,k,l}$$

and so on, decreasing in power of $n^{1/2}$. The following transformed variable $Y=(Y^1, Y^2, \ldots, Y^p)$ is distributed standard normal with error $O(n^{-3/2})$.

$$\begin{aligned}
Y^i = \; & X^i - n^{-1/2}\,\{\,3k^{i,r,s}h_{rs} + 3k^{i,i,r}h_i h_r + k^{i,i,i}h_{ii}\,\}\;/\;3! \\
& - n^{-1}\,\{4k^{i,r,s,t}h_{rst} + 6k^{i,i,r,s}h_{rs}h_i + 4k^{i,i,i,r}h_{ii}h_r + k^{i,i,i,i}h_{iii}\}\;/\;4! \\
& + n^{-1}\,\{(36k^{a,i,r}k^{a,s,t} + 18k^{i,i,r}k^{i,s,t})h_{rst} \\
& + (18k^{a,i,i}k^{a,r,s} + 12k^{i,i,i}k^{i,r,s} + 36k^{a,i,r}k^{a,i,s} + 27k^{i,i,r}k^{i,i,s})h_{rs}h_i \\
& + (36k^{a,i,i}k^{a,i,r} + 30k^{i,i,i}k^{i,i,r})h_{ii}h_r \\
& + (9k^{a,i,i}k^{a,i,i} + 8k^{i,i,i}k^{i,i,i})h_{iii}\;/\;72 \\
& + n^{-1}\,\{(36k^{a,b,i}k^{a,b,r} + 36k^{a,i,i}k^{a,i,r} + 12k^{i,i,i}k^{i,i,r})h_r \\
& + (18k^{a,b,i}k^{a,b,i} + 27k^{a,i,i}k^{a,i,i} + 10k^{i,i,i}k^{i,i,i})h_i\}\;/\;72,
\end{aligned} \qquad (3.1)$$

where $k^{a,b,i}k^{a,b,i}$ is a shorthand notation for $\sum_{a=1}^{i-1}\sum_{b=1}^{i-1} k^{a,b,i}k^{a,b,i}$ and for a fixed index number i, $k^{i,i,i,r}h_{i,i}h_r = \sum_{r=1}^{i-1} k^{i,i,i,r}h_{ii}\,h_r$. Also, $k^{a,b,c,\ldots}$ is the cumulants of multi-dimensional random vector $X=(X^1, X^2, \ldots, X^p)$ and $h_{ijk\ldots}$ is the Hermite polynomial related to the multi-dimensional normal distribution.

Example Suppose X has a lognormal distribution, meaning that for some normal random variable, Y, $X=\exp(Y)$. For convenience, assume Y is $N(\mu,1)$ in which case $E(X)=\exp(\mu+1/2)$ and the standard deviation

$$\sigma=\sqrt{Var(X)}=\sqrt{\exp\{2(\mu+1)\}-\exp\{2\mu+1\}}.$$

The MLE of parameter $E(X)$ is given by $\overline{X}$ and the asymptotic distribution of the standardized random variable $Z_1=\sqrt{n}(\overline{X}-E(X))/\sigma$ is standard normal. Fortunately, the above polynomial transformation simplifies considerably in the univariate case because of the most terms except $x^{i,i,i}h_{ii}$, $x^{i,i,i,i}h_{iii}$, $x^{i,i,i}x^{i,i,i}h_{iii}$, and $x^{i,i,i}x^{i,i,i}h_i$ are null. Also, these non-vanishing terms are more simplified as the following terms

$$\rho_3 h_2,\ \rho_4 h_3,\ \rho_3^2 h_3, \text{ and } \rho_3^2 h_1,$$

respectively, where $h_i's$ are the Hermite polynomials related to the univariate normal distribution and ρ_i (i=1,2) is the i-th standardized cumulant of Z_1. So, the modified normalizing transformation is given by

