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Cost Optimization of Ineffective Periodic Preventive Maintenance[1]

Gi Mun Jung[2] Dong Ho Park[3] Joon Keun Yum[4]

Abstract

This paper considers an imperfect repair model for which the repairable system is maintained preventively at periodic times and is replaced by a new system when a predetermined number of preventive maintenance has been applied. Our main objective of this paper is to determine the optimal number of preventive maintenances before the system is replaced and the optimal length of interval between two consecutive preventive maintenances under a new repair model, which is referred to as an ineffective preventive maintenance. Such a model assumes a periodic preventive maintenance in which the system is effectively maintained with a certain probability. Otherwise, the system is not improved at all after each maintenance and thus the failure rate remains the same as before. The criteria to determine the optimal number of preventive maintenances and length of period is the expected cost rate per unit time for an infinite time span. We give the explicit expressions for the expected cost rate per unit time. Some numerical examples are presented for illustrative purposes.

1. Introduction

Preventive maintenance(PM) is adapted to slow the deterioration of the system due to aging while the system is still operating and thus to extend the system life. Several PM policies have been proposed and investigated in the literature. Nakagawa(1979) proposes an optimum policy when the preventive maintenance is imperfect. He assumes that the unit after PM has the same failure rate as it has been before PM with probability $p(0 \le p < 1)$ and is as good as new with probability $1-p$. Murthy and Nguyen(1981) study the optimal age replacement policy with imperfect preventive maintenance. The preventive maintenance is imperfect in the sense that the system after PM is either fails instantaneously with a probability p or is like new with a probability $1-p$. Nakagawa(1988) introduces improvement factors in hazard rate and age for a sequential preventive maintenance policy and analyzes two imperfect PM models : i) PM reduces the hazard rate while it increases with the number of PM's, and ii) PM reduces the age.

Those papers mentioned above assume that the system after each PM is "as good as new"

1) This research was supported in part by Hallym University Research Fund during 1997-1998.
2) Lecturer, Department of Statistics, Dongguk University, Seoul, Korea.
3) Professor, Department of Statistics, Hallym University, Chunchon, Korea.
4) Professor, Department of Statistics, Dongguk University, Seoul, Korea.

or "as bad as old" with a certain probability. However, such assumptions give only two different levels of maintenance effects and thus it is too narrow to explain the benefit of undergoing the preventive maintenance activity, since the maintenance is usually designed to improve the state of system by some degree for a repairable system. That is, the system may not be restored to as good as new after PM. Also, it is not unusual that the system after PM has the same failure rate as it has been before PM.

In this paper, we consider a periodic PM policy under which the system is effectively maintained with a probability $p(0 \leq p \leq 1)$. Effective PM of the system reduces stress and hence slows the rate of system degradation. With a probability of $1-p$, the system after PM is not improved and thus possesses the same failure rate as it has been before PM. Such a model is referred to as 'Ineffective Preventive Maintenance(IEPM)' model throughout this paper.

The probability of effective PM at each PM intervention is implicitly assumed to be known under this model. However, it may not be true in real situations. Brown and Proschan(1982) adopt the similar ideas to propose an imperfect repair model, in which a system is perfectly repaired with probability p and minimally repaired with probability $1-p$ when the failure occurs. Lim, Lu and Park(1998) extend Brown and Proschan's(1982) imperfect repair model by assuming that the probability of perfect repair is random, instead of a constant. Some discussions on the choice of p is given in those papers.

The system is maintained preventively at periodic times kx and is replaced at the Nth PM, where $k=1,2,\cdots,N$. If the system fails between PMs, it undergoes only minimal repair and hence, the failure rate remains undisturbed by any of these minimal repairs. The expression to compute the expected cost rate per unit time is given. We also obtain the optimal period x and the optimal number N for the periodic IEPM policy, which minimize the expected cost rate per unit time for an infinite time span.

In Section 2, we define a periodic IEPM model and present the expressions for the expected cost rate for the new model. In Section 3, we consider the problem of finding the optimal period and the optimal number for the periodic IEPM policies, simultaneously. Section 4 presents the explicit solutions for the optimal periodic IEPM policies when the failure time follows a Weibull distribution.

2. Expected Cost Rate

In this paper we consider an IEPM policy for which the periodic PM is ineffective. Under such a policy, the system after each periodic PM relieves stress temporarily and hence slows the rate of system degradation with probability $p(0 \leq p \leq 1)$ and has the same failure rate as it has been without PM with probability $1-p$. The case when $p=1$ has been studied by Canfield(1986) and Park, Yum and Jung(1998) and is referred to as an effective preventive maintenance(EPM) policy.

