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# A Repair Process with Embedded Markov Chain

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ABSTRACT

A repair process of a system consisting of both perfect repairs and minimal repairs is introduced. The type of repair, when the system fails, is determined by an embedded two state Markov chain. We study several stochastic properties of the process including the preservation of ageing properties and the monotonicities of the time between successive repairs. After assigning repair costs to the process, we also show that an optimal repair policy uniquely exists, if the underlying life distribution of the system has DMRL.

*Keywords:* Life Distribution; Minimal Repair; Markov Chain; Optimization.

## 1. INTRODUCTION

Brown and Proschan (1983) introduced a definition of imperfect(minimal) repair. See also Barlow and Proschan (1965 pp. 96–98) for the earlier definition. When a system fails, a perfect repair of the system yields a system which is as good as new, however, a minimal repair yields a functioning system which is only as good as a system of age equal to its age at failure. Thus, if $F$ denotes the life distribution of the system, the time to failure of the system following a perfect repair has distribution $F$, but the time to failure following a minimal repair done at age $s$ has a survival function given by

$$\bar{F}(t|s) = \bar{F}(s+t)/\bar{F}(s), \qquad t > 0.$$

This imperfect repair model has been generalized by many authors. For the recent developments of the repair models, see Valdez-Flores and Feldman(1989), Sheu, Griffith and Nakagawa(1995), and the references there in.

In this paper, we introduce a repair process of a system, where either perfect repair or minimal repair is performed at failure of the system. The type of repair

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is determined by an embedded two state Markov chain. That is, we assume that the system is initially repaired perfectly and thereafter the type of repair at failure is determined according to the following transition probability matrix:

$$\begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{bmatrix} \alpha & 1-\alpha \\ 1-\beta & \beta \end{bmatrix} \end{array}$$

where state 0 denotes the perfect repair, state 1 denotes the minimal repair and $0 \le \alpha, \beta \le 1$. It is also assumed that each repair takes negligible time.

When $\alpha = 1 - \beta = p$, for $0 \le p \le 1$, our repair process is reduced to the imperfect repair model introduced by Brown and Proschan (1983), where the probability of performing perfect repair, when the system fails, is always $p$ regardless of the type of the previous repair. It is, however, more practical to assume that the chance of performing perfect repair depends on the type of the previous repair. That is, when the system fails, the probability of performing perfect repair is $\alpha$, if the last repair was perfect, otherwise, it is $1 - \beta$. This repair model would be very useful when no records of previous repairs are available except the type of the last repair.

In section 2, after deriving the distribution of the time between two successive perfect repairs, we show that the preservation of ageing properties no longer holds unless $\alpha = 1 - \beta = p$. We also obtain several stochastic properties of the process including the monotonicities of the time between successive repairs. In section 3, after assigning repair costs to the process, we finally show that an optimal repair policy uniquely exists, if the underlying life distribution of the system has strictly decreasing mean residual life.

## 2. STOCHASTIC RESULTS

Let $F$ denote the life distribution of the system and let $F_{\alpha,\beta}$ denote the distribution of the time between two successive perfect repairs. We assume that $F$ is absolutely continuous with density function $f$. When $\alpha = 1 - \beta = p$, Brown and Proschan (1983) show that the survival function of the time between two successive perfect repairs is given by

$$\bar{F}_p(t) = \bar{F}^p(t) = \{\bar{F}(t)\}^p.$$

They show that the preservation of ageing properties holds for their process, that is, if $F$ is in any of the life distribution classes IFR, DFR, IFRA, DFRA, NBU, NWU, DMRL, or IMRL, then $F_p$ is in the same class. In our process, however, these ageing properties are no longer preserved unless $\alpha = 1 - \beta = p$. We, first, derive the survival function $\bar{F}_{\alpha,\beta}$ of the time between two successive perfect repairs.

