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Some Stochastic Properties for Imperfect Repair Model†

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Abstract

We consider an imperfect repair model under which either a perfect repair or a minimal repair can be performed at each failure of a unit. Some stochastic properties of the number of perfect repairs and the number of minimal repairs under the imperfect repair model are investigated. We also derive the expressions for evaluating the expected numbers of perfect and minimal repairs in general and apply these formulas for certain parametric families of life distributions.

Keywords: Imperfect repair; Failure rate; NBU; NHPP.

1. INTRODUCTION

The maintenance of a system is one of the most important and practical areas in reliability theory, and appropriate maintenance policy not only reduces the cost of maintaining a system in industries but also improves the productivity of the system.

During the past three decades, a number of maintenance policies have been studied and its related results have been published. They are well summarized in Valdez-Flores and Feldman(1989). The optimal maintenance policy depends on various factors, among which the pattern of failures and the maintenance cost play an important role. Brown and Proschan(1983) propose an imperfect repair model under which a failed unit (or system) is either perfectly repaired or minimally repaired with probabilities p and $q = 1 - p$, respectively. Here $p \in [0, 1]$.

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In a perfect repair, a repaired unit is returned to the good-as-new state so that the unit repaired perfectly has effective age zero. (Replacing a failed unit by a new one is also included in this category). In a minimal repair, a repaired unit is returned to the functioning state but it is only as good as a unit of age equal to its age at failure. Block, Borges and Savits(1985) modify the imperfect repair model in such a way that the probability of a perfect repair depends on the age of the unit at its failure.

Let F be a life distribution (that is , $F(t) = 0$ for $t < 0$). Let $\bar{F} \equiv 1 - F$ be the survival function and let f be its corresponding probability density function. The failure rate of a life distribution F is defined as

$$r(t) = \frac{f(t)}{\bar{F}(t)} \tag{1.1}$$

for t such that $\bar{F}(t) > 0$ if f(t) exists. The hazard function is defined by

$$R(t) = \int_0^t r(x)dx = -\ln \bar{F}(t) \tag{1.2}$$

for t satisfying $\bar{F}(t) > 0$. A class of life distributions which arises natually in considering replacement policies is the New Better than Used (NBU) class. A life distribution F is NBU if

$$\bar{F}(s+t) \leq \bar{F}(s)\bar{F}(t) \quad for \quad s,\ t \geq 0. \tag{1.3}$$

The dual concept, a New Worse than Used(NWU) distribution is defined by reversing the inequality in (1.3).

In this paper we consider an imperfect repair model proposed by Brown and Proschan(1983), which is indexed by the probability, p, of perfect repair at each failure of a unit. In Section 2 we derive some stochastic properties of the number of perfect repairs and the number of minimal repairs with respect to the value of p. We also present the expressions to obtain the expected numbers of perfect repairs and minimal repairs during a finite time period. In Section 3, we apply our results to compute the expected numbers of repairs during a finite time period when the distributions of failure times are exponential and Weibull. Section 4 is devoted to conduct simulation studies for numerical comparisons of our results.

2. NUMBER OF REPAIRS UNDER IMPERFECT REPAIR MODEL

Under the imperfect repair model, at each failure a failed unit is perfectly repaired with a probability p or minimally repaired with a probability $q = 1 - p$, respectively. Throughout this paper, we assume that the failure time of a new unit has a distribution function F with survival function $\bar{F}$, failure rate $r(t)$ and hazard function $R(t) = -\ln \bar{F}(t)$. We also assume that it takes negligible time to perform any repair.

Let $N^P(t,p)$ and $N^M(t,p)$ be the number of perfect repairs and the number of minimal repairs during an interval (0,t], $t \geq 0$ under the imperfect repair model. The stochastic properties of $N^P(t,p)$ and $N^M(t,p)$ are summarized in Theorem 2.1.

