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A Diagnostic Method in Principal Factor Analysis

Myung Geun Kim[1] and Kang-Mo Jung[2]

Abstract

A method of detecting influential observations in principal factor analysis is suggested. It is based on a perturbation of the empirical distribution function and an adoption of the local influence method. An illustrative example is given.

1. Introduction

Principal factor analysis is one of the most widely used methods of reducing a large data (Harman, 1976). It is usually performed using the sample covariance (or correlation) matrix and does not need a distributional assumption for a descriptive purpose. It is well known that the sample covariance (or correlation) matrix is very sensitive to influential observations. Hence it is necessary to investigate the influence of observations on the parameter estimates of principal factor analysis model. Tanaka and Odaka (1989) adopted the influence function approach to investigate the influence of observations in principal factor analysis.

In this work we suggest a diagnostic method of investigating the influence of observations on the parameter estimators of principal factor analysis model, using the local influence approach by Cook (1986), Wu and Luo (1993). Jung et al. (1997) studied the local influence method in maximum likelihood factor analysis. They considered the perturbation under the normal distribution. However, principal factor analysis is free of a normal distribution assumption and so we consider a perturbation scheme in which the empirical distribution function is perturbed over all sample points. Under this perturbation scheme, the model parameters are estimated. Then the perturbation vector and the perturbed estimator form a surface in an appropriate dimensional Euclidean space. The curvatures and direction vectors associated with a certain curve on the surface yield information about individually and jointly influential observations.

In Section 2 we review principal factor analysis. In Section 3 a perturbation scheme is considered and a diagnostic procedure for investigating the influence of observations is described. Computations necessary for the diagnostic procedure are included in Section 4. An illustrative example is given in Section 5.

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1) Associate Professor, Department of Applied Statistics, Seowon University, 231 Mochung-Dong, Chongju, Chung-Buk 360-742, Korea
2) Full-time Lecturer, Department of Informatics & Statistics, Kunsan National University, 68 Miryong-Dong, Kunsan, Chollapuk-Do 573-701, Korea

2. Principal Factor Analysis

Let $\boldsymbol{x}$ be a p-variate random vector having a distribution function F with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$. The factor analysis model can be expressed as

$$\boldsymbol{x} = \boldsymbol{\mu} + \boldsymbol{\Lambda f} + \boldsymbol{e},$$

where $\boldsymbol{\Lambda}=(\lambda_{ij})$ is the p by q $(q \leq p)$ factor loading matrix, $\boldsymbol{f}$ is the q by 1 vector of common factors and $\boldsymbol{e}$ is the p by 1 vector of unique factors. We assume that the vector of common factors $\boldsymbol{f}$ has zero mean vector and identity covariance matrix, and that the vector of unique factors $\boldsymbol{e}$ has zero mean vector and diagonal covariance matrix $\boldsymbol{\Psi}= \mathrm{diag}(\psi_1, \ldots, \psi_p)$. It is further assumed that $\boldsymbol{f}$ and $\boldsymbol{e}$ are uncorrelated.

The mean vector and covariance matrix for the distribution F can be considered as statistical functionals denoted by $\boldsymbol{\mu}(F)$ and $\boldsymbol{\Sigma}(F)$, respectively. Statistical functionals for $\boldsymbol{\Lambda}$ and $\boldsymbol{\Psi}$ can be defined appropriately using the following equations for principal factor analysis model

$$\boldsymbol{\Sigma} - \boldsymbol{\Psi} = \boldsymbol{\Gamma \Phi \Gamma}^T \tag{1}$$

$$\boldsymbol{\Lambda} = \boldsymbol{\Gamma}_{(1)} \boldsymbol{\Phi}_{(1)}^{1/2} \tag{2}$$

$$\boldsymbol{\Psi} = \mathrm{diag}(\boldsymbol{\Sigma} - \boldsymbol{\Lambda \Lambda}^T), \tag{3}$$