Suppose that the system is maintained preventively at periodic times kx and is replaced at the Nth PM, where $k=1,2,\cdots,N$. If the system fails between PMs, it undergoes only minimal repair and hence, the failure rate remains unchanged by any of these minimal repairs.

To determine the optimal period x and the optimal number N for the periodic IEPM we assume that the necessary times to conduct PM, minimal repair and replacement are negligible. The failure rate of the system under the periodic IEPM can be obtained as

$$h_{ie}(t)=p\cdot h_e(t)+(1-p)h(t), \tag{1}$$

where $h_{ie}(t)$ and $h_{e(t)}$ are the failure rates of the system under the periodic IEPM and the periodic EPM, respectively and $h(t)$ is the failure rate until the first PM is conducted. The equation (1) can be rewritten as

$$h_{ie}(t)=p\{h_{ie}(kx)+h(t-k\tau)-h(k(x-\tau))\}+(1-p)h(t), \tag{2}$$

for $k=0,1,2,\cdots$, where $kx\leq t<(k+1)x$, $h_{ie}(0)=h_e(0)=h(0)$, τ is restoration interval and x is a time interval between PM interventions. The restoration interval is the length of interval by which the operational stress of a system is reduced at each EPM intervention. Thus, the EPM at time t restores the failure rate to its shape at $t-\tau$, while the level remains unchanged. Canfield(1986) gives more detailed discussions on the failure rate under the periodic EPM. The equation (2) formulates the failure rate of the system under the period IEPM model. The first part is the failure rate when the PM is effective and the formulation is the result of Canfield(1986). The second part is the failure rate due to the ineffective PM. If τ is taken to be equal to x, then by recursive substitution the equation (2) is simplified to

$$h_{ie}(t)=\begin{cases} h(t), & \text{for } 0\leq t\leq x \\ p[k\{h(x)-h(0)\}+h(t-kx)]+(1-p)h(t), & \text{for } kx<t\leq(k+1)x,\ k=1,2,3,\cdots. \end{cases} \tag{3}$$

The expected cost rate for running a periodic PM policy during $[0,Nx]$ can be obtained in the following manner :

Expected cost rate per unit time = [(expected cost of minimal repair in $[0,Nx)$)
+ (expected cost of PM in $[0,Nx)$) + (expected cost of replacement)] / Nx,

where N is the number of PM and x is the period of PM. Using the results given in Boland(1982) and the failure rate under the periodic IEPM, given in (3), the expected cost rate per unit time for running our periodic IEPM policy during $[0,Nx]$ is obtained as

$$\begin{aligned} C_p(x,N) = \frac{1}{Nx}\Big[C_{mr}\Big\{\sum_{k=0}^{N-1}\int_{kx}^{(k+1)x}\{p(k(h(x)-h(0))+h(t-kx)) \\ +(1-p)h(t)\}dt\Big\}+(N-1)C_{pm}+C_{re}\Big], \end{aligned} \tag{4}$$

where C_{mr}, C_{pm} and C_{re} are the costs for minimal repair, preventive maintenance and replacement, respectively.

The formula (4) is equivalent to

$$C_p(x,N) = \frac{1}{Nx}\Big[C_{mr}\Big\{p\Big(\frac{N(N-1)}{2}x(h(x)-h(0))+N\int_0^x h(t)dt\Big) + (1-p)\int_0^{Nx} h(t)dt\Big\}+(N-1)C_{pm}+C_{re}\Big], \tag{5}$$

which is more useful for our purposes. When $p=1$, the formula (5) becomes

$$C_p(x,N)=\frac{1}{Nx}\Big[C_{mr}\Big\{\frac{N(N-1)}{2}x(h(x)-h(0))+N\int_0^x h(t)dt\Big\}+(N-1)C_{pm}+C_{re}\Big],$$

which is the expected cost rate considered in Park, Yum and Jung(1998).