$$Z_2 = Z_1 - \rho_3(Z_1^2 - 1)/6 - \rho_4(Z_1^3 - 3Z_1)/24 + \rho_3^2(4Z_1^3 - 7Z_1)/36 \qquad (3.2)$$

where ρ_3 and ρ_4 are the third and the fourth standardized cumulant of Z_1. Figure 1 & 2 show the kernel density estimate and the normal probability plot of the standardized random variable Z_1 and the modified normalizing transformation Z_2 for $\mu = 0$, n=100 (sample size), and 1000 replications. The distribution of Z_1 is skewed to the right even for relatively large sample size but, the suggested statistic Z_2, which is a function of Z_1, is very similar to standard normal distribution. So, we conclude the suggested normalizing process converges more faster than the standardized ML estimator to normal distribution.

Figure 1. Kernel Density Estimate and Normal Probability Plot of Z_1 for $n = 100$

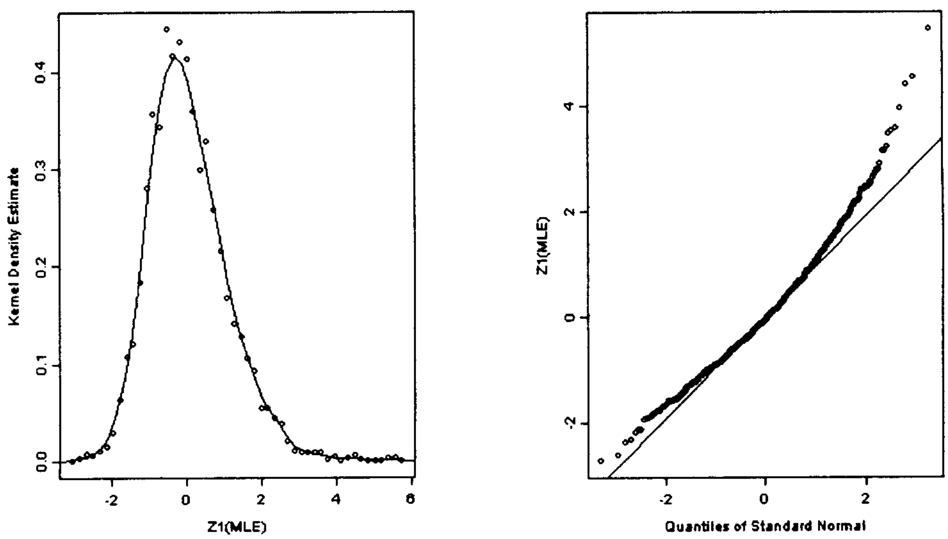
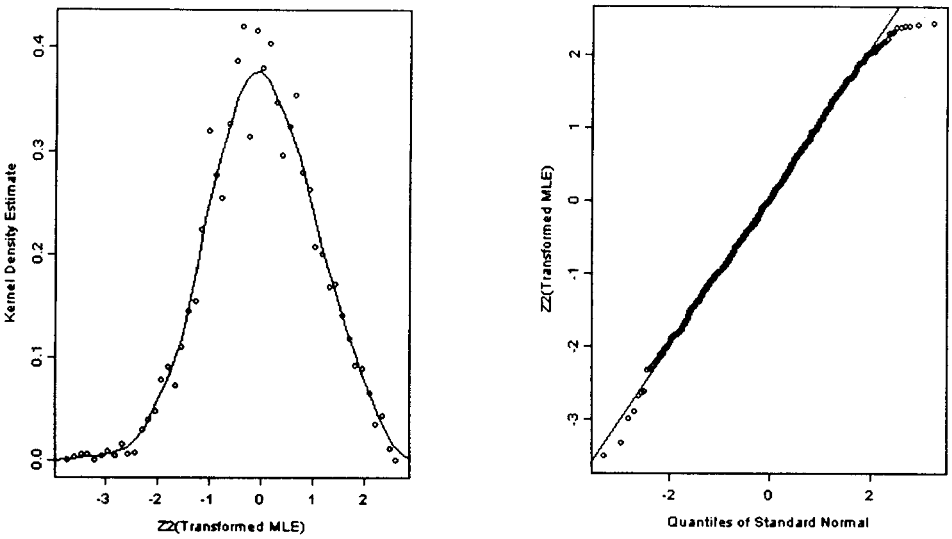


