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Robust Simple Correspondence Analysis†

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ABSTRACT

Simple correspondence analysis is a technique for giving a joint display of points representing both the rows and columns of an $n \times p$ two-way contigency table. In simple correspondence analysis, the singular value decomposition is the main algebraic tool. But, Choi and Huh (1996) pointed out the singular value decomposition is not robust. Instead, they developed a robust singular value decomposition and provided applications in principal component analysis and biplots.

In this article, by using the analogous procedures of Choi and Huh (1996), we derive a robust version of simple correspondence analysis.

Keywords: Correspondence analysis; Eigen system; Principal component analysis; Robust version; Singular value decomposition.

1. INTRODUCTION

Simple correspondence analysis is a technique for giving a joint map of points representing both the rows and columns of an $n \times p$ two-way contingency table. For a Burt table which comprises all two-way contingency tables, Greenacre (1984, Chapter 5) and Lebart et al. (1984, Chapter 4) describe the so-called multiple correspondence analysis which is an extension of simple correspondence analysis.

It is well known that the simple correspondence analysis of two-way contingency tables is similar in spirit to principal component analysis (Mardia et al., 1979, pp. 237-238; Greenacre, 1984, p. 346; Lebart et al., 1984, Chapter 2; Jolliffe, 1986, pp. 85-86). Moreover, for the simultaneous display of rows and columns of an $n \times p$ contingency table in simple correspondence analysis, the singular value decomposition is adopted as the main algebraic tool. But Choi

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and Huh (1996) pointed out that the singular value decomposition is not robust. Thus, simple correspondence analysis based on the singular value decomposition does not always give desirable results.

In Section 2, we derive a robust simple correspondence analysis based on the robust eigensystem. In Section 3, we provide a robust singular value decomposition for a simple correspondence analysis. In Section 4, a numerical illustration is given. Finally, the details of the minimization problem of Section 2 are given in the Appendix.

2. ROBUST SIMPLE CORRESPONDENCE ANALYSIS

Consider the $n \times p$ data matrix $\mathbf{O} = (o_{ij}), o_{ij} \geq 0,\ i = 1, ..., n; j = 1, ..., p$, which is in the form of two-way contingency table of counts. Each row of $\mathbf{O}$ may be viewed as the realization of a multinomial distribution, conditional on the respective row sum. We shall denote the overall total simply by $o_{++} = \mathbf{1}_n'\mathbf{O}\mathbf{1}_p$ where $\mathbf{1}_n$ and $\mathbf{1}_p$ are $n \times 1$, $p \times 1$ vectors with n and p ones respectively. Note that the correspondence matrix $\mathbf{F}$ is define as

$$\mathbf{F} = (f_{ij}),\ f_{ij} = o_{ij}/o_{++}, \qquad i = 1, ..., n; \qquad j = 1, ..., p.$$

Let $\mathbf{r}$ and $\mathbf{c}$ be row and column sums of $\mathbf{F}$ respectively;

$$\mathbf{r} = \mathbf{F}\mathbf{1}_p = (f_{1+}, ..., f_{n+})', \quad \mathbf{c} = \mathbf{F}'\mathbf{1}_n = (f_{+1}, ..., f_{+p})',$$

where $f_{i+} = \sum_{j=1}^{p} f_{ij}$ $(i = 1, ..., n)$ and $f_{+j} = \sum_{i=1}^{n} f_{ij}$ $(j = 1, ..., p)$. We shall denote the respective $n \times n$ and $p \times p$ diagonal matrices with elements f_{i+} and f_{+j} by

$$\mathbf{D}_r = diag(\mathbf{r}), \quad \mathbf{D}_c = diag(\mathbf{c}).$$

Let $\mathbf{A}$ be the $n \times p$ row profiles matrix defined as

$$\mathbf{A} = \mathbf{D}_r^{-1}\mathbf{F} = (\mathbf{a}_1, ..., \mathbf{a}_n)', \tag{2.1}$$

where $\mathbf{a}_i = (f_{i1}/f_{i+}, ..., f_{ip}/f_{i+})'$ is the i^{th} row profile vector.

