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Geometric Interpretation on Chebyshev Type Inequalities[1)]

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Abstract

We present a geometric interpretation of Chebyshev type inequalities. This uses a simple diagram which illustrates the functional bound for the indicator function of the event whose probability we want to assess. We also give a geometric interpretation of the inequalities in terms of volume in a Euclidean space of appropriate dimension. Markov's inequality and Chebyshev's inequality are treated in more detail.

1. Chebyshev Type Inequalities

There are a large number of Chebyshev type inequalities, that is, inequalities for the value of a distribution function in terms of moments of the distribution. See Godwin (1955) for an extensive survey of the subject. For an historical review of the subject, see Heyde and Senata (1977). We restrict our attention to the case where we require an upper bound of the tail probability with known high order moment. Without loss of generality, let X be a positive random variable such that $E(X^k) < \infty$, for $k \geq 1$ a given integer. Then, a generalized form of Chebyshev's inequality can be stated as

$$\Pr(X > a) \leq \min\{1, E(X^k)/a^k\} \text{ for every } a > 0. \tag{1}$$

Throughout, assume that $E(X^k)/a^k < 1$ to avoid the trivial case. For illustration, consider the case where $k = 1$, then (1) reduces to Markov's inequality. Furthermore, when $k = 2$ with $X = |Y - E(Y)|/\sigma(Y)$, where $\sigma(Y)$ is the standard deviation of Y, (1) reduces to the standard Chebyshev inequality. Its proof in most text books usually splits the interval $\{X > 0\}$ into $\{X \geq a\}$ and $\{0 < X < a\}$, then proceeds to discuss probabilities related to these. In this paper we give an alternative proof based on a simple diagram, which reflects the idea of splitting the interval through an indicator function. This can be more informative in that it gives a direct indication of when the equality holds. Second, we explain the geometric meaning of Chebyshev type inequalities by expressing the k-th moment as a

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volume of an object in k-dimensional Euclidean space.

Note that $\Pr(X \geq a)$ can be expressed as $E\{I(X \geq a)\}$, where $I(\cdot)$ denotes the indicator function. Now that $E(X^k)$ is given, we will find a linear function of x^k which is always greater than or equal to the indicator function $I(x \geq a)$ over the region $x > 0$. From Figure 1, it is easy to see that $I(x \geq a) \leq x^k/a^k$ for $a > 0$. Therefore, the inequality can be checked by the monotone property of expectation, so that

$$\Pr(X \geq a) = E\{I(X \geq a)\} \leq E(X^k/a^k).$$

In addition to the proof of the inequality itself, this figure helps determine when the equality holds. From Figure 1, it is clear that the equality holds at the points $x = 0$, or $x = a$, which are marked with "x" . Therefore, if we consider a distribution concentrated at the points 0 and a, then the equality holds in (1).

2. Geometric Interpretation

In this section, we give a geometric interpretation of Chebyshev type inequalities as expressed in (1). We may rewrite the inequality (1) as

$$(a^k/k!)\Pr(X \geq a) \leq E(X^k)/k!. \tag{2}$$

The left hand side of (2) represents the volume of the k-dimensional prism whose base is described by the region $\{(u_1,\dots,u_k) : 0 < u_1 < \cdots < u_k < a\}$ and whose height is the constant $\Pr(X \geq a)$. We may choose a among the continuity points of the distribution function of X, since the set of discontinuity points has a Lebesgue measure 0. On the other hand, for a positive random variable X with k-th moment, we can check that

$$\begin{aligned} E(X^k) &= \int_0^\infty x^k dF(x) \\ &= k\int_0^\infty \int_0^x u_k^{k-1} du_k dF(x) \\ &= k(k-1) \int_0^\infty \int_0^x \int_0^{u_k} u_{k-1}^{k-2} du_{k-1} du_k dF(x) \\ &\;\vdots \\ &= k! \int_0^\infty \int_0^x \int_0^{u_k} \cdots \int_0^{u_2} du_1 \cdots du_k dF(x) \end{aligned}$$

$$= k! \int_0^\infty \int_{u_2}^\infty \int_{u_3}^\infty \cdots \int_{u_{k-1}}^\infty \int_{u_k}^\infty dF(x) du_k \cdots du_3 du_2 du_1$$

$$= k! \int_0^\infty \cdots \int_0^\infty \Pr(X > u_k) I(u_k > \cdots > u_1) du_k \cdots du_1$$

$$= k! \int_0^\infty \cdots \int_0^\infty \Pr(X \geq u_k) I(u_k > \cdots > u_1) du_k \cdots du_1 .$$

Since $\Pr(X = u_k)$ can be positive for at most countably many values of u_k, the two integrands differ only on a set of Lebesgue measure 0 and hence the integrals are the same. Thus, $E(X^k)/k!$ represents the volume of a $(k+1)$-dimensional figure which has a k-dimensional base described by the region $\{(u_1, \ldots, u_k) : 0 < u_1 < \cdots < u_k\}$ and whose height varies as the length above the cumulative distribution function of X. This clearly includes the $(k+1)$-dimensional prism described on the left hand side of the inequality (2). Now we give a more detailed explanation for the cases where $k = 1, 2$.

EXAMPLE 1. (MARKOV'S INEQUALITY) It is well known that for $X > 0$

$$E(X) = \int_0^\infty \Pr(X \geq x) dx,$$

which is the area above the cumulative distribution function of X. Figure 2 depicts a typical situation clearly. The rectangle displayed has an area equal to $a \times \Pr(X \geq a)$. Therefore, we can write

$$a \times \Pr(X \geq a) \leq E(X),$$

which leads to Markov's inequality. This figure also shows that the equality holds when the distribution function of X is a step function with steps at $x = 0, a$, that is, a distribution with probability mass at the two points 0 and a.