Figure 2. Kernel Density Estimate and Normal Probability Plot of Z_2 for $n = 100$



4. Conclusions

A well known asymptotic property is that a multi-dimensional MLE has normal distribution. In this paper we derive the first four cumulants of multi-dimensional MLE and suggested the explicit form of the transformation of multi-dimensional MLE which converges more faster than the standardized ML estimators to normal distribution. Detailed calculations need the knowledges of the methods of cumulant computations and tensor methods in McCullagh (1987). Using these faster normalizing process we can derive more accurate asymptotic inferences based on multi-dimensional MLE even for small sample sizes. The accuracy of the suggested modified normalizing process is also considered through simple example. In a similar fashion the suggested method can be adapted to many other statistical inferences.

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Appendix

Let us prove Theorem 1 and 2 in Section 2.

Proof of Theorem 1.

(A)

$$
\begin{aligned}
E(\hat{\delta}^r) &= (n^{-1/2})^{i-1}E(a_i^r) \qquad (i\geq 1) \\
&= n^{-1/2}(\bar{x}^{r,s}\,\bar{x}^{t,u}\,\bar{x}_{st,u} + \bar{x}^{rst}\,\bar{x}_{s,t}/2) + O(n^{-3/2}) \\
&= n^{-1/2}(\bar{x}^{r,s}\,\bar{x}^{t,u}\,\bar{x}_{st,u} + \bar{x}^{r,a}\,\bar{x}^{s,b}\,\bar{x}^{t,c}\,\bar{x}_{abc}\,\bar{x}_{s,t}/2) + O(n^{-3/2}) \\
&= n^{-1/2}\,\bar{x}^{r,s}\,\bar{x}^{t,u}(\bar{x}_{st,u} + \bar{x}_{stu}/2) + O(n^{-3/2}) \\
&= -n^{-1/2}\,\bar{x}^{r,s}\,\bar{x}^{t,u}(\bar{x}_{s,t,u} + \bar{x}_{s,tu})/2 + O(n^{-3/2}) \qquad \text{by Lemma 2(3).}
\end{aligned}
$$

(B)

$$
\begin{aligned}
Cov(\hat{\delta}^r, \hat{\delta}^s) &= (n^{-1/2})^{i-1+j-1}Cov(a_i^r, a_j^s) \qquad (i, j\geq 1) \\
&= Cov(a_1^r, a_1^s) + n^{-1/2}Cov(a_1^r, a_2^s)\,[2\ ] \\
&\quad + n^{-1}Cov(a_2^r, a_2^s) + O(n^{-3/2})
\end{aligned}
$$

$$
Cov(a_1^r, a_1^s) = \bar{x}^{r,i}\,\bar{x}^{s,j}Cov(Z_i, Z_j) = \bar{x}^{r,i}\,\bar{x}^{s,j}\,\bar{x}_{i,j} = \bar{x}^{r,s}
$$

