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Design of Step-Stress Accelerated Life Tests for Weibull Distributions with a Nonconstant Shape Parameter

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ABSTRACT

This paper considers the design of step-stress accelerated life tests for the Weibull distribution with a nonconstant shape parameter under Type I censoring. It is assumed that scale and shape parameters are log-linear functions of (possibly transformed) stress and that a cumulative exposure model holds for the effect of changing stress. The asymptotic variance of the maximum likelihood estimator of a stated quantile at design stress is used as an optimality criterion. The optimum three step-stress plans are presented for selected values of design parameters and the effects of errors in pre- estimates of the design parameters are investigated.

Keywords: Accelerated life test, Step stress test, Weibull distribution, Fisher Information, Maximum likelihood, Asymptotic variance, Nonconstant shape parameter

1. INTRODUCTION

Accelerated life tests(ALTs) are used to obtain information quickly on the lifetime distribution of materials or products. The test units are run at higher than usual levels of stress to induce early failures. The test data obtained at the accelerated stresses are analyzed in terms of a model, and then extrapolated to estimate the lifetime distribution at the design stress.

The stress can be applied in various ways ; commonly used methods are constant stress and step-stress. In a constant stress test, units are subjected to a constant level of stress until failures occur or the observations are censored. A step-stress test allows the stress setting of units to be changed at prescribed

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times(time-step) or upon the occurrence of a fixed number of failures(failure-step). The step-stress ALTs have the advantage of yielding more failure data in a limited time without necessarily using a high stress to all test units . For the applications of the step-stress ALTs, see Bora(1979), Nelson(1990), etc.

Several authors have considered the problem of designing ALT plans. See, for instance, Nelson and Kielpinski(1976), Nelson and Meeker(1978), and Meeker and Hahn(1985) for constant stress ALTs. For step-stress ALTs, Miller and Nelson(1983) presented optimum simple step-stress ALT plans for exponential distributions with complete data. Bai et al.(1989) and Bai and Chun(1991) extended the results of Miller and Nelson(1983) to the cases with censoring and with competing causes of failure, respectively. Bai and Kim(1993) considered optimum designs of simple step-stress ALTs for Weibull distributions.

Most of these works assume that the log-lifetime follows a location-scale distribution for which the location parameter varies with stress, but the scale parameter is independent of stress. For example, when the lifetime follows Weibull distribution, it is assumed that the scale parameter depends on stress, but the shape parameter remains constant. In many situations, however, this assumption may not be appropriate; Li et al.(1989) found that the Weibull shape parameter decreases in electric-field stress for dielectrics, and Hiergiest et al.(1989) showed that the shape parameter depends on temperature in the test of capacitors. More examples of a nonconstant shape parameter are in the references of Meeter and Meeker(1994). Glaser(1984) presented methods of estimating the shape and scale parameters expressed as functions of the testing environments. Meeter and Meeker(1994) designed the optimum ALTs under the assumption that both the scale and shape parameters depend on stress. However, these studies considered only constant stress cases.

This paper considers the design of step-stress ALTs for the Weibull distribution with nonconstant shape and scale parameters which are log-linear functions of stress. It is assumed that a certain cumulative exposure model holds for the effect of changing stress. The asymptotic variance of the maximum likelihood estimator of a stated quantile at design stress is used as an optimality criterion. The optimum three step-stress plans are tabulated for selected combinations of design parameters, and the effects of errors in pre-estimates of the design parameters are investigated.

Notation

n total number of test units

s_D design stress level

s_H prespecified highest stress level

h number of stress levels

s_i ith stress level, $i = 1, \cdots, h$, $s_D \leq s_1 \leq \cdots \leq s_h (= s_H)$

τ_i stress change time from s_i to s_{i+1}, $i = 1, \cdots, h-1$

η censoring time

ξ_i standardized stress, $\xi_i = \frac{s_i - s_D}{s_H - s_D}$, $i = 1, \cdots, h$

x_i standardized stress change time, $x_i = \frac{\tau_i}{\eta}$

Y life of a unit under step-stress ALT

λ_i, β_i scale and shape parameters of the Weibull distribution at stress level s_i

$G(\cdot)$ cumulative distribution function(cdf) of Y

μ_{ξ_i}, σ_{ξ_i} location and scale parameters of the smallest extreme value distribution at ξ_i

$\psi(\cdot)$, $\Psi(\cdot)$ pdf and cdf of the standard smallest extreme value distribution.

α_0, α_1, α_0', α_1' parameters of log-linear relation

γ_0, γ_1, γ_0', γ_1' parameters of standardized log-linear relation

t_q qth quantile of the smallest extreme value distribution at design stress

p_D, p_H probabilities that a unit will fail by η at s_D and s_H, respectively.

$\mathbf{Asvar}(\cdot)$, $\mathbf{Ascov}(\cdot,\cdot)$ asymptotic variance and asymptotic covariance, repectively.

2. THE MODEL

Assumption

1. At any stress s, the lifetime of a test unit follows a Weibull distribution with scale parameter $\lambda(s) = \exp(\alpha_0 + \alpha_1 s)$ and shape parameter $\beta(s) = \exp(\alpha_0' + \alpha_1' s)$;
$$F(t) = 1 - e^{-(\lambda(s)t)^{\beta(s)}}$$
i.e., the log lifetimes follow a smallest extreme value distribution with location parameter $\mu(s) = -\log(\lambda(s)) = -\alpha_0 - \alpha_1 s$ and scale parameter $\sigma(s) = 1/\beta(s) = \exp(-\alpha_0' - \alpha_1' s)$.

2. A certain cumulative exposure model holds.

3. Test units are run simultaneously and their lifetimes are continuously observed.

4. The lifetimes of test units are independent and identically distributed.

Test procedure

1. h test stress levels $s_1, s_2, \cdots, s_h$ arc used.

2. Each of test units is initially placed on stress s_1.

3. If the test unit does not fail during $(\tau_{i-1}, \tau_i]$ at stress s_i, then the stress is changed to s_{i+1} at τ_i, $i = 1, 2, \cdots, h-1$, $\tau_0 \equiv 0$.