**Theorem 2.1.**

$$\bar{F}_{\alpha,\beta}(t) = (\frac{\alpha+\beta-1}{\beta})\bar{F}(t) + (\frac{1-\alpha}{\beta})\bar{F}^{1-\beta}(t).$$

**Proof:** Note that the number of failures occured after a perfect repair forms a non-homogeneous Poisson process with intensity function $r(t) = f(t)/\bar{F}(t)$, until the next perfect repair occurs. This process is called the upper record value process corresponding to $F$[Shorrock (1972)]. We denote it by $\{N(t), t \geq 0\}$.

Let $Y$ be the time between two successive perfect repairs, then conditioning on $N(t)$ gives

$$\begin{aligned}
\Pr\{Y > t\} &= \sum_{n=0}^{\infty} \Pr\{Y > t | N(t) = n\} \Pr\{N(t) = n\} \\
&= \Pr\{N(t) = 0\} + \sum_{n=1}^{\infty} (1-\alpha)\beta^{n-1} \Pr\{N(t) = n\} \\
&= (\frac{\alpha+\beta-1}{\beta}) \Pr\{N(t) = 0\} + (\frac{1-\alpha}{\beta}) \sum_{n=0}^{\infty} \beta^n \Pr\{N(t) = n\}.
\end{aligned}$$

Since $\Pr\{N(t) = 0\} = \bar{F}(t)$ and $\sum_{n=0}^{\infty} \beta^n \Pr\{N(t) = n\} = \bar{F}^{1-\beta}(t)$, the proof is completed. □

**Remark 2.1.** *Let $\mu$ be the expected time between two successive perfect repairs, then it is given by*

$$\mu = (\frac{\alpha+\beta-1}{\beta})\mu(1) + (\frac{1-\alpha}{\beta})\mu(1-\beta),$$

*where $\mu(p) = \int_0^\infty \bar{F}^p(t)dt$, for $0 < p \leq 1$.*

The following example shows that the ageing property is not preserved for $F$ in any of IFR, IFRA, NBU, DMRL, or NBUE.

**Example 2.1.** *Let $\bar{F}(t) = e^{-t}$, for $t \geq 0$, then*

$$\bar{F}_{\alpha,\beta}(t) = (\frac{\alpha+\beta-1}{\beta})e^{-t} + (\frac{1-\alpha}{\beta})e^{-(1-\beta)t}.$$

Since $F$ is an exponential distribution, it is also in IFR, IFRA, NBU, DMRL, and NBUE classes. However, it can be shown that

$$\int_t^\infty \bar{F}_{\alpha,\beta}(x)dx - \mu\bar{F}_{\alpha,\beta}(t) = \frac{(\alpha+\beta-1)(1-\alpha)}{\beta(1-\beta)}\{e^{-(1-\beta)t} - e^{-t}\} > 0,$$

if $\alpha + \beta > 1$. This shows that $F_{\alpha,\beta}$ is in NWUE class, and hence, can not be in DMRL, NBU, IFRA, and IFR classes.

**Remark 2.2.** *The above example also shows that the ageing property is not preserved for $F$ in any of DFR, DFRA, NWU, IMRL, or NWUE, since*

$$\int_t^\infty \bar{F}_{\alpha,\beta}(x)dx - \mu\bar{F}_{\alpha,\beta}(t) < 0, \qquad \text{for } \alpha+\beta < 1.$$

We, now, study the monotonicities of the time between successive repairs. First, note that the sequence of points where the perfect repair is performed forms an embedded renewal process. Let's define a cycle as the period between two successive renewal points and let $N$ be the number of repairs performed during a cycle. Then, $N$ is equal to 1 with probability $\alpha$, and is equal to $n$ with probability $(1-\alpha)\beta^{n-2}(1-\beta)$, for $n = 2, 3, \ldots$, and hence, $E(N) = (2-\alpha-\beta)/(1-\beta)$.