$$
\begin{aligned}
Cov(a_1^r, a_2^s) &= \bar{x}^{r,i}\,\bar{x}^{s,j}\,\bar{x}^{k,l}Cov(Z_i, Z_{jk}Z_l) + \bar{x}^{r,i}\,\bar{x}^{sjk}Cov(Z_i, Z_jZ_k) \\
&= \bar{x}^{r,i}\,\bar{x}^{s,j}\,\bar{x}^{k,i}E(Z_iZ_{jk}Z_l) + \bar{x}^{r,i}\,\bar{x}^{sjk}E(Z_iZ_jZ_k) \qquad \text{by Lemma 2(1)} \\
&= \bar{x}^{r,i}\,\bar{x}^{s,j}\,\bar{x}^{k,l}cum(Z_i, Z_{jk}, Z_l) + \bar{x}^{r,i}\,\bar{x}^{sjk}cum(Z_i, Z_j, Z_k) \\
&= n^{-1/2}(\bar{x}^{r,i}\,\bar{x}^{s,j}\,\bar{x}^{k,l}\,\bar{x}_{i,jk,l} + \bar{x}^{r,i}\,\bar{x}^{s,a}\,\bar{x}^{j,b}\,\bar{x}^{k,c}\,\bar{x}_{abc}\,\bar{x}_{i,j,k}) \\
&= n^{-1/2}(\bar{x}^{r,i}\,\bar{x}^{s,j}\,\bar{x}^{k,l}\,\bar{x}_{i,jk,l} + \bar{x}^{r,b,c}\,\bar{x}^{s,a}\,\bar{x}_{abc})
\end{aligned}
$$

$$
\begin{aligned}
Cov(a_2^r, a_2^s) &= \bar{x}^{r,s}\,\bar{x}^{t,u}\,\bar{x}^{i,j}\,\bar{x}^{k,l}Cov(Z_{st}Z_u, Z_{jk}Z_l) + \bar{x}^{r,s}\,\bar{x}^{t,u}\,\bar{x}^{ijk}Cov(Z_{st}Z_u, Z_jZ_k) \\
&\quad + \bar{x}^{rst}\,\bar{x}^{ijk}Cov(Z_sZ_t, Z_jZ_k)/4
\end{aligned}
$$

$$= \bar{x}^{r,s}\bar{x}^{t,u}\bar{x}^{i,j}\bar{x}^{k,l}\{Cov(Z_{st}, Z_{jk})Cov(Z_u, Z_l) + Cov(Z_{st}, Z_l)Cov(Z_{jk,u})\}$$
$$+2\bar{x}^{r,s}\bar{x}^{t,u}\bar{x}^{ijk}Cov(Z_{st}, Z_j)Cov(Z_k, Z_u)$$
$$+\bar{x}^{rst}\bar{x}^{ijk}Cov(Z_j, Z_s)Cov(Z_k, Z_t)/2 + O(n^{-1})$$

$$= \bar{x}^{r,s}\bar{x}^{t,u}\bar{x}^{i,j}\bar{x}^{k,l}\{\bar{x}_{jk,st}\bar{x}_{l,u} + \bar{x}_{l,st}\bar{x}_{u,jk}\}$$
$$+2\bar{x}^{r,s}\bar{x}^{t,u}\bar{x}^{i,a}\bar{x}^{j,b}\bar{x}^{k,c}\bar{x}_{abc}\bar{x}_{j,st}\bar{x}_{k,u}$$
$$+\bar{x}^{r,a}\bar{x}^{s,b}\bar{x}^{t,c}\bar{x}^{i,d}\bar{x}^{j,e}\bar{x}^{k,f}\bar{x}_{abc}\bar{x}_{\mathrm{def}}\bar{x}_{j,s}\bar{x}_{k,t}/2 + O(n^{-1})$$

$$= \bar{x}^{r,s}\bar{x}^{t,k}\bar{x}^{i,j}\bar{x}_{jk,st} + \bar{x}^{r,s}\bar{x}^{t,u}\bar{x}^{i,j}\bar{x}^{k,l}\bar{x}_{l,st}\bar{x}_{u,jk}$$
$$+2\bar{x}^{r,s}\bar{x}^{t,c}\bar{x}^{i,a}\bar{x}^{j,b}\bar{x}_{abc}\bar{x}_{j,st} + \bar{x}^{r,a}\bar{x}^{b,e}\bar{x}^{c,f}\bar{x}^{i,d}\bar{x}_{abc}\bar{x}_{\mathrm{def}}/2$$
$$+O(n^{-1})$$

$$= \bar{x}^{r,s}\bar{x}^{t,u}\bar{x}^{i,j}(\bar{x}_{ju,st} + \bar{x}^{k,l}\bar{x}_{l,st}\bar{x}_{u,jk} + 2\bar{x}^{b,l}\bar{x}_{bju}\bar{x}_{l,st}$$
$$+\bar{x}^{a,b}\bar{x}_{ast}\bar{x}_{bju}/2) + O(n^{-1})$$

This completes the proof of Theorem 1.