4. The test is continued until all units fail or a prespecified censoring time η at s_h.

Cumulative effect of exposure

The data analysis of a step-stress test requires a cumulative exposure (or damage) model which relates the life distribution under step stress to that of constant stress. The cumulative exposure model used here assumes, in the words of Nelson(1980), that "the remaining life of specimens depends only on the current cumulative fraction failed and current stress—regardless how the fraction accumulated. Moreover, if held at the current stress, survivors will fail according to the cdf for that stress but starting at the previously accumulated fraction failed." It has been widely used for estimation of parameters(Nelson(1980)) and in the design of test plans (Miller and Nelson(1983), Bai et al.(1989), Bai and Chun(1991), and Bai and Kim(1993)) for its mathematical convenience and plausibility. See Nelson(1990) for a discussion on the adequacy of this model.

Standardized model

Define the standardized stress ξ_i as

$$\xi_i = \frac{s_i - s_D}{s_H - s_D} \tag{2.1}$$

where s_D and s_H are design and highest stresses, respectively. Then the location and scale parameters of log-lifetime distribution of test units at stress s_i can be rewritten in terms of ξ_i as $\mu_{\xi_i} = \mu(\xi_i) = \gamma_0 + \gamma_1\xi_i$, $\sigma_{\xi_i} = \sigma(\xi_i) = \exp(\gamma_0' + \gamma_1'\xi_i)$ where $\gamma_0 = -(\alpha_0+\alpha_1 s_D)$, $\gamma_1 = -\alpha_1(s_H - s_D)$, $\gamma_0' = -(\alpha_0'+\alpha_1' s_D)$, $\gamma_1' = -\alpha_1'(s_H - s_D)$. We note that for $s = s_D$, $\xi_D = 0$ and $\mu_0 = \gamma_0$, and for $s = s_H$, $\xi_H = 1$ and $\mu_1 = \gamma_0 + \gamma_1$. Also, if $\theta = \sigma_1/\sigma_0$ is the ratio of the scale parameters at the highest and design stresses, then $\sigma_{\xi_i} = \sigma_0\theta^{\xi_i}$, $i = 1, \cdots, h$, and the relationship between shape parameter and stress can be represented by any two of the following three parameters : $\sigma_0 = \exp(\gamma_0')$, $\sigma_1 = \exp(\gamma_0' + \gamma_1')$ and $\theta = \exp(\gamma_1')$.

Lifetime distribution

From the assumption of the Weibull lifetime distribution and cumulative exposure model, the cdf of Y under step stress test is

$$G(y) = G_i(y - \tau_{i-1} + \delta_{i-1}), \quad \tau_{i-1} \le y \le \tau_i \quad i = 1, 2, \cdots, h, \tag{2.2}$$

where $G_i(\cdot)$is the cdf of Weibull distribution with scale parameter λ_i and shape parameter β_i. Then $\ln(y - \tau_{i-1} + \delta_{i-1})$ follows the smallest extreme value distribution with location parameter $\mu_{\xi_i} = -\ln\lambda_i$ and scale parameter $\sigma_{\xi_i} = 1/\beta_i$.

Let

$$z_i(y) = \frac{\ln(y - \tau_{i-1} + \delta_{i-1}) - \mu_{\xi_i}}{\sigma_{\xi_i}}, \quad i = 1, 2, \cdots, h, \tag{2.3}$$

where δ_i is the solution of $z_i(\tau_i) = z_{i+1}(\tau_i)$ by cumulative exposure model. That is, $\delta_i = (\tau_i - \tau_{i-1} + \delta_{i-1})^{\theta^{\xi_{i+1}-\xi_i}} \cdot \exp(\mu_{\xi_{i+1}} - \mu_{\xi_i}\theta^{\xi_{i+1}-\xi_i})$ with $\tau_0 \equiv \delta_0 \equiv 0$. Then, the cdf of Y is

Table 3.1: asymptotic variances of the multi-step optimum plans

	$\theta = 0.8$						$\theta = 1.0$				
p_H	3	4	5	6	7	8	3	4	5	6	7
0.25	867	*865					784	775	*766		
0.60	838	766	754	*751			665	618	605	*601	
0.90	674	619	603	598	596	*595	508	473	463	*458	
0.9999	451	417	407	402	400	*399	334	311	305	302	*301

'*' : true optimum plans.

values of design parameters, tables for finding the optimum three step- stress ALTs are given.

Optimality criterion and design parameters

The optimality criterion is to minimize Asvar($\hat{t}_q$) which is a function of γ_0, γ_1, σ_0, σ_1 and ξ_i, and x_i, $i = 1, 2, \cdots, h$.

Let p_D and p_H denote, repectively, the probabilities that a unit tested only at $s_D(\xi_D = 0)$ and $s_H(\xi_H = 1)$ will fail by censoring time η. That is,

$$p_D = \Psi\left(\frac{\ln\eta - \gamma_0}{\sigma_0}\right), p_H = \Psi\left(\frac{\ln\eta - (\gamma_0 + \gamma_1)}{\sigma_1}\right) \tag{3.1}$$

Then, ζ_i in the Fisher information matrix can be represented by functions of $\Psi^{-1}(p_D)$, $\Psi^{-1}(p_H)$, θ, σ_0, ξ_i and x_i, $i = 1, 2, \cdots, h$, as follows.

$$\zeta_i = \frac{\ln(x_i - x_{i-1} + \delta^*_{i-1}) + \sigma_0((1 - \xi_i)\Psi^{-1}(p_D) + \xi_i\theta\Psi^{-1}(p_H))}{\sigma_0\theta^{\xi_i}}, \tag{3.2}$$

where, $\delta^*_i = (x_i - x_{i-1} + \delta^*_{i-1})^{\theta^{\xi_{i+1}-\xi_i}} \exp\{\sigma_0(\theta^{\xi_{i+1}-\xi_i}((1-\xi_i)\Psi^{-1}(p_D) + \xi_i\theta\Psi^{-1}(p_H)) - ((1 - \xi_{i+1})\Psi^{-1}(p_D) + \xi_{i+1}\theta\Psi^{-1}(p_H)))\}$.