Consider the p-dimensional space defined by metric $\mathbf{D}_c^{-1}$. We call it the weighted Euclidean space $\mathcal{E}^p$. In the row profile matrix (2.1), each of the row profiles $\mathbf{a}_1, ..., \mathbf{a}_n$ can be represented by points in $\mathcal{E}^p$. The squared distance between two profile points $\mathbf{a}_i$ and $\mathbf{a}_{i'}$ is given by

$$\begin{aligned} d_p^2(\mathbf{a}_i, \mathbf{a}_{i'}) &= (\mathbf{a}_i - \mathbf{a}_{i'})'\mathbf{D}_c^{-1}(\mathbf{a}_i - \mathbf{a}_{i'}), \\ &= \sum_j (f_{ij}/f_{i+} - f_{i'j}/f_{i'+})^2/f_{+j}. \end{aligned}$$

For the same profiles $\mathbf{a}_i = \mathbf{a}_{i'}$, $d_p^2(\mathbf{a}_i, \mathbf{a}_{i'}) = 0$. The above squared distance is called the chi-square distance or chi-squared metric. The choice of this distance guarantees a certain stability of the results no matter how the variables were originally coded (Lebart et al., 1984, p. 35).

So far we have defined the set of profile points with masses, which will be defined later, in a space structured by the chi-square distance. Now we want to find the optimal s-dimensional subspace which is the closest to all the points.

In order to find the optimal subspace, we use an algorithm analogous to that of Choi and Huh(1996). We write $\mathcal{S} = span(\mathbf{v}_1, ..., \mathbf{v}_s)$, $(1 \leq s \leq p)$, where $\mathcal{S}$ is the s-dimensional subspace of $\mathcal{E}^p$ which is spanned by the $p \times s$ basis vector $\mathbf{V} = (\mathbf{v}_1, ..., \mathbf{v}_s)$ such that $\mathbf{v}_j'\mathbf{D}_c^{-1}\mathbf{v}_k = 0\,(j \neq k)$ and $\mathbf{v}_k'\mathbf{D}_c^{-1}\mathbf{v}_k = 1$.

Let $\widehat{\mathbf{a}}_i$ in $\mathcal{S}$ be the nearest point to $\mathbf{a}_i$ in $\mathcal{E}^p$. Hence we have the squared distance of $\mathbf{a}_i$ from $\widehat{\mathbf{a}}_i$ with metric $\mathbf{D}_c^{-1}$ in $\mathcal{S}$ is given by

$$d_i^2 = \|\mathbf{a}_i - \widehat{\mathbf{a}}_i\|^2_{\mathbf{D}_c^{-1}} = \mathbf{a}_i'\mathbf{D}_c^{-1}(\mathbf{D}_c - \sum_{k=1}^{s} \mathbf{v}_k\mathbf{v}_k')\mathbf{D}_c^{-1}\mathbf{a}_i.$$

Note that we set $d_i = \|\mathbf{a}_i - \widehat{\mathbf{a}}_i\|_{\mathbf{D}_c^{-1}}$ and consider the minimization of

$$D_s = \sum_i f_{i+}\,\rho(d_i) = \sum_i f_{i+}\,\rho(\|\mathbf{a}_i - \widehat{\mathbf{a}}_i\|_{\mathbf{D}_c^{-1}}), \tag{2.2}$$

with the constraint $\mathbf{v}_j'\mathbf{D}_c^{-1}\mathbf{v}_k = 0\,(j \neq k)$ and $\mathbf{v}_k'\mathbf{D}_c^{-1}\mathbf{v}_k = 1$.

Here

$$\rho(t) = \begin{cases} c^2[1 - \cos(t/c)], & \text{for } |\, t \,| \leq c\pi, \\ 2c^2, & \text{for } |\, t \,| > c\pi. \end{cases}$$

Of course, $\rho(\cdot)$ is Andrews' type and also we can use other types of $\rho(\cdot)$ given in Li (1985, p. 293). We hope that the minimization of (2.2) yields a robust version of the eigensystem and ultimately, coordinates of the row profiles for a robust simple correspondence analysis.

For our convenience, setting $\mathbf{n}_k = \mathbf{D}_c{}^{-1/2}\mathbf{v}_k$ in the constraint of (2.2), we consider the minimization of (2.2) subject to $\mathbf{n}_j'\mathbf{n}_k = 0$ $(j \neq k)$ and $\mathbf{n}_k'\mathbf{n}_k = 1$. So by using the procedure analogous to the Lagrangian method in Choi and Huh (1996, Appendix), we will obtain the $\mathbf{n}_k, k = 1, ..., s$. See the Appendix for the details.