Proof of Theorem 2.

(C)

$$cum(\hat{\delta}^r, \hat{\delta}^s, \hat{\delta}^t) = cum(a_i{}^r, a_j{}^s, a_k{}^t)(n^{-1/2})^{i-1+j-1+k-1} \qquad (i, j, k \geq 1)$$
$$= cum(a_1{}^r, a_1{}^s, a_1{}^t) + n^{-1/2}\, cum(a_1{}^r, a_1{}^s, a_2{}^t)[3] + n^{-1}$$
$$\{cum(a_1{}^r, a_1{}^s, a_3{}^t)[3] + cum(a_1{}^r, a_2{}^s, a_2{}^t)[3]\} + O(n^{-3/2})$$

$$cum(a_1{}^r, a_1{}^s, a_1{}^t) = \bar{x}^{r,i}\bar{x}^{s,j}\bar{x}^{t,k}\, cum(Z_i, Z_j, Z_k) = n^{-1/2}\bar{x}^{r,i}\bar{x}^{s,j}\bar{x}^{t,k}\bar{x}_{i,j,k}$$

$$cum(a_1{}^r, a_1{}^s, a_2{}^t) = \bar{x}^{r,i}\bar{x}^{s,j}\bar{x}^{t,a}\bar{x}^{b,c}cum(Z_i, Z_j, Z_{ab}Z_c)$$
$$+\bar{x}^{r,i}\bar{x}^{s,j}\bar{x}^{tab}cum(Z_i, Z_j, Z_aZ_b)/2$$

$$= 2\,\bar{x}^{r,i}\,\bar{x}^{s,j}\,\bar{x}^{t,a}\,\bar{x}^{b,c}\ cum(Z_i, Z_{ab})\ cum(Z_j, Z_c)$$
$$+\ \bar{x}^{r,i}\,\bar{x}^{s,j}\,\bar{x}^{tab}\ cum(Z_i, Z_a)\ cum(Z_j, Z_b) + O(n^{-1})$$

$$= 2\,\bar{x}^{r,i}\,\bar{x}^{s,b}\,\bar{x}^{t,a}\,\bar{x}_{i,ab} + \bar{x}^{r,i}\,\bar{x}^{s,j}\,\bar{x}^{t,u}\,\bar{x}^{a,v}\,\bar{x}^{b,w}\,\bar{x}_{uvw}$$
$$\cdot\ \bar{x}_{i,a}\,\bar{x}_{j,b} + O(n^{-1})$$

$$= \bar{x}^{r,j}\,\bar{x}^{s,j}\,\bar{x}^{t,k}\ \{2\,\bar{x}_{i,jk} + \bar{x}_{ijk}\}\ + O(n^{-1})$$

$$cum(a_1{}^r, a_1{}^s, a_3{}^t) = O(n^{-1/2})$$
$$cum(a_1{}^r, a_2{}^s, a_2{}^t) = O(n^{-1/2})$$

(D)

$$cum(\hat{\delta}^r, \hat{\delta}^s, \hat{\delta}^t, \hat{\delta}^u) = (n^{-1/2})^{i+j+k+l-4}\ cum(a_i{}^r, a_j{}^s, a_k{}^t, a_l{}^u)$$
$$= cum(a_1{}^r, a_1{}^s, a_1{}^t, a_1{}^u) + n^{-1/2}\ cum(a_1{}^r, a_1{}^s, a_1{}^t, a_2{}^u)[4]$$
$$+ n^{-1}\{cum(a_1{}^r, a_1{}^s, a_1{}^t, a_3{}^u)[4] + cum(a_1{}^r, a_1{}^s, a_2{}^t, a_2{}^u)[6]\}$$
$$+ O(n^{-3/2})$$