As a consequence, Asvar($\hat{t}_q$) can be rewritten in terms of p_D, p_H, θ, σ_0, ξ_i's, and x_i's, and the objective is to find x_i's and ξ_i's which minimize Asvar($\hat{t}_q$) for specified p_D, p_H, θ and σ_0. In reality, these parameters are usually unknown and thus they have to be approximated from previous experience, similar data, or preliminary experiments. These approximated parameters are called 'pre-estimates'.

Optimum plans

To find the optimum values ξ_i^*'s, and x_i^*'s, $i = 1, 2$, we used the Powell method(1964) with several different initial values, since Asvar$(\hat{t}_q)$ in (9) may have several local minimum. Table 3.2 shows optimum three-step test plans for selected combinations of $q = 0.0001, 0.001, 0.01$, $\theta = 0.8, 1.0$, $\sigma_0 = 0.5, 0.99, 2.0$, $p_H = 0.25, 0.6, 0.99, 0.9999$ and $p_D = 0.0001, 0.001, 0.01$. These are the values used by Meeter and Meeker(1994). In this table, p_L and p_M denote, respectively, the probabilities of failure by η at the low and middle stress levels, and E_k, $k = L, M, H$, is the number of failures by $\log\eta$ at each stress level per 1,000 units tested in the entire program.

We note from this table that :

1. $V_s^* = $ Asvar$(\hat{t}_q)$ becomes smaller as p_H increases for any fixed values of p_D and q, implying that the highest stress of optimum plans should be chosen to be as high as possible. Nelson and Kielpinski(1976) gave an argument for this result.

2. In a few situations the optimum plans degenerate to a test with all units at design stress. This typically occurs when p_D is large and differences between p_D and p_H are too small to provide much acceleration.

4. SENSITIVITY ANALYSIS

Incorrect choice of pre-estimates gives a non-optimum test plan and could result in a less accurate estimate of the life distribution at design stress. In the case of a step-stress ALT plan, the effect of incorrect choice of the pre-estimates of p_D, p_H and σ_0 when the value of θ is known to be one was studied by Bai and Kim(1993). Here we investigate the effects of errors in pre-estimates of θ in terms of variance ratio $V_{\theta'}^*/V_\theta^*$, where $V_{\theta'}^*$ is the asymptotic variance of the MLE of 0.1th quantile of log- lifetime when incorrect pre-estimate θ' is used and V_θ^* is the corresponding asymptotic variance when true value of θ is used.

Figure 4.1 gives $V_{\theta'}^*/V_\theta^*$ due to incorrect pre-estimate θ' of θ for given values of $p_D = 0.01$, $p_H = 0.8$ and $\sigma_0 = 0.5, 0.99, 2.0$ when the correct value of θ is 0.8, and shows that the variance increase is small if the difference between correct value of θ and its pre- estimate is not large, and that underestimating θ is more serious

Table 3.2: Optimum three-step test plans

(a) $\sigma_0 = 0.5$

q	p_D	p_H	ξ_1^*	x_1^*	p_L	ξ_2^*	x_2^*	p_M	E_L	E_M	E_H	V_r^*
			$\theta = 0.8$									
0.0001	0.0001	0.25	0.000	1.000	0.000	*	*	*	0	*	*	10013
		0.60	0.000	0.594	0.000	0.310	0.800	0.001	0	0	23	9397
		0.90	0.301	0.576	0.001	0.937	0.847	0.655	0	57	133	7940
		0.9999	0.415	0.531	0.006	0.930	0.835	0.969	1	23	389	4483
0.001	0.0001	0.25	0.278	0.132	0.006	0.430	0.628	0.002	0	0	38	8877
		0.60	0.269	0.287	0.008	0.479	0.632	0.005	0	0	104	4797
		0.90	0.237	0.362	0.007	0.498	0.651	0.008	0	1	203	3245
		0.9999	0.188	0.462	0.006	0.502	0.698	0.016	0	1	441	2122
	0.001	0.25	0.000	1.000	0.001	*	*	*	1	*	*	1000
		0.60	0.000	1.000	0.001	*	*	*	1	*	*	1000
		0.90	0.000	1.000	0.001	*	*	*	1	*	*	1000
		0.9999	0.000	0.583	0.001	0.327	0.795	0.013	0	1	222	912
0.01	0.0001	0.25	0.574	0.629	0.006	-	-	-	2	-	50	10698
		0.60	0.474	0.005	0.004	0.577	0.595	0.011	0	3	154	4397
		0.90	0.495	0.064	0.008	0.575	0.586	0.018	0	4	329	2277
		0.9999	0.459	0.251	0.010	0.585	0.605	0.041	0	7	716	981
	0.001	0.25	0.000	0.002	0.001	0.343	0.715	0.005	0	2	32	3585
		0.60	0.074	0.008	0.002	0.395	0.664	0.010	0	4	113	1742
		0.90	0.265	0.061	0.006	0.417	0.646	0.016	0	5	256	995
		0.9999	0.281	0.255	0.009	0.458	0.655	0.038	0	9	618	478
	0.01	0.25	0.000	1.000	0.010	*	*	*	0	*	*	100
		0.60	0.000	1.000	0.010	*	*	*	0	*	*	100
		0.90	0.000	1.000	0.010	*	*	*	0	*	*	100
		0.9999	0.000	1.000	0.010	*	*	*	0	*	*	100