Consequently, we have an eigensystem

$$(\mathbf{D}_r{}^{1/2}\mathbf{A}\mathbf{D}_c{}^{-1/2})'\mathbf{D}_\mathbf{w}(\mathbf{D}_r{}^{1/2}\mathbf{A}\mathbf{D}_c{}^{-1/2})\mathbf{n}_k = \lambda_{kk}\mathbf{n}_k,\ k = 1, ..., s. \tag{2.3}$$

Here the $\mathbf{n}_k, k = 1, ..., s$, are the eigenvectors corresponding to eigenvalues $\lambda_{kk} = \lambda_k^2$, $k = 1, ..., s$, of $(\mathbf{D}_r{}^{1/2}\mathbf{A}\mathbf{D}_c{}^{-1/2})'\mathbf{D}_\mathrm{w}(\mathbf{D}_r{}^{1/2}\mathbf{A}\mathbf{D}_c{}^{-1/2})$ where $\mathbf{D}_\mathrm{w} = diag(w_1, ..., w_n)$ with

$$w_i = \psi(\|\mathbf{a}_i - \widehat{\mathbf{a}}_i\|_{\mathbf{D}_c^{-1}})/\|\mathbf{a}_i - \widehat{\mathbf{a}}_i\|_{\mathbf{D}_c^{-1}},\ i = 1, ..., n, \tag{2.4}$$

and $\psi(\cdot)$ is the derivative of $\rho(\cdot)$. In general, in correspondence analysis, the eigenvalues are called inertias. We call the (2.3) a robust eigensystem.

From the the eigenvectors $\mathbf{n}_k = \mathbf{D}_c{}^{-1/2}\mathbf{v}_k$, $k = 1, ..., s$, satisfying the robust eigensystem (2.3), we obtain $\mathbf{v}_k = \mathbf{D}_c{}^{1/2}\mathbf{n}_k$, $k = 1, ..., s$. Moreover, the new $\mathbf{v}_k$ can define the k^{th} robust principal axis of the s-dimensional subspace of $\mathcal{E}^p$. The algorithm for a robust version of simple correspondence analysis can be described as follows.

STEP 1: *Take as a tentative vector* $\mathbf{v}_k$ *the* k^{th} *eigenvector from the eigensystem of*

$$(\mathbf{D}_r{}^{1/2}\mathbf{A}\ \mathbf{D}_c{}^{-1/2})'(\mathbf{D}_r{}^{1/2}\mathbf{A}\mathbf{D}_c{}^{-1/2}).$$

STEP 2: *Calculate* $\mathbf{D}_\mathrm{w} = diag(w_1, ..., w_n)$ *with* w_i *specified by* (2.4)

STEP 3: *Determine the eigenvalues and eigenvectors from the robust eigensystem of*

$$(\mathbf{D}_r{}^{1/2}\mathbf{A}\mathbf{D}_c{}^{-1/2})'\mathbf{D}_\mathrm{w}(\mathbf{D}_r{}^{1/2}\mathbf{A}\mathbf{D}_c{}^{-1/2})$$

using the w_i, $i = 1, ..., n$, *from Step 2. We call these robust eigenvalues and eigenvectors.*

STEP 4: *Repeat Steps 2 to 3 until on each successive procedure, the absolute difference between the tentative and updated eigenvector becomes not greater than some sufficiently small* ε.

In practice, in deriving (2.3), the scale parameter σ(a measure of spread) must be taken into consideration. In each iterative step of the algorithm for robust simple correspondence analysis, consider the median scale estimator $\widehat{\sigma}$ given by

$$\widehat{\sigma} = (med_i(\|\mathbf{a}_i - \widehat{\mathbf{a}}_i\|_{\mathbf{D}_c^{-1}})/0.6745). \tag{2.5}$$

Then we obtain the robust eigensystem with $\mathbf{D}_\mathrm{w} = diag(w_1, ..., w_n)$ where w_i, $i = 1, ..., n$, are specified by (2.4).

And the $n \times 1$ robust coordinates vector $\mathbf{x}_k$ is given by

$$\mathbf{x}_k = \mathbf{A}\mathbf{D}_c{}^{-1/2}\mathbf{n}_k = \mathbf{D}_r^{-1}\mathbf{F}\mathbf{D}_c^{-1}\mathbf{v}_k.$$

Therefore taking the first s columns $\mathbf{x}_1, ..., \mathbf{x}_s$ of coordinate matrix $\mathbf{X}$ and denoting $\mathbf{X}_{(s)}$ for this, $\mathbf{X}_{(s)}$ defines the projection of the row profiles onto the optimal s-dimensional subspace.

