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On Doubly Stochastically Perturbed Dynamical Systems[1)]

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Abstract

We consider a doubly stochastically perturbed dynamical system $\{X_n\}$ generated by $X_n = \Gamma_n(X_{n-1}) + W_n$ where Γ_n is a Markov chain of random functions and W_n is i.i.d. random elements. Sufficient conditions for stationarity and geometric ergodicity of X_n are obtained by considering asymptotic behaviours of the associated Markov chain. Ergodic theorem and functional central limit theorem are proved.

1. Introduction

Consider the process $\{X_n\}$ given by

$$X_{n+1} = \Gamma_{n+1}(X_n) + W_{n+1} \qquad (n \geq 0), \tag{1.1}$$

where $\{\Gamma_n\}$ is a Markov chain of nonlinear random functions , $\{W_n\}$ is a sequence of independent identically distributed random variables and $\{\Gamma_n\}$ and $\{W_n\}$ are independent.

An extensive discussion for the processes $\{X_n\}$ under $\Gamma_{n+1}(X_n) = A_{n+1} \cdot X_n$ and (A_n, W_n) are assumed to be i.i.d. is given in Vervaat(1979) and Feigin and Tweedie(1985). In Brandit(1986), i.i.d. assumption is dropped and existence of a stationary solution of (1.1) is proved under the condition that $(A_n W_n)$ is a stationary ergodic process and some mild additional assumptions.

On the other hand, the case that Γ_n, $n \geq 1$ are nonlinear random functions which is so called IFS(iterated random function systems) has been considered. Barnsley and Demko(1985), Bhattacharya and Lee(1988), and Letac(1986) investigated the ergodicity of the process and ergodic theorem when $\{\Gamma_n\}$ is i.i.d.. Elton(1990) studied the case of stationary sequence, Stenflo(1996) finite semi-Markov process with discrete time. Also a certain doubly stochastic time series model is considered by Meyn and Guo(1993).

In this paper, we consider the process $\{X_n\}$ defined by (1.1) when $\{\Gamma_n\}$ is a homogeneous Markov chain. We find sufficient conditions, under which $\{X_n\}$ is geometrically ergodic and

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functional central limit theorem holds.

The terminologies and concepts mentioned in this paper such as aperiodic, irreducible, small set, stationarity, (geometric) ergodicity etc. can be found in books on Markov chains (e.g. Nummelin(1984), Meyn and Tweedie(1993)).

Following two propositions give useful tools to determine the geometric ergodicity of the processes.

Proposition 1.1 (Tweedie(1983)) Let $\{X_n\}$ be an aperiodic, irreducible Markov chain. Suppose that there exist a small set C, a nonnegative measurable function g positive constants $c_1,$ c_2 and $\rho<1$ such that

$$E\{g(X_{n+1})|X_n=x\}\le \rho g(x)-c_1, \qquad x\in C^c, \tag{1.2}$$

and

$$E\{g(X_{n+1})|X_n=x\}\le c_2, \qquad x\in C. \tag{1.3}$$

Then $\{X_n\}$ is geometrically ergodic.

Above proposition is the so called Tweedie's drift criterion for the geometric ergodicity of Markov chains and the function g is called the (stochastic) Lyapunov function.

Proposition 1.2 (Tjostheim(1990)) If there exists a positive integer m_0 such that $\{X_{nm_0}\}$ is geometrically ergodic, then $\{X_n\}$ is geometrically ergodic.

Combining above two propositions produce the m_0-step criteria to determine the geometric ergodicity of Markov chains.

2. Main Results

Let $C(R^n, R^n)$be the space of all continuous functions endowed with the compact-open topology and let Γ be a compact subset of $C(R^n, R^n)$. Γ inherits its topology from $C(R^n, R^n)$. Let $B(\Gamma)$ be the Borel σ-field of Γ and let $B(R^n)$ be the Borel σ-field of R^n. Note that $C(R^n, R^n)$ is a complete separable metric space.

Let us , for a function $f\in C(R^n, R^n)$, define a generalized norm

$$\|f\| = \sup_{x\neq y}\frac{|f(x)-f(y)|}{|x-y|}.$$

Here $\|\cdot\|$ denotes the usual metric on R^k $(k\ge 1)$.

Let $\{\Gamma_n\}_{n=1}^{\infty}$ be a homogeneous Markov chain with Γ as its state space and $\{W_n\}_{n=1}^{\infty}$ be a sequence of independent and identically distributed random elements with common distribution Q and $E|W_1| < \infty$. Assume $\{\Gamma_n\}$ and $\{W_n\}$ are independent.