q	p_D	p_H	ξ_1^*	x_1^*	p_L	ξ_2^*	x_2^*	p_M	E_L	E_M	E_H	V_r^*
			$\theta = 1.0$									
0.0001	0.0001	0.25	0.000	0.577	0.000	0.281	0.860	0.001	0	0	7	8924
		0.60	0.292	0.592	0.001	0.927	0.866	0.376	0	42	65	7513
		0.90	0.372	0.543	0.004	0.924	0.851	0.657	1	116	168	5180
		0.9999	0.427	0.536	0.013	0.913	0.839	0.967	3	317	387	2809
0.001	0.0001	0.25	0.332	0.184	0.001	0.423	0.684	0.003	0	1	39	5047
		0.60	0.295	0.356	0.002	0.463	0.686	0.007	0	1	108	2846
		0.90	0.254	0.424	0.001	0.478	0.705	0.012	0	1	214	2025
		0.9999	0.198	0.524	0.001	0.473	0.757	0.022	0	2	461	1444
	0.001	0.25	0.000	1.000	0.001	*	*	*	1	*	*	1000
		0.60	0.000	1.000	0.001	*	*	*	1	*	*	1000
		0.90	0.000	0.560	0.001	0.560	0.874	0.008	0	1	51	917
		0.9999	0.243	0.647	0.009	0.647	0.873	0.987	3	241	322	791
0.01	0.0001	0.25	0.551	0.687	0.008	-	-	-	3	-	47	5902
		0.60	0.551	0.645	0.015	-	-	-	6	-	154	2422
		0.90	0.502	0.054	0.015	0.539	0.637	0.022	0	8	331	1270
		0.9999	0.444	0.312	0.016	0.549	0.665	0.052	1	12	711	576
	0.001	0.25	0.342	0.760	0.007	-	-	-	3	-	32	2086
		0.60	0.325	0.002	0.009	0.377	0.710	0.013	0	6	113	1008
		0.90	0.327	0.101	0.013	0.390	0.697	0.020	0	9	257	587
		0.9999	0.350	0.255	0.014	0.433	0.715	0.051	1	13	608	301
	0.01	0.25	0.000	1.000	0.010	*	*	*	10	*	*	100
		0.60	0.000	1.000	0.010	*	*	*	10	*	*	100
		0.90	0.000	1.000	0.010	*	*	*	10	*	*	100
		0.9999	0.000	0.950	0.010	0.101	0.996	0.020	9	1	2	99

'-' : test with two stress levels.
'*' : degenerate tests.

(b) $\sigma_0 = 0.99$

			$\theta = 0.8$									
q	p_D	p_H	ξ_1^*	x_1^*	p_L	ξ_2^*	x_2^*	p_M	E_L	E_M	E_H	V_s^*
0.0001	0.0001	0.25	0.228	0.713	0.000	0.908	0.916	0.110	0	16	18	8976
		0.60	0.385	0.583	0.002	0.915	0.877	0.297	1	76	83	6148
		0.90	0.455	0.544	0.005	0.917	0.860	0.563	2	183	193	4175
		0.9999	0.501	0.558	0.016	0.907	0.852	0.919	8	428	375	2288
0.001	0.0001	0.25	0.234	0.338	0.001	0.427	0.784	0.002	0	0	42	3105
		0.60	0.201	0.429	0.000	0.408	0.841	0.003	0	0	88	2365
		0.90	0.432	0.661	0.004	0.941	0.872	0.672	2	151	180	1969
		0.9999	0.493	0.627	0.014	0.924	0.857	0.958	8	406	379	1190
	0.001	0.25	0.000	0.643	0.001	0.170	0.945	0.002	0	0	8	976
		0.60	0.000	0.662	0.001	0.204	0.950	0.003	0	0	22	918
		0.90	0.000	0.680	0.001	0.204	0.955	0.004	0	0	47	886
		0.9999	0.334	0.647	0.013	0.893	0.884	0.939	8	383	342	652
0.01	0.0001	0.25	0.592	0.646	0.007	-	-	-	4	-	79	2054
		0.60	0.508	0.201	0.006	0.586	0.674	0.012	0	5	206	989
		0.90	0.456	0.298	0.005	0.578	0.718	0.019	1	7	379	636
		0.9999	0.395	0.414	0.005	0.546	0.800	0.026	1	9	701	406
	0.001	0.25	0.414	0.700	0.008	-	-	-	5	-	66	811
		0.60	0.312	0.194	0.006	0.436	0.727	0.013	1	6	171	442
		0.90	0.267	0.303	0.006	0.443	0.768	0.020	1	8	313	309
		0.9999	0.219	0.423	0.005	0.424	0.839	0.028	2	11	602	218
	0.01	0.25	0.000	1.000	0.010	*	*	*	10	*	*	100
		0.60	0.000	1.000	0.010	*	*	*	10	*	*	100
		0.90	0.000	0.615	0.010	0.184	0.940	0.023	6	7	73	97
		0.9999	0.000	0.650	0.010	0.227	0.945	0.036	6	10	221	89

			$\theta = 1.0$									
q	p_D	p_H	ξ_1^*	x_1^*	p_L	ξ_2^*	x_2^*	p_M	E_L	E_M	E_H	V_s^*
0.0001	0.0001	0.25	0.321	0.615	0.001	0.898	0.892	0.120	0	34	29	6625
		0.60	0.413	0.552	0.004	0.904	0.871	0.316	2	112	98	4205
		0.90	0.459	0.534	0.010	0.903	0.860	0.582	5	243	205	2781
		0.9999	0.485	0.574	0.025	0.887	0.862	0.920	14	504	344	1520
0.001	0.0001	0.25	0.264	0.366	0.001	0.425	0.841	0.003	0	1	43	1963
		0.60	0.220	0.470	0.001	0.401	0.896	0.004	0	1	89	1568
		0.90	0.438	0.660	0.008	0.929	0.877	0.675	5	212	191	1294
		0.9999	0.477	0.650	0.023	0.906	0.869	0.957	14	486	347	789
	0.001	0.25	0.000	0.589	0.001	0.214	0.960	0.003	0	1	11	862
		0.60	0.000	0.652	0.001	0.219	0.974	0.004	0	1	22	825
		0.90	0.278	0.663	0.008	0.886	0.902	0.615	5	200	159	730
		0.9999	0.352	0.641	0.025	0.869	0.886	0.938	15	481	324	478
0.01	0.0001	0.25	0.588	0.710	0.011	-	-	-	7	-	78	1261
		0.60	0.512	0.193	0.011	0.566	0.735	0.017	2	9	210	600
		0.90	0.449	0.298	0.009	0.553	0.780	0.026	2	12	387	393
		0.9999	0.382	0.430	0.008	0.515	0.862	0.035	3	15	698	263
	0.001	0.25	0.418	0.765	0.011	-	-	-	8	-	64	521
		0.60	0.330	0.227	0.009	0.424	0.790	0.018	2	10	170	290
		0.90	0.275	0.336	0.008	0.423	0.833	0.026	2	12	310	211
		0.9999	0.220	0.463	0.007	0.395	0.900	0.036	3	15	581	159
	0.01	0.25	0.000	0.965	0.010	0.090	0.997	0.014	9	0	0	100
		0.60	0.000	0.535	0.010	0.175	0.951	0.022	5	9	41	93
		0.90	0.000	0.575	0.010	0.209	0.957	0.031	5	11	90	87
		0.9999	0.000	0.650	0.010	0.218	0.974	0.043	6	14	202	82