3. Optimal Number and Period of Periodic IEPM Policy

We first consider the case when the period x is fixed. To find an optimal number of PM, N^*, for a given $x>0$ which minimizes $C_p(x,N)$ in (5), we form the inequalities $C_p(x,N+1)\geq C_p(x,N)$ and $C_p(x,N)< C_p(x,N-1)$. These two inequalities yield the relations,

$$p\frac{N(N+1)}{2}x(h(x)-h(0))+(1-p)\Big\{N\int_{Nx}^{(N+1)x} h(t)dt-\int_0^{Nx} h(t)dt\Big\}\geq \frac{C_{re}-C_{pm}}{C_{mr}} \tag{6}$$

and

$$p\frac{N(N-1)}{2}x(h(x)-h(0))+(1-p)\Big\{(N-1)\int_{(N-1)x}^{Nx} h(t)dt-\int_0^{(N-1)x} h(t)dt\Big\}< \frac{C_{re}-C_{pm}}{C_{mr}}. \tag{7}$$

Combining (6) and (7), we have

$$L_p(x,N)\geq \frac{C_{re}-C_{pm}}{C_{mr}} \quad \text{and} \quad L_p(x,N-1)< \frac{C_{re}-C_{pm}}{C_{mr}}, \tag{8}$$

where

$$L_p(x,N)=\begin{cases} p\frac{N(N+1)}{2}x(h(x)-h(0))+(1-p)\Big\{N\int_{Nx}^{(N+1)x} h(t)dt-\int_0^{Nx} h(t)dt\Big\}, & N=1,2,\cdots \\ 0, & N=0. \end{cases}$$

Theorem 3.1. If F is an IFR distribution with strictly increasing failure rate function $h(t)$, then there exists a unique $N^*<\infty$ which satisfies (8) for any $x>0$.

Proof. Since $h(t)$ is strictly increasing to ∞, we have

$$L_p(x,N)-L_p(x,N-1) = pNx(h(x)-h(0))+(1-p)N\Big\{\int_{Nx}^{(N+1)x} h(t)dt-\int_{(N-1)x}^{Nx} h(t)dt\Big\} > 0.$$

Thus, $L_p(x,N)$ is strictly increasing in N and tends to ∞ as $N\to\infty$, and hence the result follows. ∎

Next, we find an optimal period x^* for our periodic IEPM policy when N is fixed. If such an x^* exists, then $C_p(x,N)$, given in (5), is minimized at x^* for a fixed N. Differentiating $C_p(x,N)$ with respect to x and setting it equal to 0, we have

$$g(x)=\{(N-1)C_{pm}+C_{re}\}\ /\ C_{mr}, \tag{9}$$

where

$$g(x)=p\left\{\frac{N(N-1)}{2}x^2h'(x)+Nxh(x)-N\int_0^x h(t)dt\right\}+\ (1-p)\left\{Nxh(Nx)-\int_0^{Nx}h(t)dt\right\}.$$

Here $g(0)=0$ and the right hand side of (9) is greater than zero. Thus, if $g(x)$ is strictly increasing to ∞ as x increases and $C_p'(x_0,N)=0$, $C_p(x,N)$ decreases initially, then achieves the minimum at x_0 and increases for $x>x_0$.

Hence our objective is to determine the value of x which satisfies (9). If such x exists, it is the optimal period for our periodic IEPM policy. Straightforward arguments show

Theorem 3.2. If F is an IFR distribution with strictly increasing and convex failure rate function $h(t)$, then there exists a unique $x^*<\infty$ which satisfies (9) for any integer $N\geq 1$.

We now consider the optimal period x^* and the optimal number of PM, N^*, prior to the replacement of the system which minimize $C_p(x,N)$, given in (5), simultaneously. Neither x nor N is assumed to be fixed.

First, we can determine x_N as a function of N satisfying (9). Then, the expected cost rate per unit time for our periodic IEPM policy can be written as a function of x_N as follows.

$$\begin{aligned}C_p(x_N,N)=\frac{1}{Nx_N}\Big[C_{mr}\Big\{p\Big(\frac{N(N-1)}{2}x_N(h(x_N)-h(0))+N\int_0^{x_N}h(t)dt\Big)\\+\ (1-p)\int_0^{Nx_N}h(t)dt\Big\}+(N-1)C_{pm}+C_{re}\Big].\end{aligned} \tag{10}$$

Since the formula (10) is a function of N alone, we can find N^* so that

$$N^*=\min_N\ C_p(x_N,N),\qquad N=1,2,\cdots. \tag{11}$$

Theorem 3.3. Assume that F is an IFR distribution with strictly increasing and convex failure rate function. For a given $N\geq 1$, there exists a finite and unique x_N which satisfies (9). Thus the value of N^* which satisfies (11) is the optimal number of PM which minimizes the expected cost rate per unit time, given in (5).

Once the value of N^* is determined, the optimal period x^* is obtained by Theorem 3.2. The optimal period and the optimal number of PM which minimize the expected cost rate per unit time, given in (5), are x^* and N^*, respectively.