Since the repair process regenerates itself after a cycle, it can be shown that with probability 1, $E(N)/\mu = \lim_{t\to\infty} M(t)/t$, where $M(t)$ is the total number of repairs up to time $t$. See, for example, Ross (1983 p.78). Hence, $\mu/E(N)$ can be interpreted as the long-run average time between successive repairs.

**Theorem 2.2.** *If $F$ is in NBUE class, then*

*(i)* $\mu \le E(N)\mu(1)$.

*(ii)* $\mu/E(N)$ *is increasing in* $\alpha$.

*Dual results hold for $F$ in NWUE class.*

**Proof:** (2.2) follows from (2.2), and (2.2) can be shown by direct calculation and the fact that $\mu(1) \ge p\mu(p)$, for $0 < p \le 1$, if $F$ is in NBUE class. See Brown and Proschan(1983). □

To prove the monotonicity of $\mu/E(N)$ with respect to $\beta$, we need the following lemma:

**Lemma 2.1.** *Let $\{a_n, n = 0, 1, 2, \ldots\}$ be a sequence of positive real numbers and let $g(p) = p\sum_{n=0}^{\infty} a_n(1-p)^n$ for $0 < p < 1$. Then, $g(p)$ is concavely increasing in $p$, if $\{a_n, n = 0, 1, 2, \ldots\}$ is a decreasing sequence.*

**Proof:** Note that $g(p)$ is formed by assigning the sequence $\{a_n, n = 0, 1, 2, \ldots\}$ as weights to the geometric series $p\sum_{n=0}^{\infty}(1-p)^n$ which converges to 1, for $0 < p < 1$. Differentiating $g(p)$ with respect to $p$ gives

$$g'(p) = \sum_{n=1}^{\infty} a_n(1 - p - np)(1 - p)^{n-1}.$$

Let $n'$ be the largest integer less than or equal to $(1-p)/p$ so that terms in $g'(p)$ are non-negative as long as $n \leq n'$. Then, we see that

$$\begin{aligned} g'(p) &= \sum_{n=1}^{n'} a_n(1-p-np)(1-p)^{n-1} + \sum_{n=n'+1}^{\infty} a_n(1-p-np)(1-p)^{n-1} \\ &\geq \sum_{n=1}^{n'} a_{n'}(1-p-np)(1-p)^{n-1} + \sum_{n=n'+1}^{\infty} a_{n'}(1-p-np)(1-p)^{n-1} \\ &= a_{n'}\frac{d}{dp}\{p\sum_{n=0}^{\infty}(1-p)^n\} = 0. \end{aligned}$$

This proves that $g(p)$ is increasing in $p$. To show the concavity of $g(p)$, we differentiate $g(p)$ twice and obtain

$$g''(p) = \sum_{n=2}^{\infty} a_n n(np + p - 2)(1 - p)^{n-2}.$$

An argument similar to the above shows that

$$g''(p) \leq a_{n''}\frac{d^2}{dp^2}\{p\sum_{n=0}^{\infty}(1-p)^n\} = 0,$$

where, $n''$ is the largest integer less than or equal to $(2-p)/p$ so that terms in $g''(p)$ are non-positive as long as $n \leq n''$. This completes the proof. □

**Remark 2.3.**

*(i) In the above Lemma 2.1, dual result holds, if $\{a_n, n = 0, 1, 2, \ldots\}$ is an increasing sequence.*

*(ii) When $\alpha = 1 - \beta = p$, note that by conditioning on $N$, the expected time between two successive perfect repairs can be rewritten as*

$$\begin{aligned}\mu(p) &= \sum_{n=1}^{\infty} E(X_0 + X_1 + X_2 + \cdots + X_{n-1})p(1-p)^{n-1} \\ &= \sum_{n=0}^{\infty} E(X_n)(1-p)^n,\end{aligned}$$

*where $X_0$ is the time to failure after a perfect repair and $X_n$ is the time to failure after $n$ consecutive minimal repairs, for $n = 1, 2, \ldots$.*