where $\boldsymbol{\Phi}= \mathrm{diag}(\phi_1, \ldots, \phi_p)$ is the diagonal matrix whose diagonal elements are the eigenvalues of $\boldsymbol{\Sigma} - \boldsymbol{\Psi}$ with $\phi_1 > \cdots > \phi_p$, $\boldsymbol{\Gamma} = (\boldsymbol{\gamma}_1, \ldots, \boldsymbol{\gamma}_p)$ is the orthogonal matrix of the eigenvectors corresponding to $\boldsymbol{\Phi}$, $\boldsymbol{\Phi}_{(1)}$ is the q by q diagonal matrix consisting of the largest q eigenvalues and $\boldsymbol{\Gamma}_{(1)}$ is the p by q matrix of the eigenvectors associated with $\boldsymbol{\Phi}_{(1)}$. The identifiability of $\boldsymbol{\Lambda}$ is automatically ensured since $\boldsymbol{\Lambda}^T\boldsymbol{\Lambda}$ becomes a diagonal matrix.

Let $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\}$ be a random sample of size n from the distribution F. The empirical distribution function $\hat{F}$ based on $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\}$ is defined by putting mass $1/n$ on each $\boldsymbol{x}_u$ $(u=1, \ldots, n)$. Then $\boldsymbol{\mu}(\hat{F})$ and $\boldsymbol{\Sigma}(\hat{F})$ are the sample mean vector $\bar{\boldsymbol{x}}$ and covariance matrix $S=(s_{ij})$ with its divisor n, respectively. Then estimates of the factor loading matrix $\boldsymbol{\Lambda}$ and the unique variance matrix $\boldsymbol{\Psi}$ are obtained by iteratively solving equations (1) to (3) with $\boldsymbol{\Sigma}$ replaced by S.

Let $\boldsymbol{D}$ be the p by p diagonal matrix whose ith diagonal element is the reciprocal of square root of the ith diagonal element of S. Then the sample correlation matrix is given by $\boldsymbol{R}=\boldsymbol{DSD}$ which can be also considered as the value of an appropriate functional evaluated at $\hat{F}$. It can be used for performing principal factor analysis instead of the sample covariance matrix.

3. Local Influence

A perturbation scheme can be characterized by a perturbation vector $\boldsymbol{w}=(w_1, \ldots, w_n)^T$ in which the perturbation vector is expressed as $w_u = 1 + ac_u$ for $u=1, \ldots, n$. The scalar a represents the magnitude of the perturbation and the vector $\boldsymbol{c}=(c_1, \ldots, c_n)^T$ of unit length its direction. Let $\theta=\theta(F)$ be a scalar functional of interest that can be an element of the factor loading matrix $\boldsymbol{\Lambda}$ or a diagonal element of the unique variance matrix $\boldsymbol{\Psi}$. The estimator of θ is written as $\hat{\theta}$ for the unperturbed model and as $\hat{\theta}(\boldsymbol{w})$ for the perturbed model. The perturbation scheme is chosen such that $\hat{\theta}$ equals $\hat{\theta}(\boldsymbol{1}_n)$, where $\boldsymbol{1}_n$ denotes the unit vector of size n with all elements equal to one.

The $(n+1)$ by 1 vector $(\boldsymbol{w}^T, \hat{\theta}(\boldsymbol{w}))^T$ forms a surface in the $(n+1)$-dimensional Euclidean space as $\boldsymbol{w}$ varies over a certain space. The direction vectors of the surface at $\boldsymbol{w} = \boldsymbol{1}_n$ corresponding to the large absolute curvatures yield information about individually and jointly influential observations. A plot of direction cosines of the first two direction vectors in the plane is usually helpful for detecting influential observations. Observations separated from the main body are locally influential around the point $\boldsymbol{w} = \boldsymbol{1}_n$ and the global influence of those observations can be confirmed by using single and multiple case deletion diagnostics (Cook, 1986, p. 137).