(c) $\sigma_0 = 2.0$

			$\theta = 0.8$									
q	p_D	p_H	ξ_1^*	x_1^*	p_L	ξ_2^*	x_2^*	p_M	E_L	E_M	E_H	V_s^*
0.0001	0.0001	0.25	0.381	0.650	0.001	0.904	0.922	0.106	1	48	31	6163
		0.60	0.477	0.584	0.004	0.913	0.899	0.291	3	152	103	3904
		0.90	0.523	0.569	0.011	0.913	0.890	0.545	7	316	206	2598
		0.9999	0.547	0.633	0.027	0.895	0.904	0.881	20	596	300	1465
0.001	0.0001	0.25	0.092	0.233	0.000	0.365	0.983	0.001	0	0	21	2045
		0.60	0.462	0.736	0.004	0.928	0.920	0.335	3	131	92	1760
		0.90	0.517	0.670	0.010	0.924	0.901	0.593	7	298	200	1277
		0.9999	0.545	0.700	0.026	0.905	0.908	0.914	21	587	302	776
	0.001	0.25	0.000	0.533	0.001	0.152	0.996	0.002	0	0	8	867
		0.60	0.223	0.782	0.004	0.888	0.957	0.303	3	114	59	838
		0.90	0.338	0.689	0.009	0.890	0.929	0.558	7	282	164	674
		0.9999	0.412	0.693	0.026	0.873	0.925	0.896	20	585	282	451
0.01	0.0001	0.25	0.334	0.061	0.001	0.607	0.810	0.008	0	6	90	699
		0.60	0.367	0.166	0.002	0.565	0.907	0.010	0	7	179	455
		0.90	0.358	0.235	0.002	0.531	0.964	0.011	0	8	242	365
		0.9999	0.334	0.289	0.003	0.482	0.995	0.013	1	9	277	290
	0.001	0.25	0.000	0.078	0.001	0.467	0.844	0.010	0	8	78	306
		0.60	0.089	0.171	0.002	0.428	0.930	0.013	0	10	150	223
		0.90	0.122	0.234	0.002	0.395	0.973	0.014	1	10	205	191
		0.9999	0.135	0.288	0.003	0.351	0.996	0.015	1	11	238	163
	0.01	0.25	0.000	0.449	0.010	0.168	0.949	0.016	6	6	34	93
		0.60	0.000	0.506	0.010	0.188	0.981	0.020	7	8	61	88
		0.90	0.000	0.533	0.010	0.178	0.994	0.022	7	9	80	85
		0.9999	0.000	0.532	0.010	0.150	0.992	0.023	7	9	351	83

			$\theta = 1.0$									
q	p_D	p_H	ξ_1^*	x_1^*	p_L	ξ_2^*	x_2^*	p_M	E_L	E_M	E_H	V_s^*
0.0001	0.0001	0.25	0.403	0.594	0.003	0.896	0.912	0.118	1	66	36	4411
		0.60	0.473	0.560	0.008	0.902	0.900	0.312	5	190	108	2686
		0.90	0.507	0.563	0.016	0.899	0.897	0.567	12	371	202	1765
		0.9999	0.518	0.655	0.036	0.874	0.922	0.887	29	645	264	1008
0.001	0.0001	0.25	0.173	0.253	0.000	0.370	0.995	0.002	0	1	19	1356
		0.60	0.459	0.728	0.007	0.920	0.922	0.356	5	171	97	1170
		0.90	0.501	0.677	0.015	0.912	0.909	0.614	12	355	196	858
		0.9999	0.516	0.732	0.036	0.886	0.926	0.919	30	638	265	533
	0.001	0.25	0.000	0.472	0.001	0.155	0.971	0.002	0	1	46	784
		0.60	0.288	0.688	0.007	0.879	0.937	0.330	5	175	80	665
		0.90	0.359	0.650	0.016	0.876	0.925	0.587	12	358	171	508
		0.9999	0.400	0.700	0.038	0.849	0.937	0.902	31	645	247	334
0.01	0.0001	0.25	0.358	0.045	0.002	0.613	0.854	0.013	0	11	92	451
		0.60	0.373	0.140	0.003	0.558	0.943	0.016	1	13	183	287
		0.90	0.358	0.203	0.004	0.514	0.986	0.017	1	13	224	232
		0.9999	0.327	0.255	0.004	0.457	0.999	0.018	2	13	225	187
	0.001	0.25	0.130	0.072	0.002	0.470	0.894	0.014	0	12	77	218
		0.60	0.178	0.179	0.003	0.419	0.965	0.017	1	13	143	160
		0.90	0.176	0.240	0.004	0.380	0.992	0.019	1	14	174	140
		0.9999	0.163	0.294	0.004	0.329	0.999	0.020	2	14	173	122
	0.01	0.25	0.000	0.391	0.010	0.210	0.968	0.020	6	10	36	85
		0.60	0.000	0.485	0.010	0.202	0.993	0.025	6	11	55	81
		0.90	0.000	0.468	0.010	0.163	0.967	0.024	6	11	323	79
		0.9999	0.207	0.817	0.040	0.829	0.966	0.943	36	633	197	74