Since the $\mathbf{D_w}$ norm of $(\mathbf{D}_r{}^{1/2}\mathbf{A}\mathbf{D}_c{}^{-1/2})\mathbf{n}_k$ is equal to λ_k, i.e.,

$$\|(\mathbf{D}_r{}^{1/2}\mathbf{A}\mathbf{D}_c{}^{-1/2})\mathbf{n}_k\|^2_{\mathbf{D_w}} = \mathbf{n}_k'(\mathbf{D}_r{}^{1/2}\mathbf{A}\mathbf{D}_c{}^{-1/2})'\mathbf{D_w}(\mathbf{D}_r{}^{1/2}\mathbf{A}\mathbf{D}_c{}^{-1/2})\mathbf{n}_k = \lambda_k^2,$$

the normalized coordinate vector $\mathbf{m}_k$ such that $\mathbf{m}_j'\mathbf{D_w}\mathbf{m}_k = 0$ $(j \neq k)$ and $\|\mathbf{m}_k\|_{\mathbf{D_w}} = 1$ $j, k = 1, ..., s$, is given by

$$\begin{aligned} \mathbf{m}_k &= (\mathbf{D}_r{}^{1/2}\mathbf{A}\mathbf{D}_c{}^{-1/2})\mathbf{n}_k / \|(\mathbf{D}_r{}^{1/2}\mathbf{A}\mathbf{D}_c{}^{-1/2})\mathbf{n}_k\|_{\mathbf{D_w}}, \\ &= (1/\lambda_k)\mathbf{D}_r{}^{-1/2}\mathbf{F}\mathbf{D}_c{}^{-1/2}\mathbf{n}_k. \end{aligned} \tag{2.6}$$

Therefore we obtain an optimal s-dimensional map of a robust simple correspondence analysis by using the robust coordinates of rows and columns.

3. ROBUST SINGULAR VALUE DECOMPOSITION FOR A ROBUST CORRESPONDENCE ANALYSIS

From (2.5), if p is the rank of $\mathbf{D}_r{}^{-1/2}\mathbf{F}\mathbf{D}_c{}^{-1/2}$, we have the relationship between the unitary vectors $\mathbf{m}_k$ and $\mathbf{n}_k$ is given by

$$\mathbf{D}_r{}^{-1/2}\mathbf{F}\mathbf{D}_c{}^{-1/2}\mathbf{n}_k = \lambda_k\mathbf{m}_k, \; k = 1, ..., p.$$

Postmultiplying both sides by $\mathbf{n}_k'$ and summing over k, we have

$$\mathbf{D}_r{}^{-1/2}\mathbf{F}\mathbf{D}_c{}^{-1/2}\sum_{k=1}^{p}\mathbf{n}_k\mathbf{n}_k' = \sum_{k=1}^{p}\lambda_k\mathbf{m}_k\mathbf{n}_k',$$

which leads to

$$\mathbf{D}_r{}^{-1/2}\mathbf{F}\mathbf{D}_c{}^{-1/2} = \sum_{k=1}^{p}\lambda_k\mathbf{m}_k\mathbf{n}_k'. = \mathbf{M}\mathbf{D}_\lambda\mathbf{N}', \tag{3.1}$$

where $\mathbf{M}$ is the $n \times p$ matrix whose columns are the $n \times 1$ left singular vectors $\mathbf{m}_k$ of $(\mathbf{D}_r{}^{-1/2}\mathbf{F}\mathbf{D}_c{}^{-1/2})'\mathbf{D_w}(\mathbf{D}_r{}^{-1/2}\mathbf{F}\mathbf{D}_c{}^{-1/2})$ such that $\mathbf{M}'\mathbf{D_w}\mathbf{M} = \mathbf{I}_p$, $\mathbf{N}$ is a $p \times p$ matrix whose columns are the $p \times 1$ right singular vectors $\mathbf{n}_k$ of

$(\mathbf{D}_r{}^{-1/2}\mathbf{F}\mathbf{D}_c{}^{-1/2})'\mathbf{D}_w(\mathbf{D}_r{}^{-1/2}\mathbf{F}\ \mathbf{D}_c{}^{-1/2})$ such that $\mathbf{N}'\mathbf{N} = \mathbf{N}\mathbf{N}' = \mathbf{I}_p$ and $\mathbf{D}_\lambda$ is a $p \times p$ diagonal matrix where k^{th} positive diagonal element λ_k called robust singular value of $\mathbf{D}_r{}^{-1/2}\mathbf{F}\mathbf{D}_c{}^{-1/2}$.