The curvature and its associated direction vector of the surface at $\boldsymbol{w} = \boldsymbol{1}_n$ are obtained by solving the following generalized eigenvalue problem

$$|\boldsymbol{E} - a\boldsymbol{B}| = 0 , \tag{4}$$

where $\boldsymbol{E}$ is the n by n matrix having $\partial^2\hat{\theta}(\boldsymbol{w})/\partial w_u \partial w_v|_{\boldsymbol{w}=\boldsymbol{1}_n}$ as its (u, v)th element, $\boldsymbol{B}$ is the n by n matrix given by $(1+\dot{\boldsymbol{\eta}}^T\dot{\boldsymbol{\eta}})^{1/2}(\boldsymbol{I}_n+\dot{\boldsymbol{\eta}}\dot{\boldsymbol{\eta}}^T)$ and $\dot{\boldsymbol{\eta}}$ is the n by 1 column vector whose uth element is $\partial\hat{\theta}(\boldsymbol{w})/\partial w_u|_{\boldsymbol{w}=\boldsymbol{1}_n}$. The signed curvature of the surface is given by the eigenvalue in (4) and the direction vector $\boldsymbol{c}$ is its associated eigenvector of unit length. This comes from the fact that the curvature is equivalent to the value of $\boldsymbol{c}^T\boldsymbol{E}\,\boldsymbol{c}/\,\boldsymbol{c}^T\boldsymbol{B}\,\boldsymbol{c}$. For more details, refer to equations (2.2) to (2.5) of Wu and Luo (1993) or equation (20) of Cook (1986).

We consider a perturbation $\hat{F}_{\boldsymbol{w}}$ of the empirical distribution function defined as follows:

$$\hat{F}_{\boldsymbol{w}} \quad \text{puts mass } \frac{1}{n} \quad \text{on } w_u \boldsymbol{x}_u$$

for a given perturbation $\boldsymbol{w}$. When $w_u = 1$ for all u, $\hat{F}_{\boldsymbol{w}}$ reduces to $\hat{F}$. The mean vector and covariance matrix for $\hat{F}_{\boldsymbol{w}}$ are

$$\mu(\hat{F}_{\boldsymbol{w}}) = \frac{1}{n}\sum_{u=1}^{n} w_u \boldsymbol{x}_u$$

$$\Sigma(\hat{F}_{\boldsymbol{w}}) = \frac{1}{n} X H(\boldsymbol{w}) X^T,$$

where $X = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$ and $H(\boldsymbol{w}) = \mathrm{diag}(\boldsymbol{w}\,\boldsymbol{w}^T) - \boldsymbol{w}\,\boldsymbol{w}^T/n$. When $w_i = 0$ and $w_u = 1$ for all $u \neq i$, observation $\boldsymbol{x}_i$ is deleted in computing $\mu(\hat{F}_{\boldsymbol{w}})$ and $\Sigma(\hat{F}_{\boldsymbol{w}})$. Hence the above perturbation scheme includes the case deletion as a special case. Note that observation $\boldsymbol{x}_i$ is deleted whenever $a = 1$, $c_i = -1$ and $c_u = 0$ for all $u \neq i$.

4. Derivation

First we derive the first and second order partial derivatives of the perturbed sample covariance and correlation matrices with respect to w_u and w_u, w_v. The first and second order partial derivatives of $S(\boldsymbol{w}) = \Sigma(\hat{F}_{\boldsymbol{w}})$ evaluated at $\boldsymbol{w} = \mathbf{1}_n$ are computed as

$$\left.\frac{\partial S(\boldsymbol{w})}{\partial w_u}\right|_{\boldsymbol{w}=\mathbf{1}_n} = \frac{1}{n}(2\,\boldsymbol{x}_u \boldsymbol{x}_u^T - \boldsymbol{x}_u \bar{\boldsymbol{x}}^T - \bar{\boldsymbol{x}}\,\boldsymbol{x}_u^T)$$

$$\left.\frac{\partial^2 S(\boldsymbol{w})}{\partial w_u \partial w_v}\right|_{\boldsymbol{w}=\mathbf{1}_n} = \begin{cases} \dfrac{2(n-1)}{n^2}\,\boldsymbol{x}_u \boldsymbol{x}_u^T & \text{if } u = v \\ -\dfrac{1}{n^2}(\boldsymbol{x}_u \boldsymbol{x}_v^T + \boldsymbol{x}_v \boldsymbol{x}_u^T) & \text{if } u \neq v. \end{cases}$$

For each u and v, we will denote by $S_u = (s_{ij,u})$ and $S_{uv} = (s_{ij,uv})$ the first and second order partial derivatives obtained above, respectively.