In this paper, we consider the stochastic process $\{X_n\}$ given by

$$X_0, \qquad X_{n+1} = \Gamma_{n+1}(X_n) + W_{n+1} \qquad (n \geq 0). \tag{2.1}$$

X_n derived by (2.1) is not a Markov chain. In order to study the asymptotic properties of $\{X_n\}$, it is convenient to consider the associated Markov chain $\Phi_n = (\Gamma_n, X_n)$, $n \geq 0$ with state space $\Gamma \times R^n$ and homogeneous transition probability function

$$\tilde{p}((\gamma, x), C) = \int\int I_C(g, g(x)+s) p(\gamma, dg) Q(ds),$$

where $C \in B(\Gamma) \times B(R^n)$, and p is the transition probability function for $\{\Gamma_n\}$.

Lemma 2,1 If $\{\Gamma_n\}$ is weak Feller, i.e. for any real-valued bounded uniformly continuous function f, $\int f(g) p(\cdot, dg)$ is continuous, then so is $\{\Phi_n\}$.

Proof. Suppose $H : \Gamma \times R^n \to R$ is a bounded and uniformly continuous function and $(\gamma_n, x_n) \to (\gamma, x)$ as $n \to \infty$. Then

$$\begin{aligned} &\left| \int\int H(g, g(x_n) + s) p(\gamma_n, dg) dQ(s) - \int\int H(g, g(x)+s) p(\gamma, dg) dQ(s) \right| \\ &\leq \int\int |H(g, g(x_n)+s) - H(g, g(x)+s)| p(\gamma_n, dg) dQ(s) \\ &\qquad + \int \left| \int H(g, g(x)+s) p(\gamma_n, dg) - \int H(g, g(x)+s) p(\gamma, dg) \right| dQ(s) \\ &\to 0, \end{aligned}$$

as $n \to \infty$ by uniform continuity of H and compactness of Γ.

Next lemma gives sufficient condition which guarantees that every compact set is a small set.

Lemma 2.2 If a Markov chain $\{\Phi_n\}$ is aperiodic, φ-irreducible Feller chain such that supp φ has nonempty interior, then every compact set is small.

Proof. See theorem 6.25(p.134) in Meyn and Tweedie(1993).

For simplicity of the notation, we define

$$F_n(x) = \Gamma_n(x) + W_n.$$

We make the following assumptions:

(A1) $\{\Gamma_n\}$ is aperiodic and φ-irreducible Feller chain such that supp φ has nonempty interior.

(A2) The distribution of W_n is absolutely continuous with respect to the Lebesgue measure λ and has a density function q which is positive a.e.(λ).

(A3) There exist x_0 in R^n, constants $b<\infty$, $\rho<1$, and positive integer m_0 such that for each initial $\phi_0=(\gamma, x)$,

$$E_\gamma \|F_{m_0}\cdots F_1\| \leq \rho \tag{2.2}$$

and

$$E_\gamma |F_{m_0}\cdots F_1(x_0)| \leq b. \tag{2.3}$$

Theorem 2.1 Assume (A1), (A2) and (A3). Then Φ_n is geometrically ergodic with invariant probability, say π and the distribution of X_n converges in norm to the measure π_2 exponentially fast, where

$$\pi_2(B)=\pi(\{(\gamma, x)|x\in B\}), \qquad B\in B(R^n). \tag{2.4}$$

Proof. Under (A1) and (A2), it can be easily seen that Φ_n is a $\varphi\times\lambda$-irreducible.

Define a (stochastic) Lyapunov function $v: \Gamma\times R^n \to R$ by

$$v((\gamma, x)) = |x-x_0|+1, \tag{2.5}$$

where x_0 is given in (A3). Then for any n,

$$\begin{aligned}
&E[v(\Phi_{(n+1)m_0}) \mid \Phi_{nm_0}=(\gamma, x)] \\
&= E[|X_{(n+1)m_0}-x_0|+1 \mid \Gamma_{nm_0}=\gamma, X_{nm_0}=x] \\
&\leq E[|F_{(n+1)m_0}\cdots F_{nm_0+1}(x) - F_{(n+1)m_0}\cdots F_{nm_0+1}(x_0)| \\
&\qquad +|F_{(n+1)m_0}\cdots F_{nm_0+1}(x_0)-x_0|+1 \mid \Gamma_{nm_0}=\gamma, X_{nm_0}=x] \\
&\leq E_\gamma \|F_{(n+1)m_0}\cdots F_{nm_0+1}\| \; |x-x_0| + E_\gamma|F_{(n+1)m_0}\cdots F_{nm_{0-1}}(x_0)| + |x_0|+1 \\
&\leq \rho|x-x_0| + b + |x_0| + 1.
\end{aligned}$$