Let us premultiply and postmultiply the singular value decomposition (3.1) by $\mathbf{D}_r{}^{1/2}$ and by $\mathbf{D}_c{}^{1/2}$ respectively. So the generalized robust singular value decomposition is given by

$$\mathbf{F} = \mathbf{D}_r{}^{1/2}\mathbf{M}\mathbf{D}_\lambda\mathbf{N}'\mathbf{D}_c{}^{1/2} = \mathbf{U}\mathbf{D}_\lambda\mathbf{V}'.$$

where $\mathbf{U} = \mathbf{D}_r{}^{1/2}\mathbf{M}$ and $\mathbf{V} = \mathbf{D}_c{}^{1/2}\mathbf{N}$ have columns which are orthogonalized with respect to metric $\mathbf{D}_r^{-1}$ and $\mathbf{D}_c^{-1}$ such that $\mathbf{U}'\mathbf{D}_w\mathbf{D}_r^{-1}\mathbf{U} = \mathbf{I}_p$, $\mathbf{V}'\mathbf{D}_c^{-1}\mathbf{V} = \mathbf{V}\mathbf{D}_c^{-1}\mathbf{V}' = \mathbf{I}_p$, respectively and $\mathbf{D}_\lambda = diag(\lambda_1, ..., \lambda_p)$ in which λ_k^2 are the eigenvalues of $(\mathbf{D}_r{}^{-1/2}\mathbf{F}\mathbf{D}_c{}^{-1/2})'\mathbf{D}_w\ (\mathbf{D}_r{}^{-1/2}\mathbf{F}\mathbf{D}_c{}^{-1/2})$. So we know that the principal axes as well as the coordinates of the row profiles with respect to these axes can be easily obtained from a robust generalized singular value decomposition of $\mathbf{F}$ and ultimately from a robust singular value decomposition of $\mathbf{D}_r{}^{-1/2}\mathbf{F}\mathbf{D}_c{}^{-1/2}$.

Finally, we need a measure for goodness of a robust simple correspondence analysis. For this, only using $\mathbf{D}_r{}^{-1/2}\mathbf{F}\mathbf{D}_c{}^{-1/2}$ and $(\mathbf{D}_r{}^{-1/2}\mathbf{F}\mathbf{D}_c{}^{-1/2})_{(s)}$ instead of $\widetilde{\mathbf{X}}^*$ and $\widetilde{\mathbf{X}}^*_{(s)}$ in Choi and Huh(1996, Appendix), a goodness of robust approximation in robust simple correspondence analysis can be given by

$$\rho_s = 1 - \parallel \mathbf{P} - \mathbf{P}_{(s)} \parallel^2_{\mathbf{D}_w} / \parallel \mathbf{P} \parallel^2_{\mathbf{D}_w} = \sum_{k=1}^{s} \lambda_k^2 / \sum_{k=1}^{p} \lambda_k^2.$$

where p is the rank of $\mathbf{P} = \mathbf{D}_r{}^{-1/2}\mathbf{F}\mathbf{D}_c{}^{-1/2}$.

4. NUMERICAL ILLUSTRATIONS

The data are from Greenacre (1993, p. 75) and consist of ten rows(scientific discipline) and five columns(funding category). The columns are labeled in order from highest to lowest categories of funding. They are A(most supported),..., D(least supported) and E(no funding yet). The 796 scientific researchers are cross-classified according to row and column markers of data.

In simple correspondence analysis, Fig.4.1 shows an optimal two-dimensional map and the principal inertias and their percentages of total inertia are 0.039 (47.20%), 0.030 (36.66%), 0.011 (13.11%) and 0.003 (3.03%). Thus the first two principal axes of Fig. 4.1 account for almost 83.86% . The interpretation of this map is detailed in Greenacre (1993, pp. 74-93). In particular, Greenacre pointed out that among the rows the points **Ge**ology, **Bi**ochemistry and **En**gineering have been the most influential in finding the principal plane formed from the first two princiapl axes. And the row **Ma**thematics is poorly displayed, with over two-thirds of its inertia lying off the plane. Having observed this result, we should be more careful when interpreting the position of **Ma**thematics in the map. In terms of its position it looks quite similar to the profile of **St**atistics, but this projected position is not an accurate reflection of its true position.