The perturbation of the sample correlation matrix $R(\boldsymbol{w})$ is naturally obtained as

$$R(\boldsymbol{w}) = D(\boldsymbol{w})\,S(\boldsymbol{w})\,D(\boldsymbol{w}),$$

where $D(\boldsymbol{w})$ is determined by $S(\boldsymbol{w})$ in the same way as D. A little algebra yields

$$\left.\frac{\partial R(\boldsymbol{w})}{\partial w_u}\right|_{\boldsymbol{w}=\mathbf{1}_n} = D_u S D + D S_u D + D S D_u$$

$$\begin{aligned}\left.\frac{\partial^2 R(\boldsymbol{w})}{\partial w_u \partial w_v}\right|_{\boldsymbol{w}=\mathbf{1}_n} = {} & D_{uv} S D + D S D_{uv} + D_u S_v D \\ & + D S_v D_u + D_u S D_v + D_v S D_u \\ & + D_v S_u D + D S_u D_v + D S_{uv} D,\end{aligned}$$

where

$$D_u = \left.\frac{\partial D(w)}{\partial w_u}\right|_{w=1_n} = \mathrm{diag}\left(-\frac{1}{2}s_{ii}^{-3/2}s_{ii,u}\right)$$

$$D_{uv} = \left.\frac{\partial^2 D(w)}{\partial w_u \partial w_v}\right|_{w=1_n} = \mathrm{diag}\left(\frac{3}{4}s_{ii}^{-5/2}s_{ii,u}s_{ii,v} - \frac{1}{2}s_{ii}^{-3/2}s_{ii,uv}\right).$$

Here $\mathrm{diag}(\cdot)$ indicates the diagonal matrix whose ith diagonal element is given by that surrounded by parentheses.

The investigation of the influence of observations in principal factor analysis needs the first and second order partial derivatives of the perturbed estimators of the model parameters obtained by solving

$$S(w) - \widehat{\Psi}(w) = \widehat{\Gamma}(w)\widehat{\Phi}(w)\widehat{\Gamma}^T(w) \tag{5}$$

$$\widehat{\Lambda}(w) = \widehat{\Gamma}_{(1)}(w)\widehat{\Phi}_{(1)}^{1/2}(w) \tag{6}$$

$$\widehat{\Psi}(w) = \mathrm{diag}\left(S(w) - \widehat{\Lambda}(w)\widehat{\Lambda}^T(w)\right). \tag{7}$$

The order of the column vectors $\widehat{\gamma}_i(w)$ of $\widehat{\Gamma}(w)$ is determined by that of the column vectors $\widehat{\gamma}_i$ of $\widehat{\Gamma}$ such that $\widehat{\gamma}_i = \widehat{\gamma}_i(w)|_{w=1_n}$ and accordingly the order of the diagonal elements $\widehat{\phi}_i(w)$ of $\widehat{\Phi}(w)$ is determined, that is $\widehat{\phi}_i = \widehat{\phi}_i(w)|_{w=1_n}$. When $w = 1_n$, the above equations reduce to those for the unperturbed model. In what follows all derivatives are evaluated at $w = 1_n$.