Let for $r>0$, $K_r=\{x\in R^n \mid |x-x_0|\leq r\}$. Then for some $c_1>0$, we may choose ρ', $\rho<\rho'<1$ and $r>00$ such that

$$E[v(\Phi_{(n+1)m_0})|\Phi_{nm_0}=(\gamma, x)] \leq \rho' v((\gamma, x)) - c_1, \quad (\gamma, x)\in\Gamma\times K_r^{c}. \tag{2.6}$$

Clearly, we have

$$E[v(\Phi_{(n+1)m_0})|\Phi_{nm_0}=(\gamma, x)] \leq \rho r + b + |x_0| + 1 < \infty, \quad (\gamma, x)\in\Gamma\times K_r \tag{2.7}$$

By lemma 2.1, Φ_n is weak Feller and hence $\Gamma\times K_r$ is a small set by lemma 2.2.

Therefore, (2.7) together with (2.6), by applying the proposition 1.1, ensures the geometric ergodicity of $\{\Phi_{nm_0}\}$, and hence by the proposition 1.2, geometric ergodicity of $\{\Phi_n\}$ follows.

Geometric ergodicity of Φ_n implies that the existence of π which is the unique invariant probability and a constant θ, $0<\theta<1$ such that

$$\sup_C \{\theta^n | \tilde{p}^n((\gamma, x), C) \to \pi(C)|\} \to 0, \quad \text{as} \quad n \to \infty, \quad (\gamma, x) \in \Gamma \times R^n.$$

Therefore π_2 defined by (2.4) is the stationary distribution for X_n of (1.1) and the distribution of X_n converges in norm to π_2 exponentially fast.

Following is an additional assumption.

(A4) $\sup_{1 \le k < m_0} E_\gamma \|\Gamma_k\| \cdots \|\Gamma_1\| \le d < \infty$, for any initial γ.

Before stating the next theorem, we give a lemma which is easy to check.

Lemma 2.3 For each initial γ, $E_\gamma \|F_{m_0} \cdots F_1\| \le E_\gamma \|\Gamma_{m_0}\| \cdots \|\Gamma_1\|$.

Theorem 2.2 Let conditions (A1) and (A4) hold. Suppose $E_\gamma \|\Gamma_{m_0}\| \cdots \|\Gamma_1\| \le \rho$ for some m_0 and $\rho < 1$. Then there exists a unique invariant probability π for Φ_n such that

$$\tilde{p}^n(\mathbf{x}, d\mathbf{y}) \to \pi(d\mathbf{y}), \quad \text{weakly} \quad \mathbf{x}, \mathbf{y} \in \Gamma \times R^n.$$

Proof. The proof follows essentially the same line of Meyn(1989). We give the sketch of the proof. For each $\varepsilon \in [0,1]$, define a perturbed process $\{X_n^\varepsilon, \ n \ge 0\}$,

$$X_0^\varepsilon = X_0, \qquad X_{n+1}^\varepsilon = \Gamma_{n+1}(X_n^\varepsilon) + W_{n+1} + \varepsilon N_{n+1}$$

where $\{N_n\}$ is a sequence of i.i.d. $N(0,1)$, and $\{\Gamma_n\}$, $\{W_n\}$ and $\{N_n\}$ are independent.

By theorem 2.1 and lemma 2.3, for each $\varepsilon > 0$, $\Phi_n^\varepsilon = (\Gamma_n, X_n^\varepsilon)$ is geometrically ergodic with invariant probability, say π^ε, from which we have as $n \to \infty$,

$$E_{\phi_0}[h(\Phi_n^\varepsilon)] \to \int h \, d\pi^\varepsilon, \tag{2.8}$$

for every bounded uniformly continuous function h on $\Gamma \times R^n$. From assumption $E_\gamma \|\Gamma_{m_0}\| \cdots \|\Gamma_1\| \le \rho$, we have that

$$E_{\phi_0}[|X_n - X_n^\varepsilon|] \le \frac{\varepsilon K m_0 d}{1-\rho} \tag{2.9}$$

for every initial $\phi_0 = (\gamma, x) \in \Gamma \times R^n$ and $K = E|N_1|$, and hence we get

$$\lim_{\varepsilon \to 0} \sup_{n \ge 0} E_{\phi_0}[|X_n - X_n^\varepsilon|] = 0. \tag{2.10}$$

Moreover for any bounded uniformly continuous function f on $\Gamma\times R^n$, by applying Chebyshev's inequality and (2.10), we have

$$\lim_{\varepsilon\to 0}\sup_{n\geq 0} E_{\phi_0}[f(X_n)-f(X_n^{\varepsilon})]=0 \tag{2.11}$$