Theorem 2.1. *(i) For every fixed t≥0 and every F,* $\{ N^P(t,p) \ : \ 0 \leq p \leq 1 \}$ *is a stochastically increasing family in p.*
(ii) Let Z be the time of first perfect repair under the imperfect repair model. Then conditional on the set $t < Z$, $\{N^M(t,p) \ : \ 0 \leq p \leq 1 \}$ *is a stochastically decreasing family in p.*

Proof: (i) Let H_p be the distribution of times between two successive perfect repairs under the model. Brown and Proschan(1983) show that $\bar{H}_p(t) = \bar{F}^p(t)$. Let $k \geq 1$ be an integer. Then it is obvious that

$$P(N^P(t,p) \geq k) = H_p^{(k)}(t),$$

where $0 \leq p \leq 1$ and $H_p^{(k)}(\cdot)$ is the k-th convolution of H_p.
Let $0 \leq p_1 < p_2 \leq 1$. Then

$$H_{p_1}^{(1)}(t) = 1 - \bar{F}^{p_1}(t) \leq 1 - \bar{F}^{p_2}(t) = H_{p_2}^{(1)}(t).$$

Assuming that $H_{p_1}^{(n-1)}(t) \leq H_{p_2}^{(n-1)}(t)$, we have

$$\begin{aligned} H_{p_1}^{(n)}(t) &= \int_0^t H_{p_1}^{(n-1)}(t-x)\, dH_{p_1}(x) \\ &\leq \int_0^t H_{p_2}^{(n-1)}(t-x)\, dH_{p_2}(x) \\ &= H_{p_2}^{(n)}(t). \end{aligned}$$

This completes the proof.

(ii) Since the minimal repairs generate a nonhomogeneous Poisson process, it follows from the result of Fontenot and Brown(1984) (see Lemma 1.1 for details) that

$$P(N^M(t,p) = k | Z > t) = \frac{(qR(t))^k e^{-qR(t)}}{k!}. \tag{2.1}$$

Let $0 \leq p_1 < p_2 \leq 1$. Then, it is sufficient to prove that the inequalities

$$P(N^M(t,p_1) \geq k|Z > t) \geq P(N^M(t,p_2) \geq k|Z > t) \tag{2.2}$$

hold for all positive integer k. Since given that $Z > t$, $N^M(t,p)$ follows a Poisson distribution with a mean of $qR(t)$ from (2.1) and $q_1R(t) > q_2R(t)$, where $q_i = 1 - p_i$, i=1, 2, (2.2) immediately follows. This completes the proof. □

By taking $p_1 = p,\ p_2 = 1$ for result (i) and $p_1 = 0,\ p_2 = p$ where $0 < p < 1$, we obtain Corollary 2.1.

Corollary 2.1. *Let $0 < p < 1$.*
(i) For $t \geq 0$, $N^P(t,1) \overset{\text{st.}}{\geq} N^P(t,p)$
and
(ii) Let Z be the time of first perfect repair under the imperfect model. Then given that $t < Z$, $N^M(t,p) \overset{\text{st.}}{\leq} N^M(t,0)$,

where the notation, st.$\geq$, represents "stochastically larger than".

Here $N^P(t,1)$ and $N^M(t,0)$ are the number of perfect repairs and minimal repairs when a failed unit is perfectly repaired or minimally repaired at each failure, respectively. Since $\{N^P(t,1), t \geq 0\}$ is a renewal process with interoccurrence distribution F and $\{N^M(t,0), t \geq 0\}$ is a nonhomogeneous Poisson process with intensity function $r(t)$, we have Theorem 2.2.

Theorem 2.2. *Let $t > 0$ be given. Then*
(i) $EN^P(t,1) = M_F(t) = \sum_{k=1}^{\infty} F^{(k)}(t)$
and
(ii) $EN^M(t,0) = -\ln \bar{F}(t)$,
where $F^{(k)}(\cdot)$ is the k-th convolution of F and $M_F(t)$ is the renewal function.

The following lemma is needed to derive the formulas for the expected number of perfect repairs and the expected number of minimal repairs under the imperfect repair model. The result is due to Lim and Park(1999).

Lemma 2.1. *For the repair process based on the imperfect repair model, let T_i be the time of the i-th perfect repair, i=1,2,...,$N^P(t,p)$, and $t > 0$. Let $W_i = T_i - T_{i-1}$ and let $w_i = t_i - t_{i-1}$ be the realizations of $W_i = T_i - T_{i-1}$, where i=1,2,...,$N^P(t,p)$*

and $T_0 = 0$. *The conditional expectation for the number of minimal repairs in (0,t], given that* $N^P(t,p) = n, T_1 = t_1, ..., T_n = t_n$ *is*

$$E[N^M(t,p)|N^P(t,p) = n, T_1 = t_1, ..., T_n = t_n] = (1-p)\{\sum_{i=1}^{n} R(w_i) + R(t - t_n)\}.$$

It is noted that $\{N^P(t,p), t \geq 0\}$ is a renewal process with interoccurrence distribution H. Taking expectations with respect to $(T_1, T_2, ..., T_n)$ on both sides of the result of Lemma 2.1 yields Theorem 2.3.