4.1. Perturbed Principal Component Analysis

The subscripts u and v in a matrix have the same meaning as in S_u and S_{uv}. Let $\widehat{\gamma}_{i,u}$ be the ith column vector of $\widehat{\Gamma}_u$ and $\widehat{\phi}_{i,u}$ the ith diagonal element of $\widehat{\Phi}_u$. Differentiation of both sides in (5) with respect to w_u gives

$$\widehat{\Gamma}^T(S_u - \widehat{\Psi}_u)\widehat{\Gamma} = \widehat{\Gamma}^T\widehat{\Gamma}_u\widehat{\Phi} + \widehat{\Phi}\widehat{\Gamma}_u^T\widehat{\Gamma} + \widehat{\Phi}_u. \tag{8}$$

From the orthonormality condition on $\widehat{\gamma}_i(w)$, we have

$$\widehat{\gamma}_{i,u}^T\widehat{\gamma}_j + \widehat{\gamma}_i^T\widehat{\gamma}_{j,u} = 0. \tag{9}$$

The comparison of the ith diagonal element of both sides in (8) with the help of (9) yields

$$\widehat{\phi}_{i,u} = \widehat{\gamma}_i^T(S_u - \widehat{\Psi}_u)\widehat{\gamma}_i. \tag{10}$$

Similarly the comparison of the (i,j)th element of both sides in (8) and then a simple algebra give

$$\hat{\gamma}_{i,u} = \sum_{j\neq i, j=1}^{p} (\hat{\phi}_i - \hat{\phi}_j)^{-1} \{ \hat{\gamma}_i^T (S_u - \hat{\Psi}_u) \hat{\gamma}_j \} \hat{\gamma}_j. \tag{11}$$

Further differentiation of both sides in (8) with respect to w_v yields

$$\hat{\Gamma}^T S_* \hat{\Gamma} = \hat{\Gamma}^T \hat{\Gamma}_{uv} \hat{\Phi} + \hat{\Phi} \hat{\Gamma}_{uv}^T \hat{\Gamma} + \hat{\Phi}_{uv}, \tag{12}$$

where $S_* = S_0 - \hat{\Psi}_{uv}$ and

$$\begin{aligned} S_0 = & S_{uv} - \hat{\Gamma}_u \hat{\Phi}_v \hat{\Gamma}^T - \hat{\Gamma}_u \hat{\Phi} \hat{\Gamma}_v^T - \hat{\Gamma}_v \hat{\Phi}_u \hat{\Gamma}^T \\ & - \hat{\Gamma} \hat{\Phi}_u \hat{\Gamma}_v^T - \hat{\Gamma}_v \hat{\Phi} \hat{\Gamma}_u^T - \hat{\Gamma} \hat{\Phi}_v \hat{\Gamma}_u^T. \end{aligned}$$

Let $\hat{\phi}_{i,uv}$ be the ith diagonal element of $\hat{\Phi}_{uv}$ and $\hat{\gamma}_{i,uv}$ the ith column of $\hat{\Gamma}_{uv}$. Since equation (12) has the same form as (8), it can be solved analogously to the first order case and therefore we have

$$\hat{\phi}_{i,uv} = \hat{\gamma}_i^T S_* \hat{\gamma}_i + 2\hat{\phi}_i \hat{\gamma}_{i,u}^T \hat{\gamma}_{i,v} \tag{13}$$

$$\begin{aligned} \hat{\gamma}_{i,uv} = & -(\hat{\gamma}_{i,u}^T \hat{\gamma}_{i,v}) \hat{\gamma}_i + \sum_{j\neq i, j=1}^{p} (\hat{\phi}_i - \hat{\phi}_j)^{-1} \\ & \times \{ \hat{\gamma}_i^T S_* \hat{\gamma}_j + \hat{\phi}_j (\hat{\gamma}_{i,u}^T \hat{\gamma}_{j,v} + \hat{\gamma}_{i,v}^T \hat{\gamma}_{j,u}) \} \hat{\gamma}_j. \end{aligned} \tag{14}$$

4.2. First Order Derivatives

Differentiation of both sides in (6) and (7) with respect to w_u gives

$$\hat{\Lambda}_u = \hat{\Gamma}_{(1),u} \hat{\Phi}_{(1)}^{1/2} + \frac{1}{2} \hat{\Gamma}_{(1)} \hat{\Phi}_{(1)}^{-1/2} \hat{\Phi}_{(1),u} \tag{15}$$

$$\hat{\Psi}_u = \text{diag}(S_u - \hat{\Lambda}_u \hat{\Lambda}^T - \hat{\Lambda} \hat{\Lambda}_u^T). \tag{16}$$