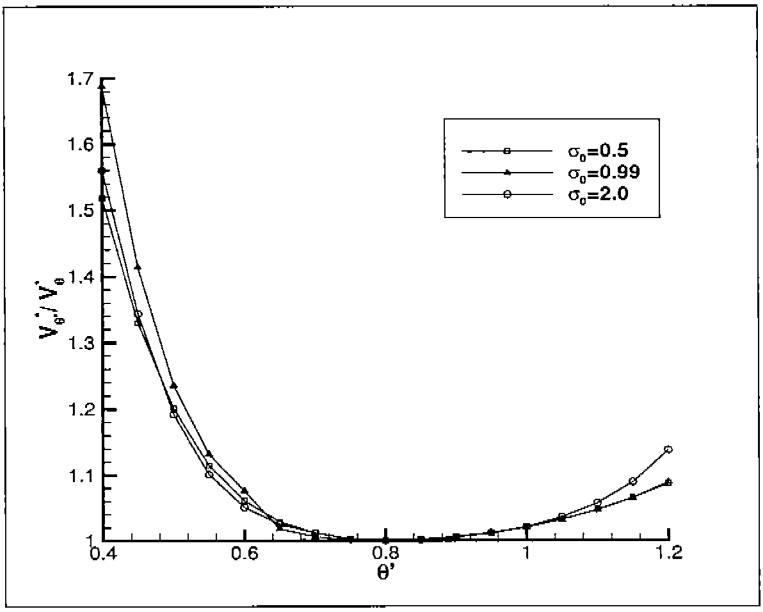

Figure 4.1: Variance ratio vs. pre-estimates of θ' when $\theta = 0.8$

than overestimating θ. Similar results are obtained for other combinations of p_D, p_H and σ_0.

5. THE EFFECTS OF θ

In this section, optimum step-stress ALTs with a nonconstant shape parameter are compared with corresponding ones with a constant shape parameter(Bai and Kim(1993)) in terms of relative efficiency. When θ is known, partial derivatives and Fisher information matrix are as follows.

The first and second partial derivatives of l with respect to γ_0, γ_1 are the same as (16) and (18) in Appendix A, respectively. The first partial derivative of l with respect to σ_0 is

$$\frac{\partial l}{\partial \sigma_0} = \frac{1}{\sigma_0}\left[-\sum_{i=1}^{h} U_i(y)\left\{1 + z_i(y) - z_i(y)e^{z_i(y)}\right\} + U_{h+1}(y)e^{z_h(\eta)}v_{hj}(\eta)\right], \quad (5.1)$$

and the second partial derivatives of l with respect to σ_0 are

$$\frac{\partial^2 l}{\partial\gamma_j\partial\sigma_0} = -\frac{1}{\sigma_0}\frac{\partial l}{\partial\gamma_j} - \frac{1}{\sigma_0}\sum_{i=1}^{h} U_i(y)\left\{\frac{l_{ij}}{b_i(y)} - u_{ij}(y)z_i(y)e^{z_i(y)}\right\} + \frac{1}{\sigma_0}U_{h+1}(y)e^{z_h(\eta)}v_{hj}(\eta), \tag{5.2}$$

$$\frac{\partial^2 l}{\partial\sigma_0^2} = -\frac{2}{\sigma_0}\frac{\partial l}{\partial\sigma_0} + \frac{1}{\sigma_0^2}\left[-\sum_{i=1}^{h} U_i(y)\left\{1 + z_i^2(y)e^{z_i(y)}\right\} - U_{h+1}(y)e^{z_h^2(\eta)}v_{hj}(\eta)\right], \tag{5.3}$$

The Fisher information matrix for single observation can be obtained by taking the negative expectations of the second partial derivatives of log-likelihood function.

$$F = \begin{bmatrix} A_{00} & A_{01} & B'_0 \\ A_{10} & A_{11} & B'_1 \\ B'_0 & B'_1 & C' \end{bmatrix} \tag{5.4}$$

where $A_{jk} = E\left(-\frac{\partial^2 l}{\partial\gamma_j\partial\gamma_k}\right)$, $B'_j = E\left(-\frac{\partial^2 l}{\partial\gamma_j\partial\sigma_0}\right)$, $C' = E\left(-\frac{\partial^2 l}{\partial\sigma_0^2}\right)$, $j, k = 0, 1$.

When $\theta \neq 1$, models with constant σ are not appropriate and analysis of ALT data using a model with constant σ results in a poor estimate of the specified quantile of log-life at design stress. The effects of θ in terms of the asymptotic variance ratio were studied for $\theta \leq 1.0$.

Let V' and V^* be the asymptotic variances of the MLEs of 0.01th quantile of log-lifetime at design stress under incorrect assumption that $\theta = 1.0$ and correct value of $\theta \neq 1.0$, respectively. When $p_D = 0.0001, 0.001$, $p_H = 0.25, 0.40, 0.60, 0.80, 0.90, 0.99$ and $\sigma_0 = 0.5, 0.99, 2.0$, the ratios V'/V^* are given in Table 3.2. The above values of p_D, p_H and σ_0 are the values used by Bai and Kim(1993). Table 5.1 shows that V'/V^* becomes larger as θ decreases for any fixed values of p_D and q, and if the true θ is far from 1.0, the variance increase is large.