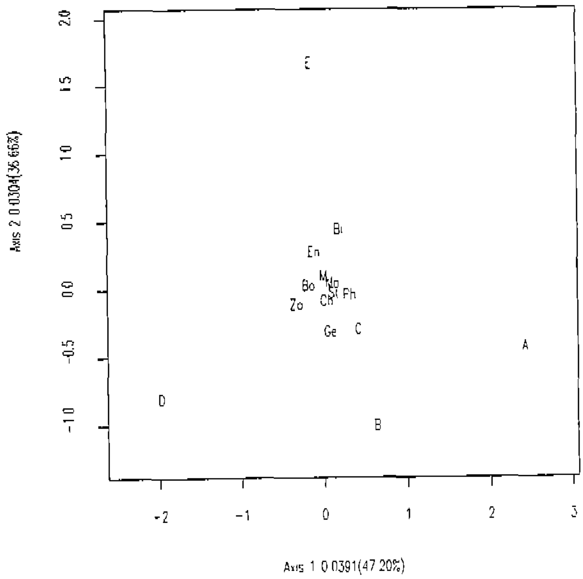

Figure 4.1: Optimal 2-dimensional map by simple correspondence analysis

So to resolve this problem, now consider the robust simple correspondence analysis with $\psi(\cdot)$ function which is the derivative of Andrews' $\rho(\cdot)$ described in Section 2. For this, we use $c = 1/\pi$ and 0.194 as the median scale estimator of (2.5) respectively. The final weights and in computing robust eigensystem (2.3) are in the diagonal matrix,

$$\mathbf{D_W} = \text{diag}(0.000,\ 0.000,\ 0.966,\ 0.944,\ 1.000,\ 0.000,\ 0.421,\ 0.421,\ 0.981,\ 0.131).$$

In $\mathbf{D_W}$, we note the first, second and sixth diagonal elements have 0.000, 0.000 and 0.000 respectively. As pointed out by Greenacre previously, they are **Ge**ology, **Bi**ochemistry and **En**gineering respectively.

Now by reducing their influence, we give the robust two-dimensional map Fig. 4.2 with the goodness of a robust approxiamtion 99.15%. Moreover, to match Fig. 4.1, the new rotated two axes in Fig. 4.2 are determined from the counterclockwise rotation with an angle 90^o. From Fig. 4.2, we note that **St**atistics and **Ma**thematics have the same direction as Axis 1 and Axis 2. Also **Ma**thematics does have less type A researchers than **St**atistics and more type Cs than is apparent from the Fig. 4.1. So this robust version improved the two problems which were pointed out by Greenacre.

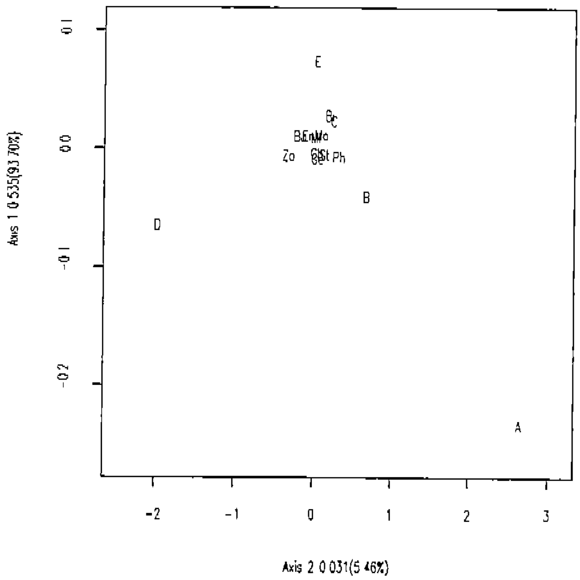

Figure 4.2: Optimal 2-dimensional map by robust correspondence analysis

5. CONCLUDING REMARKS

Here we have developed a robust simple correspondence analysis and a robust singular value decomposition for this. In the numerical illustration described in Section 4, our method produces a more reliable map by reducing the influence of several peculiar rows. Thus we could resolve the problems pointed out by Greenacre (1993, pp. 74-93). Of course, because there exit duality between row analysis and column analysis, we can also discuss the influence of columns. Finally, future valuable research will be provided by the application of the robust approach discussed in simple correspondence analysis to multiple correspondence analysis. This is possible because multiple correspondence is an extension of simple correspondence analysis.