Putting (15), (10) and (11) into (16) and using $\hat{\Lambda} = \hat{\Gamma}_{(1)} \hat{\Phi}_{(1)}^{1/2}$, we get for all $m = 1, \ldots, p$

$$\begin{aligned} \hat{\psi}_{m,u} = & s_{mm,u} - \sum_{i=1}^{q} (\hat{\gamma}_i^T S_u \hat{\gamma}_i) \hat{\gamma}_{mi}^2 \\ & - 2\sum_{i=1}^{q} \sum_{j\neq i, j=1}^{p} \hat{\phi}_i (\hat{\phi}_i - \hat{\phi}_j)^{-1} (\hat{\gamma}_i^T S_u \hat{\gamma}_j) \hat{\gamma}_{mi} \hat{\gamma}_{mj} + \sum_{k=1}^{p} \hat{\psi}_{k,u} \xi_{mk}, \end{aligned} \tag{17}$$

where $\xi_{mk} = \sum_{i=1}^{q} \hat{\gamma}_{ki}^2 \hat{\gamma}_{mi}^2 + 2\sum_{i=1}^{q} \sum_{j\neq i, j=1}^{p} \hat{\phi}_i (\hat{\phi}_i - \hat{\phi}_j)^{-1} \hat{\gamma}_{ki} \hat{\gamma}_{kj} \hat{\gamma}_{mi} \hat{\gamma}_{mj}$, $\hat{\psi}_{m,u}$ is the mth diagonal element of $\hat{\Psi}_u$ and $\hat{\gamma}_{mi}$ is the mth element of $\hat{\gamma}_i$. Thus we can solve the linear equation (17) to get $\hat{\psi}_{k,u}$, and then get $\hat{\phi}_{i,u}$ using (10), $\hat{\gamma}_{i,u}$ using (11) and finally $\hat{\Lambda}_u$ using (15).

4.3. Second Order Derivatives

The second order derivatives are obtained in the same way as the first order case. Further differentiation of both sides in (15) and (16) with respect to w_v yields

$$\begin{aligned}\widehat{\Lambda}_{uv} &= \widehat{\Gamma}_{(1),uv}\ \widehat{\Phi}_{(1)}^{\ 1/2} + \frac{1}{2}\widehat{\Gamma}_{(1),u}\ \widehat{\Phi}_{(1)}^{\ -1/2}\ \widehat{\Phi}_{(1),v} + \frac{1}{2}\widehat{\Gamma}_{(1),v}\ \widehat{\Phi}_{(1)}^{\ -1/2}\ \widehat{\Phi}_{(1),u} \\ &\quad -\frac{1}{4}\widehat{\Gamma}_{(1)}\ \widehat{\Phi}_{(1)}^{\ -3/2}\ \widehat{\Phi}_{(1),u}\ \widehat{\Phi}_{(1),v} + \frac{1}{2}\widehat{\Gamma}_{(1)}\ \widehat{\Phi}_{(1)}^{\ -1/2}\ \widehat{\Phi}_{(1),uv}\end{aligned} \tag{18}$$

$$\widehat{\Psi}_{uv} = \operatorname{diag}(\ S_{uv} - \widehat{\Lambda}_{uv}\ \widehat{\Lambda}^{T} - \widehat{\Lambda}_{u}\ \widehat{\Lambda}_{v}^{\ T} - \widehat{\Lambda}_{v}\ \widehat{\Lambda}_{u}^{\ T} - \widehat{\Lambda}\ \widehat{\Lambda}_{uv}^{\ T}). \tag{19}$$