EXAMPLE 2. (CHEBYSHEV'S INEQUALITY) Consider the standard Chebyshev inequality as the second example. We need to check that Chebyshev's inequality can be reduced to the following;

$$\Pr(|X| \geq a) \leq E(X^2)/a^2,$$

where $E(X) = 0$. Let $H(x)$ be the distribution function of the random variable $|X|$. Then, we can write

$$
\begin{aligned}
E(X^2) &= \int_0^\infty y^2 dH(y) \\
&= 2\int_0^\infty \int_0^y v dv dH(y) \\
&= 2\int_0^\infty \int_0^y \int_0^v du dv dH(y) \\
&= 2\int_0^\infty \int_u^\infty \int_v^\infty dH(y) dv du \\
&= 2\int_0^\infty \int_0^\infty \Pr(|X| > v) I(v > u) dv du \\
&= 2\int_0^\infty \int_0^\infty \Pr(|X| \geq v) I(v > u) dv du .
\end{aligned}
$$

Therefore, in Figure 3, $E(X^2)/2$ represents the volume of the wedge-shaped object that consists of the part above the plane formed by distribution function $H(\cdot)$ stretched over the u direction, but restricted to the region $\{v > u\}$. The four vertices of the wedge-shaped object are given in the order (u, v, H) by $(0,0,0)$, $(0,0,1)$, $(0, H^{-1}(1), 1)$, and $(H^{-1}(1), H^{-1}(1), 1)$. Now, the wedge inscribed has the volume $(a^2/2)\Pr(|X| \geq a)$. Therefore, we have

$$
(a^2/2)\Pr(|X| \geq a) \leq E(X^2)/2,
$$

which reduces to the standard Chebyshev inequality.

References

[1] Godwin, H. J. (1955), On Generalizations of Tchebychef's Inequality, Journal of the American Statistical Association, **50**, 923-945.

[2] Heyde, C. C. and Seneta, E. (1977). I. J. Bienaymé: Statistical Theory Anticipated. Springer-Verlag, New York.

Figure 1: Diagram for the proof of Chebyshev type inequalities. The black line denotes the indicator function $I(x \geq a)$, and the gray line denotes the upper bound x^k / a^k.

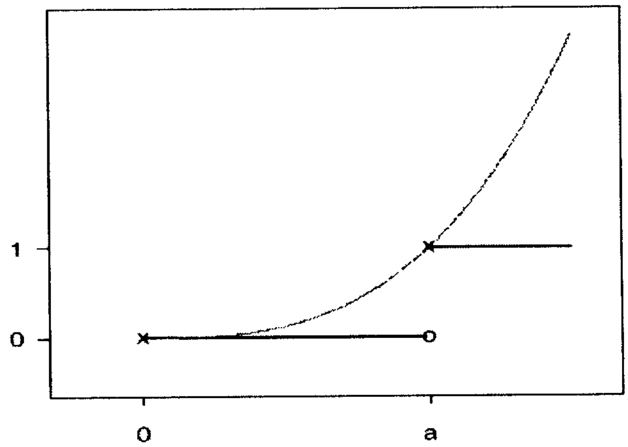

Figure 1.

Figure 2: Geometric interpretation of Makov's inequality. The area above the curve equals $E(X)$, while the area of the rectangle inscribed equals $a \times \Pr(X \geq a)$.

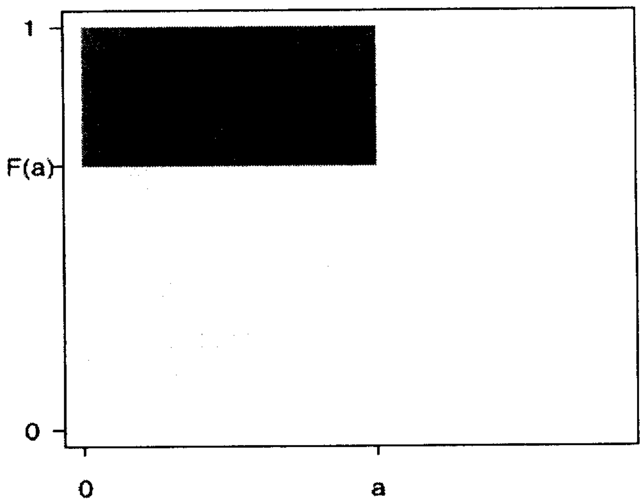

Figure 2.

Figure 3: Geometric interpretation of Chebyshev's inequality. The volume of the wedge-shaped object equals $E(X^2)/2$, where the volume of the wedge inscribed equals $(a^2/2)\Pr(|X| \geq a)$.

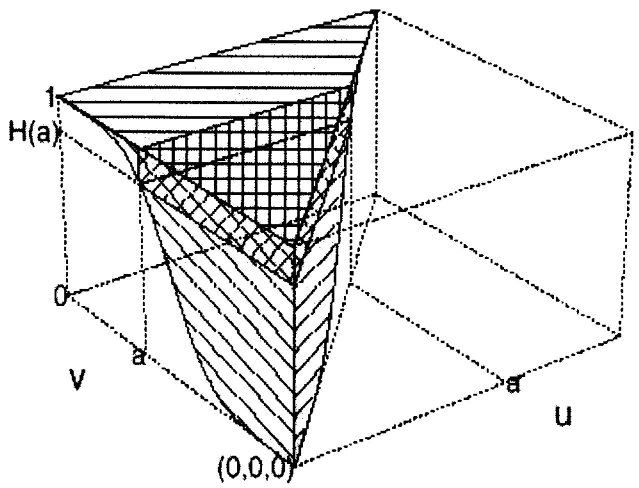

Figure 3.