Since π^{ε} is tight , there exists a sequence $\{\varepsilon_n\}$, $\varepsilon_n\to 0$ and π such that

$$\pi^{\varepsilon_n}\to\pi \quad \text{weakly as } n\to\infty. \tag{2.12}$$

Hence combining (2.8), and (2.10)-(2.12), we have

$$\lim_{n\to\infty} E_{\phi_0}[f(\Phi_n)]=\int f d\pi. \tag{2.13}$$

This implies π is an invariant probability for Φ_n and also π is the unique limit point of the probabilities $\{\pi^{\varepsilon},\ \varepsilon>0\}$ and hence π is the unique invariant probability for Φ_n .

Corollary 2.1 Suppose that $\{\Gamma_n\}$ is weak Feller and that (A3) holds with $m_0=1$. Then there exists a stationary solution of the process generated by (2.1).

Proof. In the proof of theorem 2.1, we only use the condition (A3) to get the eqn.(2.5) and the eqn.(2.6). Therefore the conclusion follows from the compactness of $\Gamma\times K_r$ and the weak Feller property of $\{\Phi_n\}$ (see Tweedie(1988)).

Theorem 2.3 Suppose that the assumptions of theorem 2.1 or theorem 2.2 hold and that π is the unique invariant probability for Φ_n. If the distribution of Φ_0 is π, then for a measurable function f: $R^n\to R$ with $E|f(X_0)|<\infty$,

$$\frac{1}{n}\sum_{k=1}^{n} f(X_n)\to E[f(X_0)].$$

Proof. If the distribution of Φ_0 is π, then $\{\Phi_n\}$ is a stationary ergodic Markov chain and hence by Birkhoff's ergodic theorem, for a measurable function f, $E|f(\Phi_0)|<\infty$,

$$\frac{1}{n}\sum_{k=1}^{n} f(\Phi_n)\to E[f(\Phi_0)] \tag{2.14}$$

If we take $f(\gamma,x)=f(x)$, then from (2.14),

$$\frac{1}{n}\sum_{k=1}^{n} f(X_n)\to E[f(X_0)], \quad \text{as } n\to\infty.$$

In the end of this section, we consider the functional central limit theorem for

$$Y_n(t) = \frac{1}{\sqrt{n}} \sum_{k=0}^{[nt]} (f(\Phi_k) - \int f d\pi), \qquad t \geq 0 \tag{2.15}$$

which is essential in evaluating asymptotic efficiencies of estimators. Deriving functional central limit theorem for Markov processes has been discussed in many papers such as Glynn and Meyn(1996) and the references therein.

In order to state the next theorem, let for $g \in L^2(\Gamma \times R^n, \pi)$, $\boldsymbol{x}, \boldsymbol{y} \in \Gamma \times R^n$.

$$Pg(\boldsymbol{x}) = \int g(\boldsymbol{y}) p(\boldsymbol{x}, d\boldsymbol{y}), \qquad \bar{f} = \int f d\pi, \qquad \|g\|_2^2 = \int g^2(\boldsymbol{y}) d\pi(\boldsymbol{y})$$

and π denote the invariant initial probability.

Theorem 2.4 Let the hypotheses of theorem 2.1 hold with $m_0 = 1$ and let $v((\gamma, x)) = |x - x_0| + K$ where K is any positive constant greater than 1. Then for any $f^2 \leq v$, $Y_n(\cdot)$ converges in distribution to a Brownian motion with mean 0 and variance parameter $\|g\|_2^2 - \|Pg\|_2^2$, where $Pg - g = f - \bar{f}$. In particular, the functional central limit theorem holds for every bounded measurable function f.

Proof. For given $v((\gamma, x))$, by the same arguments as those in the proof of theorem 2.1, there exist constants r, $M < \infty$ and $0 < \lambda < 1$ such that

$$Pv((\gamma, x)) \leq \lambda v((\gamma, x)) + MI_{\{(\gamma, x): (\gamma, x) \in \Gamma \times K_r\}}.$$

Then theorem 4.1 in Glynn and Meyn(1996) ensures that if $f^2 \leq v$, then $f \in L^2(\Gamma \times R^n, \pi)$ is in the range of $P - I$ and hence the functional central limit theorem holds for f with mean 0 and variance parameter $\|g\|_2^2 - \|Pg\|_2^2$. Suppose f is bounded measurable with $|f| \leq K_0$ for some $K_0 < \infty$. Then by taking $K = K_0^2$ in $v((\gamma, x)) = |x - x_0| + K$, we have $f^2 \leq v$ and therefore f holds the functional central limit theorem.

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