By (18), (13) and (14), for $m=1,\ldots,p$ equation (19) becomes

$$\begin{aligned}\widehat{\psi}_{m,uv} &= s_{mm,uv} - \sum_{i=1}^{q}(\ \widehat{\gamma}_{i}^{\ T}\ S_0\ \widehat{\gamma}_{i})\ \widehat{\gamma}_{mi}^{\ 2} \\ &\quad -2\sum_{i=1}^{q}\ \sum_{j\neq i,j=1}^{p}\ \widehat{\phi}_{i}(\ \widehat{\phi}_{i} - \widehat{\phi}_{j})^{-1}\{\ \widehat{\gamma}_{i}^{\ T}\ S_0\ \widehat{\gamma}_{j} \\ &\qquad + \widehat{\phi}_{j}(\ \widehat{\gamma}_{i,u}^{\ T}\ \widehat{\gamma}_{j,v} + \widehat{\gamma}_{i,v}^{\ T}\ \widehat{\gamma}_{j,u})\}\ \widehat{\gamma}_{mi}\ \widehat{\gamma}_{mj} \\ &\quad -2\zeta_{m,uv} + \sum_{k=1}^{p}\ \widehat{\psi}_{k,uv}\xi_{mk}\ ,\end{aligned} \tag{20}$$

where $\zeta_{m,uv} = \sum_{i=1}^{q}(\ \widehat{\phi}_{i}\ \widehat{\gamma}_{mi,u}\ \widehat{\gamma}_{mi,v} + \widehat{\phi}_{i,u}\ \widehat{\gamma}_{mi}\ \widehat{\gamma}_{mi,v} + \widehat{\phi}_{i,v}\ \widehat{\gamma}_{mi}\ \widehat{\gamma}_{mi,u})$, $\widehat{\psi}_{m,uv}$ is the mth diagonal element of $\widehat{\Psi}_{uv}$ and $\widehat{\gamma}_{mi,u}$ is the mth element of $\widehat{\gamma}_{i,u}$. Similarly to the first order case, we first solve the linear equation (20) to get $\widehat{\psi}_{k,uv}$, and then $\widehat{\phi}_{i,uv}$ using (13), $\widehat{\gamma}_{i,uv}$ using (14) and finally $\widehat{\Lambda}_{uv}$ using (18).

5. Numerical Example

The local influence method is applied to the stiffness data set (Johnson and Wichern, 1992, p. 162, Table 4.3) consisting of 30 measurements on four variables. The analysis will be limited to the third unique variance based on the correlation matrix. Note that the q-factor model makes sense if the degrees of freedom $(p-q)^2/2-(p+q)/2 \geq 0$ (Harman, 1976, p. 205). For this data set the degrees of freedom is not negative only if $q = 1$. Thus the number of loading factors is assumed to be one.

The model parameter estimates are obtained first by setting $\psi_i = 1/r^{ii}$ for $i=1,\ldots,4$, where r^{ii} is the ith diagonal element of R^{-1} and then by iteratively solving equations (1) to (3). The estimates of unique variances based on the full data are $\widehat{\psi}_1=0.0335$, $\widehat{\psi}_2=0.2343$, $\widehat{\psi}_3=0.1360$ and $\widehat{\psi}_4=0.1235$.

Figure 1 shows a scatter plot of 30 observations with respect to the first direction vector versus the second direction vector for $\widehat{\psi}_3$. The first two large curvatures including the sign are 1.0386 and -0.7062. Points separated from the main body around the origin in the scatter plot may be noteworthy since they can be individually or jointly influential. Observations 2, 9, 21 and 29 are locally influential.

Next, the index plot of the values of the empirical influence function for $\hat{\psi}_3$ is included in Figure 2 following Tanaka and Odaka (1989). In this case the influence function method is modified by substituting the empirical influence function of the sample correlation matrix for that of the sample covariance matrix (Fung and Kwan, 1995). The visual inspection of Figure 2 indicates that observations 9 and 21 are influential for $\hat{\psi}_3$.