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APPENDIX: Details of the minimization problem in Section 2.

Consider the minimization of $D_s = \sum_{i=1}^{n} f_{i+}\rho(d_i) = \sum_{i=1}^{n} f_{i+}\rho(\|\mathbf{a}_i - \widehat{\mathbf{a}}_i\|_{\mathbf{D}_c^{-1}})$, with the constraint $\mathbf{v}_j'\mathbf{D}_c^{-1}\mathbf{v}_k = 0\,(j \neq k)$ and $\mathbf{v}_k'\mathbf{D}_c^{-1}\mathbf{v}_k = 1$.

Since

$$\partial\|\mathbf{a}_i - \widehat{\mathbf{a}}_i\|^2_{\mathbf{D}_c^{-1}}/\partial\mathbf{v}_k \;=\; -2\mathbf{D}_c^{-1}\mathbf{a}_i(\mathbf{a}_i'\mathbf{D}_c^{-1}\mathbf{v}_k),\ k = 1, ..., s,$$

we have

$$\begin{aligned}\partial\|\mathbf{a}_i - \widehat{\mathbf{a}}_i\|_{\mathbf{D}_c^{-1}}/\partial\mathbf{v}_k &= (\partial\|\mathbf{a}_i - \widehat{\mathbf{a}}_i\|^2_{\mathbf{D}_c^{-1}}/\partial\mathbf{v}_k)/(2\|\mathbf{a}_i - \widehat{\mathbf{a}}_i\|_{\mathbf{D}_c^{-1}}),\\ &= -\mathbf{D}_c^{-1}\mathbf{a}_i(\mathbf{a}_i'\mathbf{D}_c^{-1}\mathbf{v}_k)/\|\mathbf{a}_i - \widehat{\mathbf{a}}_i\|_{\mathbf{D}_c^{-1}}.\end{aligned}$$

We set $w_i = \psi(\|\mathbf{a}_i - \widehat{\mathbf{a}}_i\|_{\mathbf{D}_c^{-1}})/\|\mathbf{a}_i - \widehat{\mathbf{a}}_i\|_{\mathbf{D}_c^{-1}}$, $i = 1, ..., n$, in which $\psi(\cdot)$ is the derivative of $\rho(\cdot)$ with respect to $\mathbf{v}_k$. So by using the analogous procedure to the Lagrangian method of Choi and Huh (1996, Appendix), we have

$$\sum_i w_i f_{i+}\mathbf{D}_c^{-1}\mathbf{a}_i(\mathbf{a}_i'\mathbf{D}_c^{-1}\mathbf{v}_k) = \lambda_{kk}\mathbf{D}_c^{-1}\mathbf{v}_k,$$

which leads to

$$\mathbf{D}_c^{-1}(\mathbf{A}'\mathbf{D}_\mathbf{w}\mathbf{D}_r\mathbf{A})\mathbf{D}_c^{-1}\mathbf{v}_k = \lambda_{kk}\mathbf{D}_c^{-1}\mathbf{v}_k,\ k = 1, ..., s,$$

where $\mathbf{D}_\mathbf{w} = diag(w_1, ..., w_n)$. Premultiplying this equation by $\mathbf{D}_c{}^{1/2}$, we have

$$(\mathbf{D}_c{}^{-1/2}\mathbf{A}'\mathbf{D}_\mathbf{w}\mathbf{D}_r\mathbf{A}\mathbf{D}_c{}^{-1/2})\mathbf{D}_c{}^{-1/2}\mathbf{v}_k = \lambda_{kk}\mathbf{D}_c{}^{-1/2}\mathbf{v}_k,\ k = 1, ..., s.$$

Setting $\mathbf{n}_k = \mathbf{D}_c{}^{-1/2}\mathbf{v}_k$, we have the resistant eigensystem as

$$(\mathbf{D}_r{}^{1/2}\mathbf{A}\mathbf{D}_c{}^{-1/2})'\mathbf{D}_\mathbf{w}(\mathbf{D}_r{}^{1/2}\mathbf{A}\mathbf{D}_c{}^{-1/2})\mathbf{n}_k = \lambda_{kk}\mathbf{n}_k,\ k = 1, ..., s,$$

where $\mathbf{n}_k'\mathbf{n}_k = \mathbf{v}_k'\mathbf{D}_c^{-1}\mathbf{v}_k = 1$.

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