$$cum(a_1{}^r, a_1{}^s, a_1{}^t, a_1{}^u) = \bar{x}^{r,i}\,\bar{x}^{s,j}\,\bar{x}^{t,k}\,\bar{x}^{u,l}\ cum(Z_i, Z_j, Z_k, Z_l)$$
$$= n^{-1}\,\bar{x}^{r,i}\,\bar{x}^{s,j}\,\bar{x}^{t,k}\,\bar{x}^{u,l}\,\bar{x}_{i,j,k,l}$$

$$cum(a_1{}^r, a_1{}^s, a_1{}^t, a_2{}^u) = \bar{x}^{r,i}\,\bar{x}^{s,j}\,\bar{x}^{t,k}\,\bar{x}^{a,b}\,\bar{x}^{c,d}\ cum(Z_i, Z_j, Z_k, Z_{bc}Z_d)$$
$$+\ \bar{x}^{r,i}\,\bar{x}^{s,j}\,\bar{x}^{t,k}\,\bar{x}^{abc}\ cum(Z_i, Z_j, Z_k, Z_bZ_c)/2$$
$$= 6\,\bar{x}^{r,i}\,\bar{x}^{s,j}\,\bar{x}^{t,k}\,\bar{x}^{a,b}\ cum(Z_{bc}, Z_i, Z_j)\ cum(Z_d, Z_k)$$
$$+ 3\,\bar{x}^{r,i}\,\bar{x}^{s,j}\,\bar{x}^{t,k}\,\bar{x}^{a,u}\,\bar{x}^{b,v}\,\bar{x}^{c,w}\,\bar{x}_{uvw}$$
$$cum(Z_b, Z_i, Z_j)\ cum(Z_c, Z_k) + O(n^{-1})$$

$$= 6\,\bar{x}^{r,i}\,\bar{x}^{s,j}\,\bar{x}^{t,c}\,\bar{x}^{a,b}\ n^{-1/2}\,\bar{x}_{i,j,bc}$$
$$+ 3\,\bar{x}^{r,i}\,\bar{x}^{s,j}\,\bar{x}^{t,w}\,\bar{x}^{a,u}\,\bar{x}^{b,v}\,\bar{x}_{uvw}\,\bar{x}_{b,i,j}\ n^{-1/2} + O(n^{-1})$$

$$= n^{-1/2}\, \bar{x}^{r,i}\, \bar{x}^{s,j}\, \bar{x}^{t,k}\, \bar{x}^{u,l}\, \{6\, \bar{x}_{i,j,kl} + 3\, \bar{x}^{a,v}\, \bar{x}_{kuv}\, \bar{x}_{a,i,j}\} + O(n^{-1})$$

$$\begin{aligned}
cum(a_1{}^r,\ a_1{}^s,\ a_1{}^t,\ a_3{}^u) &= \bar{x}^{r,i}\, \bar{x}^{s,j}\, \bar{x}^{t,k}\, \bar{x}^{a,b}\, \bar{x}^{c,d}\, \bar{x}^{e,f}\, cum(Z_i, Z_j, Z_k, Z_{bc}Z_{de}Z_f) \\
&+ \bar{x}^{r,i}\, \bar{x}^{s,j}\, \bar{x}^{t,k}\, \bar{x}^{abc}\, \bar{x}^{d,e}\, cum(Z_i, Z_j, Z_k, Z_b, Z_{cd}Z_e) \\
&+ \bar{x}^{r,i}\, \bar{x}^{s,j}\, \bar{x}^{t,k}\, \bar{x}^{a,b}\, \bar{x}^{cde}\, cum(Z_i, Z_j, Z_k, Z_{bc}Z_dZ_e)/2 \\
&+ \bar{x}^{r,i}\, \bar{x}^{s,j}\, \bar{x}^{t,k}\, \bar{x}^{abc}\, \bar{x}^{\mathrm{def}}\, \bar{x}_{c,e}\, cum(Z_i, Z_j, Z_k, Z_bZ_dZ_f)/2 \\
&+ \bar{x}^{r,i}\, \bar{x}^{s,j}\, \bar{x}^{t,k}\, \bar{x}^{a,b}\, \bar{x}^{c,d}\, \bar{x}^{e,f}\, cum(Z_i, Z_j, Z_k, Z_{bdf}Z_cZ_e)/2 \\
&+ \bar{x}^{r,i}\, \bar{x}^{s,j}\, \bar{x}^{t,k}\, \bar{x}^{abcd}\, cum(Z_i, Z_j, Z_k, Z_bZ_cZ_d)/6
\end{aligned}$$