**Theorem 2.3.** *If $F$ is in DMRL class, then $\mu/E(N)$ is decreasing in $\beta$. Dual result holds for $F$ in IMRL class.*

**Proof:** By making use of the above Remark 2.3, (2.3), $\mu(1-\beta)$ can be expressed as

$$\mu(1-\beta) = \mu(1) + \beta g(1-\beta)/(1-\beta),$$

with $a_n = E(X_{n+1})$, for $n = 0, 1, 2, \ldots$. Note that $\mu(1) = E(X_0)$. Hence, it can be shown that

$$\frac{\mu}{E(N)} = \frac{(1-\beta)\mu(1) + (1-\alpha)g(1-\beta)}{2-\alpha-\beta}.$$

The numerator of $\frac{\partial}{\partial\beta}\{\mu/E(N)\}$ is, now, given by

$$(1-\alpha)\{g(1-\beta) - \mu(1) - (2-\alpha-\beta)g'(1-\beta)\},$$

which is negative, since $g'(1-\beta) > 0$, from the Lemma 2.1, and $g(1-\beta) = (1-\beta)\sum_{n=0}^{\infty} E(X_{n+1})\beta^n \leq E(X_0)$, if $F$ is in DMRL class. This completes the proof. □

## 3. OPTIMIZATION

Let $C_1$ be the cost of perfect repair and let $C_2$ be the cost of minimal repair. Both costs may slightly vary from time to time, in which case we consider them as the long-run average costs. Since the sequence of points where the perfect repair is performed forms an embedded renewal process, by applying the renewal reward theorem[Ross (1983 p. 78)], we can see that the long-run average repair cost per unit time is given by

$$C(\alpha, \beta) = \{C_1 + (1-\alpha)C_2/(1-\beta)\}/\mu,$$

where $\mu = (\frac{\alpha+\beta-1}{\beta})\mu(1) + (\frac{1-\alpha}{\beta})\mu(1-\beta)$.

**Theorem 3.1.** *For a given $\alpha$, if $F$ is in strictly DMRL class, then there exists a unique $\beta$ which minimizes $C(\alpha, \beta)$.*

**Proof:** Again, by making use of Remark 2.3, (2.3), we can show that $C(\alpha, \beta)$ is rewritten as

$$C(\alpha, \beta) = \frac{(1-\beta)C_1 + (1-\alpha)C_2}{(1-\beta)\mu(1) + (1-\alpha)g(1-\beta)},$$

where $g(1-\beta) = (1-\beta)\{\mu(1-\beta) - \mu(1)\}/\beta$. The numerator of $\frac{\partial}{\partial\beta}C(\alpha, \beta)$ is given by

$$A(\beta) = (1-\alpha)[C_2\mu(1) - C_1 g(1-\beta) + \{(1-\beta)C_1 + (1-\alpha)C_2\}g'(1-\beta)].$$

Now, we see, from the Lemma, that

$$A'(\beta) = -(1-\alpha)\{(1-\beta)C_1 + (1-\alpha)C_2\}g''(1-\beta)$$

is positive, for $0 < \beta < 1$. Hence, there exists at most one solution for equation $A(\beta) = 0$, if $F$ is in strictly DMRL class. This completes the proof. □

**Remark 3.1.** *It can be easily shown that for a given $\beta$, $C(\alpha, \beta)$ is minimized at either $\alpha = 0$ or $\alpha = 1$, for any life distribution $F$. Hence, when $F$ is in strictly DMRL class and when we control $\alpha$ and $\beta$ simultaneously, the minimum value of $C(\alpha, \beta)$ is to be either $C(1, \beta) = C_1/\mu(1)$ or $C(0, \beta^*)$, where $\beta^*$ is the unique solution of equation*

$$C_2\mu(1) - C_1 g(1-\beta) + \{(1-\beta)C_1 + C_2\}g'(1-\beta) = 0.$$

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