4. Numerical Examples

Suppose that the failure time of a system has a Weibull distribution, i.e., $h(t)=\beta\lambda^{\beta}t^{\beta-1}$ for $\beta>1$ and $t\geq 0$. As a special case, we take $\beta=3$ and $\lambda=1$ for $t\geq 0$.

It can easily be shown that (9) and (10) yield

$$x_N=\left(\frac{(N-1)C_{pm}+C_{re}}{(pN(3N-1)+(1-p)2N^3)C_{mr}}\right)^{1/3} \tag{12}$$

and

$$C_p(x_N,N)=\frac{1}{Nx_N}\left[C_{mr}\left\{p\left(\frac{N(N-1)}{2}3\,x_N^3+Nx_N^3\right)+(1-p)N^3x_N^3\right\}+(N-1)C_{pm}+C_{re}\right]. \tag{13}$$

Using (12) and (13), we determine N^* so that

$$N^*=\min_N C(x_N,N),\qquad N=1,2,\cdots. \tag{14}$$

Once the value of N^* is determined, the optimal period x^* is obtained by replacing N of (12) by N^*. Replacing x and N of (5) by x^* and N^*, we obtain the corresponding expected cost rate as follows.

$$\begin{aligned}C_p(x^*,N^*)=\frac{1}{N^*x^*}\Big[C_{mr}\Big\{p\Big(&\frac{3N(N-1)}{2}(x^*)^3+N^*(x^*)^3\Big)\\&+(1-p)(N^*)^3(x^*)^3\Big\}+(N^*-1)C_{pm}+C_{re}\Big].\end{aligned} \tag{15}$$

Tables 4.1 lists the values of N^*, x^* and $C_p(x^*,N^*)$ for various choice of C_{re} and p when $C_{mr}=1$ and $C_{pm}=1.5$. This table shows that as the amount of C_{re} increases, the value of N^* increases.

It also shows that for a fixed N^*, the value of x^* increases monotonically and its corresponding expected cost is increasing. It means that as the replacement cost gets higher, the greater number of PMs are needed before the replacement takes place and thus, the expected cost rate becomes higher. On the other hand, when C_{re} is fixed, $C_p(x^*,N^*)$ gets smaller as the value of p increases. That is, the more effective the PM is, the less amount of expected cost for running the PM policy is required before the replacement.

Table 4.1. Optimal time x^*, number N^* and its expected cost rate $C_p(x^*, N^*)$ with $C_{mr} = 1$ and $C_{pm} = 1.5$.

p	C_{re}	N^*	x^*	$C_p(x^*, N^*)$	p	C_{re}	N^*	x^*	$C_p(x^*, N^*)$
0.2	3	1	1.14471	3.93111	0.4	3	1	1.14471	3.93111
	5	1	1.35721	5.52605		5	1	1.35721	5.52605
	7	1	1.51829	6.91565		7	1	1.51829	6.91565
	9	1	1.65096	8.17704		9	1	1.65096	8.17704
	15	1	1.95743	11.4946		15	1	1.95743	11.4946
	20	1	2.15443	13.9248		20	2	1.16493	13.8420
	30	1	2.46621	18.2466		30	2	1.32309	17.8559
	50	2	1.51536	25.4890		50	3	1.08063	24.5228
	70	2	1.69050	31.7214		70	4	0.92412	30.2315
	90	3	1.24666	37.2997		90	5	0.81399	35.3814
	100	3	1.28983	39.9278		100	5	0.84132	37.7977
p	C_{re}	N^*	x^*	$C_p(x^*, N^*)$	p	C_{re}	N^*	x^*	$C_p(x^*, N^*)$
0.6	3	1	1.14471	3.93111	0.8	3	1	1.14471	3.93111
	5	1	1.35721	5.52605		5	1	1.35721	5.52605
	7	1	1.51829	6.91565		7	1	1.51829	6.91565
	9	1	1.65096	8.17704		9	2	0.97872	8.04625
	15	2	1.09990	11.2510		15	3	0.84343	10.6707
	20	3	0.86127	13.3523		20	4	0.73862	12.4387
	30	4	0.76322	16.9511		30	6	0.60661	15.4548
	50	5	0.73333	22.9092		50	8	0.55600	20.4026
	70	6	0.69188	28.0036		70	10	0.50932	24.5917
	90	7	0.65126	32.5741		90	12	0.46978	28.3375
	100	8	0.59674	34.7198		100	12	0.48405	30.0847

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