Table 5.1: Asymptotic variance ratio V'/V^* when $\theta \leq 1.0$

		$\sigma_0 = 0.5$					
p_D	p_H	$\theta = 0.5$	0.6	0.7	0.8	0.9	1.0
0.0001	0.25	1.352	1.169	1.076	1.029	1.007	1.000
	0.40	1.393	1.187	1.084	1.031	1.006	1.000
	0.60	1.424	1.198	1.086	1.030	1.005	1.000
	0.80	1.515	1.246	1.112	1.043	1.010	1.000
	0.90	1.572	1.275	1.127	1.050	1.013	1.000
	0.99	1.671	1.320	1.147	1.057	1.014	1.000
0.001	0.25	1.056	1.034	1.018	1.007	1.002	1.000
	0.40	1.104	1.059	1.030	1.012	1.003	1.000
	0.60	1.144	1.078	1.038	1.016	1.004	1.000
	0.80	1.204	1.109	1.054	1.022	1.006	1.000
	0.90	1.226	1.117	1.056	1.022	1.006	1.000
	0.99	1.284	1.142	1.065	1.024	1.005	1.000
		$\sigma_0 = 0.99$					
p_D	p_H	$\theta = 0.5$	0.6	0.7	0.8	0.9	1.0
0.0001	0.25	1.343	1.182	1.071	1.034	1.006	1.000
	0.40	1.325	1.181	1.062	1.030	1.003	1.000
	0.60	1.381	1.173	1.084	1.025	1.009	1.000
	0.80	1.343	1.207	1.070	1.034	1.005	1.000
	0.90	1.368	1.225	1.078	1.039	1.007	1.000
	0.99	1.414	1.249	1.087	1.041	1.006	1.000
0.001	0.25	1.176	1.076	1.039	1.016	1.003	1.000
	0.40	1.152	1.074	1.031	1.012	1.002	1.000
	0.60	1.165	1.079	1.039	1.013	1.005	1.000
	0.80	1.163	1.081	1.037	1.012	1.003	1.000
	0.90	1.166	1.089	1.036	1.014	1.002	1.000
	0.99	1.229	1.111	1.051	1.018	1.003	1.000
		$\sigma_0 = 2.0$					
p_D	p_H	$\theta = 0.5$	0.6	0.7	0.8	0.9	1.0
0.0001	0.25	1.371	1.162	1.085	1.026	1.009	1.000
	0.40	1.379	1.149	1.082	1.020	1.007	1.000
	0.60	1.373	1.183	1.074	1.033	1.004	1.000
	0.80	1.437	1.160	1.092	1.025	1.008	1.000
	0.90	1.474	1.174	1.102	1.029	1.009	1.000
	0.99	1.530	1.196	1.111	1.031	1.009	1.000
0.001	0.25	1.140	1.086	1.038	1.014	1.004	1.000
	0.40	1.148	1.072	1.033	1.011	1.002	1.000
	0.60	1.161	1.082	1.035	1.016	1.003	1.000
	0.80	1.170	1.081	1.035	1.014	1.002	1.000
	0.90	1.186	1.081	1.039	1.012	1.002	1.000
	0.99	1.228	1.114	1.050	1.018	1.004	1.000

APPENDIX A : Partial derivatives of likelihood function

The first partial derivatives of l with respect to γ_0, γ_1, σ_0 and σ_1 are

$$\begin{aligned}\frac{\partial l}{\partial \gamma_j} &= \sum_{i=1}^{h} U_i(y)\left\{-\frac{l_{ij}}{b_i(y)} + u_{ij}(y)\left(1-e^{z_i(y)}\right)\right\} \\ &\quad -U_{h+1}(y)e^{z_h(\eta)}u_{hj}(\eta), \quad j=0,1\end{aligned} \tag{A.1}$$

$$\begin{aligned}\frac{\partial l}{\partial \sigma_j} &= \sum_{i=1}^{h} U_i(y)\left\{-\frac{n_{ij}}{\sigma_{\xi_i}} - \frac{m_{ij}}{b_i(y)} + v_{ij}(y)\left(1-e^{z_i(y)}\right)\right\} \\ &\quad -U_{h+1}(y)e^{z_h(\eta)}v_{hj}(\eta), \quad j=0,1\end{aligned} \tag{A.2}$$

where $l_{ij} = \frac{\partial \delta_{i-1}}{\partial \gamma_j}$, $m_{ij} = \frac{\partial \delta_{i-1}}{\partial \sigma_j}$, $n_{ij} = \frac{(1-\xi_i)^{1-j}\xi_i^j}{\sigma_j}\sigma_{\xi_i}$, $u_{ij}(y) = \frac{1}{\sigma_{\xi_i}}\left(\frac{l_{ij}}{b_i(y)} - \xi_i^j\right)$, $v_{ij}(y) = \frac{1}{\sigma_{\xi_i}}\left(\frac{m_{ij}}{b_i(y)} - n_{ij}z_i(y)\right)$.

The second partial derivatives of l with respect to γ_0, γ_1, σ_0 and σ_1 are

$$\begin{aligned}\frac{\partial^2 l}{\partial \gamma_j \partial \gamma_k} &= \sum_{i=1}^{h} U_i(y)\left\{u'_{ijk}(y)\rho_i(y) - u_{ij}(y)u_{ik}(y)e^{z_i(y)}\right\} \\ &\quad -U_{h+1}(y)e^{z_h(\eta)}\left\{u_{hj}(\eta)u_{hk}(\eta) + u'_{hjk}(\eta)\right\}, \quad j,k=0,1\end{aligned} \tag{A.3}$$

$$\begin{aligned}\frac{\partial^2 l}{\partial \gamma_j \partial \sigma_k} &= \sum_{i=1}^{h} U_i(y)\left\{w'_{ijk}(y)\rho_i(y) - u_{ij}(y)\left(n_{ik} + v_{ik}(y)e^{z_i(y)}\right)\right\} \\ &\quad -U_{h+1}(y)e^{z_h(\eta)}\{u_{hj}(\eta)v_{hk}(\eta) + w'_{hjk}(\eta)\}, \quad j,k=0,1\end{aligned} \tag{A.4}$$

$$\begin{aligned}\frac{\partial^2 l}{\partial \sigma_j \partial \sigma_k} &= \sum_{i=1}^{h} U_i(y)\left\{v'_{ijk}(y)\rho_i(y) - v_{ik}(y)\left(n_{ij} + v_{ij}(y)e^{z_i(y)}\right)\right. \\ &\quad \left. -\frac{m_{ij}m_{ik}}{\sigma_{\xi_i}b_i(y)} + \left(\frac{n_{ij}}{\sigma_k\sigma_{\xi_i}}\vartheta_i(y)+1\right)^{1-|j-k|} - 1\right\} \\ &\quad -U_{h+1}(y)e^{z_h(\eta)}\left\{v_{hj}(\eta)v_{hk}(\eta) + v'_{hjk}(\eta)\right. \\ &\quad \left. + \left(\frac{n_{hj}z_h(\eta)}{\sigma_k\sigma_{\xi_h}}+1\right)^{1-|j-k|} - 1\right\}, \quad j,k=0,1\end{aligned} \tag{A.5}$$