Theorem 2.3. *Let* $0 < p < 1$ *and* $t > 0$ *be given. Then*
(i) $EN^P(t,p) = M_H(t) = \sum_{k=1}^{\infty} H^{(k)}(t)$,
and
(ii) $EN^M(t,p) = (1-p)\{EE[\sum_{i=1}^{N^P(t,p)} R(W_i)|N^P(t,p)] + EE[R(t - T_{N^P(t,p)})|N^P(t,p)]\}$.

In the followings, we investigate the distribution of $\sum_{i=1}^{N^P(t,p)} R(W_i)$ and $R(t - T_{N^P(t,p)})$. The results are summarized in the following theorem.

Theorem 2.4. *Let* $\{X_n\}_{n=1}^{\infty}$ *be a sequence of i.i.d. random variables from* $H(\cdot)$. *Let* $N_H(t)$ *be the number of renewals in [0,t) with interoccurrence distribution* $H(\cdot)$, *where* $H(t) = 1 - \bar{F}^p(t)$ *for* $0 < p < 1$. *Let* $Y_i = -\ln \bar{F}(X_i)$, *i=1,2,... and let* $S_n = \sum_{i=1}^{n} Y_i$. *Then*
(i) $\{Y_n\}_{n=1}^{\infty}$ *is a sequence of i.i.d. random variables from* $G(t) = 1 - e^{-pt}$ *for* $t \geq 0$.
(ii) If F is NBU, then

$$P(S_{N_H(t)} \geq x) = \sum_{n=0}^{\infty} [G_n(t^*) - G_n(x)] P[N_H(t) = n],$$

where $t^* = -\ln \bar{F}(t)$ *and* $G_n(u) = \int_0^u \frac{p^n}{\Gamma(n)} y^{n-1} e^{-py}\, dy$.
(iii) $ES_{N_H(t)} = \sum_{n=0}^{\infty} [G_n(t^*) - \int_0^{t^*} G_n(x)\, dx] P(N_H(t) = n)$.

Proof: (i) It is obvious that $Y_i's$ are i.i.d. random variables. And the distribution of Y_i is easily obtained by using the technique of the probability integral transformation.
(ii) It is clear that $\sum_{i=1}^{N_H(t)} X_i \leq t$. Since F is NBU and $\bar{F}$ is decreasing,

$$\prod_{i=1}^{N_H(t)} \bar{F}(X_i) \geq \bar{F}(\sum_{i=1}^{N_H(t)} X_i) \geq \bar{F}(t)$$

and

$$S_{N_H(t)} = \sum_{i=1}^{N_H(t)} -\ln \bar{F}(X_i) \le -\ln \bar{F}(t) = t^*.$$

Hence

$$\begin{aligned} P(S_{N_H(t)} \ge x) &= P(x \le S_{N_H(t)} \le t^*) \\ &= \sum_{n=0}^{\infty} P(x \le S_n \le t^* | N_H(t) = n) P(N_H(t) = n) \\ &= \sum_{n=0}^{\infty} [G_n(t^*) - G_n(x)] P(N_H(t) = n). \end{aligned}$$

(iii) As a consequence of (ii), it follows that

$$\begin{aligned} ES_{N_H(t)} &= \int_0^{t^*} P(S_{N_H(t)} \ge x)\, dx \\ &= \sum_{n=0}^{\infty} [G_n(t^*) - \int_0^{t^*} G_n(x)\, dx] P(N_H(t) = n). \end{aligned}$$