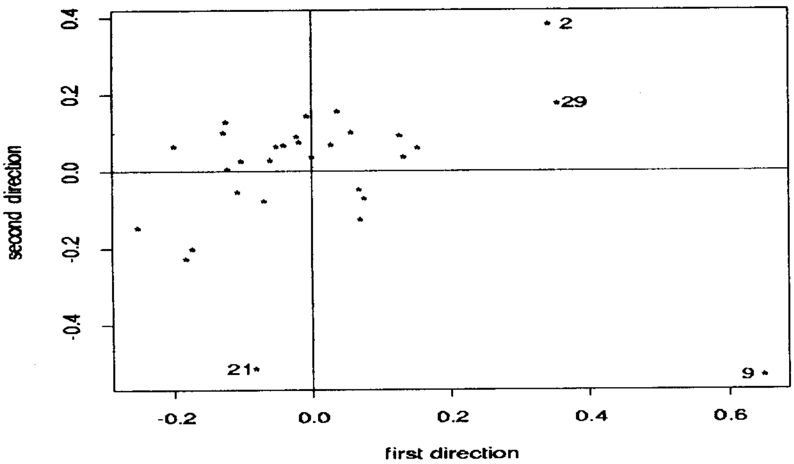

Figure 1. The scatter plot of the first direction vector versus the second direction vector for $\hat{\psi}_3$.

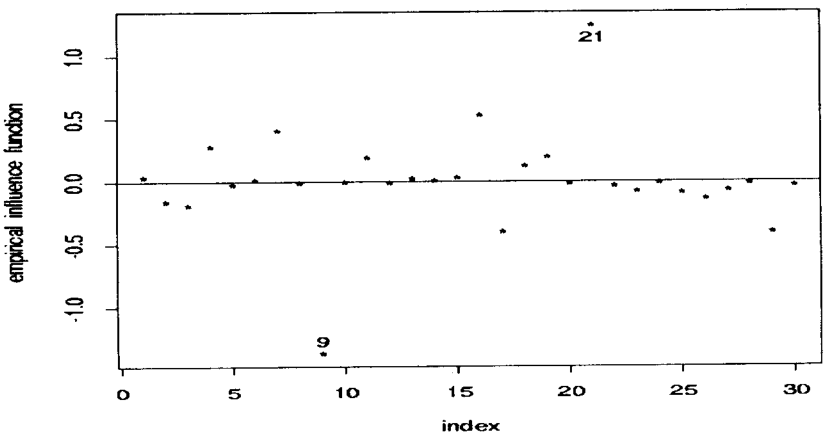

Figure 2. The index plot of the empirical influence function for $\hat{\psi}_3$.

Table 1 shows some results of single case (I), double case (II) and triple case (III) deletions for $\hat{\psi}_3$. The number in parentheses is the estimate of ψ_3 based on the data set after removing the corresponding observations. In each column observations are arranged in the decreasing order of the absolute difference between $\hat{\psi}_3$ and $\hat{\psi}_{3,(J)}$, where $\hat{\psi}_{3,(J)}$ is the estimate of ψ_3 based on the deleted observations in the index set J.

I	II	III
9 (0.207)	9, 29 (0.254)	2, 9, 29 (0.305)
21 (0.090)	9, 17 (0.239)	2, 17, 29 (0.300)
16 (0.117)	2, 9 (0.238)	2, 9, 17 (0.281)

Table 1. Multiple case deletions.

The single case deletion results are in parallel with those from the empirical influence function values of Tanaka and Odaka (1989). They indicate that removal of observation 9 increases the value of $\hat{\psi}_3$ while deletion of observation 21 decreases the value of $\hat{\psi}_3$. Note that observation 9 is locally influential along the first direction vector with positive curvature while observation 21 along the second direction vector with negative curvature. Thus the sign of curvature plays an interesting role. A similar phenomenon in the linear hypothesis testing problem can be found in Kim (1998). The double and triple case deletions show that observations 9, 29 and 2 are jointly influential. Note that these observations are locally influential along the first direction vector. Observation 21 is not included in the sets of jointly influential observations for the double and triple case deletions.

From this example we see that the local influence method provides information about individually and jointly influential observations. However, the influence function method is not sufficient for detecting jointly influential observations. For the other model parameters the above method also yields useful information.

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