$$\begin{aligned}
&= 6\, \bar{x}^{r,i}\, \bar{x}^{s,j}\, \bar{x}^{t,e}\, \bar{x}^{a,b}\, \bar{x}^{c,d}\, \bar{x}_{i,bc}\, \bar{x}_{j,de} + 6\, \bar{x}^{r,v}\, \bar{x}^{s,j}\, \bar{x}^{t,d}\, \bar{x}^{a,u}\, \bar{x}^{c,w}\, \bar{x}_{uvw}\, \bar{x}_{j,cd} \\
&\quad + 3\, \bar{x}^{r,i}\, \bar{x}^{j,v}\, \bar{x}^{t,w}\, \bar{x}^{a,b}\, \bar{x}^{c,u}\, \bar{x}_{uvw}\, \bar{x}_{i,bc} + 3\, \bar{x}^{r,v}\, \bar{x}^{s,\mathrm{x}}\, \bar{x}^{t,z}\, \bar{x}^{a,u}\, \bar{x}_{uvw}\, \bar{x}_{xyz} \\
&\quad + 3\, \bar{x}^{r,i}\, \bar{x}^{s,d}\, \bar{x}^{t,f}\, \bar{x}^{a,b}\, \bar{x}_{i,bdf} + \bar{x}^{r,v}\, \bar{x}^{s,w}\, \bar{x}^{t,\mathrm{x}}\, \bar{x}^{a,u}\, \bar{x}_{uvw\mathrm{x}} + O(n^{-1/2})
\end{aligned}$$

$$\begin{aligned}
&= \bar{x}^{r,i}\, \bar{x}^{s,j}\, \bar{x}^{t,k}\, \bar{x}^{u,l}\, \{6\, \bar{x}^{a,b}\, \bar{x}_{i,bu}\, \bar{x}_{j,lk} + 6\, \bar{x}^{a,b}\, \bar{x}_{ubi}\, \bar{x}_{j,ak} \\
&\quad + 3\, \bar{x}^{a,b}\, \bar{x}_{sku}\, \bar{x}_{i,bl} + 3\, \bar{x}_{iua}\, \bar{x}_{jkb} + 3\, \bar{x}_{i,jkl} + \bar{x}_{uijk}\} + O(n^{-1/2})
\end{aligned}$$

$$\begin{aligned}
cum(a_1{}^r,\ a_1{}^s,\ a_2{}^t,\ a_2{}^u) = cum(&\bar{x}^{r,i} Z_i,\ \bar{x}^{s,j} Z_j,\ \bar{x}^{a,b}\, \bar{x}^{c,d} Z_{bc}Z_d + \bar{x}^{abc} Z_bZ_c/2, \\
&\bar{x}^{t,u}\, \bar{x}^{v,w} Z_{uv}Z_w + \bar{x}^{tuv} Z_uZ_v/2)
\end{aligned}$$