□

Next we discuss the distribution of $R(t - T_{N^P(t,p)})$, where $T_{N^P(t,p)}$ is stochastically equal to the sum of $N^P(t,p)$ i.i.d. random variables, each having a distribution $H(\cdot)$. Let $\delta_t = t - T_{N^P(t,p)}$. Then δ_t is known to be an age random variable (shortage random variable or current life random variable) and the distribution of δ_t is given by

$$\begin{aligned} P(\delta_t \le x) &= P(t - T_{N^P(t,p)} \le x) \\ &= P(t - x \le T_{N^P(t,p)}) \\ &= \begin{cases} \int_{t-x}^{t} \bar{H}(t-u)\, dM_H(u) & \text{if } 0 < x < t \\ 1 & \text{otherwise,} \end{cases} \end{aligned}$$

where $\bar{H}(t) = \bar{F}^p(x)$ and $M_H(u) = \sum_{k=1}^{\infty} H^{(k)}(u)$.
Hence, we obtain

$$\begin{aligned} P(R(t - T_{N^P(t,p)}) \le x) &= P(-\ln \bar{F}(\delta_t) \le x) \\ &= P(\delta_t \le \bar{F}^{-1}(e^{-x})) \\ &= \begin{cases} \int_{t-\bar{F}^{-1}(e^{-x})}^{t} \bar{H}(t-u)\, dM_H(u) & \text{if } x < t^* \\ 1 & \text{if } x \ge t^*. \end{cases} \end{aligned}$$

3. COMPUTATION OF EXPECTED NUMBER OF REPAIRS

In this section, we apply our results for the cases when the underlying life distributions of a unit are exponential and Weibull.

3.1. Exponential distribution

Although an exponential distribution is a special case of Weibull distribution, we consider it separately from the general Weibull distribution because of its analytical tractability. The survival function is given by $\bar{F}(t) = e^{-\lambda t},\ t \geq 0$ and $\lambda > 0$ and hazard function is obtained as $R(t) = \lambda t$. It is well known (Brown and Proschan, 1983) that $H(t) = 1 - \bar{F}^p(t) = 1 - e^{-p\lambda t}$, which is the exponential distribution. Thus, $EN^P(t,p) = p\lambda t$. The expected number of minimal repairs under the imperfect repair model can be obtained by using Theorem 2.3 as follows:

$$\begin{aligned} EN^M(t,p) &= (1-p)\{EE[\sum_{i=1}^{N^P(t,p)} R(W_i)|N^P(t,p)] + EE[R(t-T_{N^P(t,p)})|N^P(t,p)]\} \\ &= (1-p)\{EE[\sum_{i=1}^{N^P(t,p)} \lambda W_i|N^P(t,p)] + EE[\lambda(t-T_{N^P(t,p)})|N^P(t,p)]\} \\ &= (1-p)EE[\lambda(T_1 - T_0 + T_2 - T_1 + \cdots + t - T_{N^P(t,p)})|N^P(t,p)] \\ &= (1-p)EE[\lambda t|N^P(t,p)] \\ &= (1-p)\lambda t. \end{aligned}$$

3.2. Weibull distribution with a shape parameter α

The survival function of Weibull distribution with a scale parameter 1 and a shape parameter α is given by $\bar{F}(t) = exp[-t^\alpha],\ t \geq 0$ and $\alpha > 0$ and its failure rate and hazard function are obtained as $r(t) = \alpha t^{\alpha-1}$ and $R(t) = t^\alpha$, respectively. The expected numbers of perfect repairs and minimal repairs under the imperfect repair model can be obtained by using the results of Theorem 2.3.

$$EN^P(t,p) = \sum_{k=1}^{\infty} H^{(k)}(t)$$

and

$$EN^M(t,p) = (1-p)\{EE[\sum_{i=1}^{N^P(t,p)} W_i^\alpha|N^P(t,p)] + EE[(t-T_{N^P(t,p)})^\alpha|N^P(t,p)]\},$$

where W_i's are i.i.d. from $H(\cdot)$, $T_{N^P(t,p)} = \sum_{i=1}^{N^P(t,p)} W_i$ and $H(t) = 1 - exp(-pt^\alpha)$.

Evaluation of these expectations is not feasible analytically. In Section 4, we, however, perform a simulation study for numerical evaluation of these expectations under several situations.