where, $\rho_i(y) = 1 - \sigma_{\xi_i} - e^{z_i(y)}$, $\vartheta_i(y) = 1 + \sigma_{\xi_i}z_i(y)$,
$u'_{ijk} = \frac{\partial u_{ij}(y)}{\partial \gamma_k} = \frac{1}{\sigma_{\xi_i}b_i(y)}\left(\frac{\partial^2\delta_{i-1}}{\partial\gamma_j\partial\gamma_k} - \frac{l_{ij}l_{ik}}{b_i(y)}\right)$,
$w'_{ijk} = \frac{\partial u_{ij}(y)}{\partial \sigma_k} = \frac{1}{\sigma_{\xi_i}b_i(y)}\left(\frac{\partial^2\delta_{i-1}}{\partial\gamma_j\partial\sigma_k} - \frac{l_{ij}m_{ik}}{b_i(y)} - n_{ik}b_i(y)u_{ij}(y)\right)$,
$v'_{ijk} = \frac{\partial v_{ij}(y)}{\partial \sigma_k} = \frac{1}{\sigma_{\xi_i}b_i(y)}\left(\frac{\partial^2\delta_{i-1}}{\partial\sigma_j\partial\sigma_k} - \frac{m_{ij}m_{ik}}{b_i(y)} - n_{ij}b_i(y)v_{ik}(y) - \frac{m_{ij}n_{ik}}{\sigma_{\xi_i}}\right)$.

APPENDIX B : Fisher Information Matrix

The Fisher information matrix for single observation can be obtained by taking the negative expectations of the second partial derivatives of log-likelihood function;

$$F = \begin{bmatrix} A_{00} & A_{01} & B_{00} & B_{01} \\ A_{10} & A_{11} & B_{10} & B_{11} \\ B_{00} & B_{01} & C_{00} & C_{01} \\ B_{10} & B_{11} & C_{10} & C_{11} \end{bmatrix} \tag{B.1}$$

where,
$A_{jk} = E\left(-\frac{\partial^2 l}{\partial\gamma_j\partial\gamma_k}\right)$, $B_{jk} = E\left(-\frac{\partial^2 l}{\partial\gamma_j\partial\sigma_k}\right)$, $C_{jk} = E\left(-\frac{\partial^2 l}{\partial\sigma_j\partial\sigma_k}\right)$, $j, k = 0, 1$.
$E(U_{h+1}(y)) = \bar{\Psi}(\zeta_h)$, $E(U_i(y)\vartheta_i(y)) = D_3(i) + D_5(i)$,
$E(U_i(y)\rho_i(y)) = (1-\sigma_{\xi_i})D_5(i) - D_4(i)$,
$E(U_i(y)\rho_i(y)z_i(y)) = (1-\sigma_{\xi_i})D_3(i) - \sigma_{\xi_i}d_3(i)$,
$E\left(U_i(y)\frac{\rho_i(y)}{b_i^2(y)}\right) = D_2(i) + \sigma_{\xi_i}d_2(i)$,
$E\left(U_i(y)\frac{e^{z_i(y)}}{b_i(y)}\right) = (1-\sigma_{\xi_i})d_1(i) - D_1(i)$,
$E\left(U_i(y)\frac{e^{z_i(y)}}{b_i^2(y)}\right) = (1-2\sigma_{\xi_i})d_2(i) - D_2(i)$,
where, $d_1(i) = e^{-\mu_{\xi_i}}\int_{e^{\zeta_{i-1}}}^{e^{\zeta_i}} x^{-\sigma_{\xi_i}}e^{-x}dx$, $d_2(i) = e^{-2\mu_{\xi_i}}\int_{e^{\zeta_{i-1}}}^{e^{\zeta_i}} x^{-\sigma_{\xi_i}}e^{-x}dx$,
$d_3(i) = e^{-\mu_{\xi_i}}\int_{e^{\zeta_{i-1}}}^{e^{\zeta_i}}(\ln x)x^{1-\sigma_{\xi_i}}e^{-x}dx$, $d_4(i) = e^{-\mu_{\xi_i}}\int_{e^{\zeta_{i-1}}}^{e^{\zeta_i}}(\ln x)xe^{-x}dx$,
$d_5(i) = e^{-\mu_{\xi_i}}\int_{e^{\zeta_{i-1}}}^{e^{\zeta_i}}(\ln x)^2xe^{-x}dx$,
$D_1(i) = e^{-\mu_{\xi_i}}\left(e^{-e^{\zeta_i}+\zeta_i(1-\sigma_{\xi_i})} - e^{-e^{\zeta_{i-1}}+\zeta_{i-1}(1-\sigma_{\xi_i})}\right)$,
$D_2(i) = e^{-2\mu_{\xi_i}}\left(e^{-e^{\zeta_i}+\zeta_i(1-2\sigma_{\xi_i})} - e^{-e^{\zeta_{i-1}}+\zeta_{i-1}(1-2\sigma_{\xi_i})}\right)$,
$D_3(i) = \left(e^{-e^{\zeta_i}}\left(\zeta_ie^{\zeta_i}+1\right) - e^{-e^{\zeta_{i-1}}}\left(\zeta_{i-1}e^{\zeta_{i-1}}+1\right)\right)$,
$D_4(i) = \left(-e^{-e^{\zeta_i}}\left(e^{\zeta_i}+1\right) + e^{-e^{\zeta_{i-1}}}\left(e^{\zeta_{i-1}}+1\right)\right)$,
$D_5(i) = \left(-e^{-e^{\zeta_i}} + e^{-e^{\zeta_{i-1}}}\right)$.

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