$$\begin{aligned}
&= \bar{x}^{r,i}\, \bar{x}^{s,j}\, \bar{x}^{a,b}\, \bar{x}^{c,d}\, \bar{x}^{t,u}\, \bar{x}^{v,w}\, cum(Z_i, Z_j, Z_{bc}Z_d, Z_{uv}Z_w) \\
&\quad + \bar{x}^{r,i}\, \bar{x}^{s,j}\, \bar{x}^{a,b}\, \bar{x}^{c,d}\, \bar{x}^{t,l}\, \bar{x}^{u,m}\, \bar{x}^{v,n}\, \bar{x}_{lmn} \times cum(Z_i, Z_j, Z_{bc}Z_d, Z_uZ_v) \\
&\quad + \bar{x}^{r,i}\, \bar{x}^{s,j}\, \bar{x}^{a,l}\, \bar{x}^{b,m}\, \bar{x}^{c,n}\, \bar{x}^{t,o}\, \bar{x}^{u,p}\, \bar{x}^{w,q}\, \bar{x}_{lmn}\, \bar{x}_{opq} \times cum(Z_i, Z_j, Z_bZ_c, Z_uZ_v)/4
\end{aligned}$$

$$= 8\,\bar{x}^{r,i}\,\bar{x}^{s,j}\,\bar{x}^{a,b}\,\bar{x}^{c,d}\,\bar{x}^{t,u}\,\bar{x}^{v,w} \times cum(Z_{bc}, Z_{uv})cum(Z_d, Z_i)cum(Z_j, Z_w)$$

$$+8\,\bar{x}^{r,i}\,\bar{x}^{s,j}\,\bar{x}^{a,b}\,\bar{x}^{c,d}\,\bar{x}^{t,l}\,\bar{x}^{u,m}\,\bar{x}^{v,n}\,\bar{x}_{lmn} \times cum(Z_{bc}, Z_u)cum(Z_d, Z_i)cum(Z_j, Z_v)$$

$$+2\,\bar{x}^{r,i}\,\bar{x}^{s,j}\,\bar{x}^{a,l}\,\bar{x}^{b,m}\,\bar{x}^{c,n}\,\bar{x}^{t,o}\,\bar{x}^{u,p}\,\bar{x}^{v,q}\,\bar{x}_{lmn}\,\bar{x}_{opq}$$

$$\times cum(Z_b, Z_u)cum(Z_c, Z_i)cum(Z_j, Z_v) + O(n^{-1/2})$$

$$= 8\,\bar{x}^{r,c}\,\bar{x}^{s,v}\,\bar{x}^{a,b}\,\bar{x}^{t,u}\,\bar{x}_{bc,uv} + 8\,\bar{x}^{r,c}\,\bar{x}^{s,n}\,\bar{x}^{a,b}\,\bar{x}^{t,l}\,\bar{x}^{u,m}\,\bar{x}_{lmn}\,\bar{x}_{u,bc}$$

$$+2\,\bar{x}^{r,m}\,\bar{x}^{s,q}\,\bar{x}^{a,l}\,\bar{x}^{m,p}\,\bar{x}^{t,o}\,\bar{x}_{lmn}\,\bar{x}_{opq} + O(n^{-1/2})$$

$$= \bar{x}^{r,i}\,\bar{x}^{s,j}\,\bar{x}^{t,k}\,\bar{x}^{u,l}\,\{8\,\bar{x}_{il,jk} + 8\,\bar{x}^{a,b}\,\bar{x}_{jkl}\,\bar{x}^{u,bi} + 2\,\bar{x}^{m,p}\,\bar{x}_{lmi}\,\bar{x}_{pkj}\} + O(n^{-1/2})$$

$$= \bar{x}^{r,i}\,\bar{x}^{s,j}\,\bar{x}^{t,k}\,\bar{x}^{u,l}\,\{8\,\bar{x}_{ij,kl} + 8\,\bar{x}^{a,b}\,\bar{x}_{jkl}\,\bar{x}_{u,bi} + 2\,\bar{x}^{a,b}\,\bar{x}_{lia}\,\bar{x}_{kjb}\} + O(n^{-1/2})$$

This completes the proof of Theorem 2.