4. SIMULATION RESULTS

In this section, we consider three distributions for conducting a simulation study. They are exponential distribution, Weibull distribution with $\alpha < 1$ and Weibull distribution with $\alpha > 1$. Operating time is specified by $t = 10$ and the number of iterations for each simulation is 10,000. Firstly, we consider an exponential distribution with $\lambda = 1$. Because the true value of each expectation is known, simulated values can be compared with true values. Secondly, we consider Weibull distributions with $\alpha = .5$ and 2. When $\alpha = .5$, the failure rate of the distribution is decreasing and when $\alpha = 2$, the distribution has an increasing failure rate. Note that $EN^P(t,1)$ does not depend on the value of p and $EN^P(t,p) + EN^M(t,p)$ denotes the expected number of failures under the imperfect repair model.

Table 4.1 shows the results of simulation when the underlying distribution is exponential with $\lambda = 1$. The values in the parentheses in Table 4.1 denote the true values and it is shown from tables that the simulated values are very close to the true values except when p is small. The values on the second and the third columns of each table are obtained on the basis of the formulas in Theorem 2.3. The simulated values for $EN^M(t,p)$ in Table 4.1 are exactly the same as the true values since $\sum_{i=1}^{N^P(t,p)} R(W_i) = \sum_{i=1}^{N^P(t,p)} W_i$ when F is exponential distribution. Table 4.1 also shows that the expected number of failures stays close to 10, which is of course expected for the exponential distribution. Tables 4.2 and 4.3 show the simulated values of expected number of repairs when F is Weibull distribution with α =.5 and 2. In both tables, as p increases the expected number of perfect repairs tends to increase while the expected number of minimal repairs decreases. It is of great interest to note that for Weibull distribution with α=.5 which shows the decreasing failure rate, the total number of expected failures increases as the value of p increases. This is expected since for DFR life distribution, the aging benefits the unit. For Weibull distribution with α= 2 which is IFR, the aging affects the unit adversely and thus the total number of expected failures decreases as p increases, which agrees with the results shown in Table 4.3.

Table 4.1: Simulated Values of $N^P(t,1)$, $N^P(t,p)$ and $N^M(t,p)$ when $\bar{F}(x) = e^{-\lambda x}$ with $\lambda = 1$ and t=10.

p	$EN^P(t,1)$	$EN^P(t,p)$	$EN^M(t,p)$	$EN^P(t,p) + EN^M(t,p)$
.1	10.036(10)	1.518(1.0)	9.0(9.0)	10.518
.2		2.296(2.0)	8.0(8.0)	10.296
.3		3.166(3.0)	7.0(7.0)	10.166
.4		4.078(4.0)	6.0(6.0)	10.078
.5		5.049(5.0)	5.0(5.0)	10.049
.6		6.032(6.0)	4.0(4.0)	10.032
.7		7.004(7.0)	3.0(3.0)	10.004
.8		7.913(8.0)	2.0(2.0)	9.913
.9		8.984(9.0)	1.0(1.0)	9.984

Table 4.2: Simulated Values of $N^P(t,1)$, $N^P(t,p)$ and $N^M(t,p)$ when $\bar{F}(x) = e^{-x^\alpha}$ with $\alpha = .5$ and t=10.

p	$EN^P(t,1)$	$EN^P(t,p)$	$EN^M(t,p)$	$EN^P(t,p) + EN^M(t,p)$
.1	6.479	.358	3.119	3.477
.2		.768	3.021	3.789
.3		1.242	2.869	4.111
.4		1.771	2.656	4.427
.5		2.372	2.378	4.750
.6		3.060	2.043	5.103
.7		3.822	1.639	5.461
.8		4.677	1.170	5.847
.9		5.614	.624	6.238

Table 4.3: Simulated Values of $N^P(t,1)$, $N^P(t,p)$ and $N^M(t,p)$ when $\bar{F}(x) = e^{-x^\alpha}$ with $\alpha = 2.0$ and t=10.

p	$EN^P(t,1)$	$EN^P(t,p)$	$EN^M(t,p)$	$EN^P(t,p) + EN^M(t,p)$
.1	10.941	4.050	21.212	25.262
.2		5.063	16.538	21.601
.3		5.974	12.879	18.853
.4		6.845	9.919	16.764
.5		7.651	7.522	15.173
.6		8.391	5.565	13.956
.7		9.108	3.882	12.990
.8		9.734	2.432	12.166
.9		10.373	1.146	11.519

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