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Block Bootstrapped Empirical Process for Dependent Sequences

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Abstract

Conditinal weakly convergence of the blockwise bootstrapped empirical process for stationary sequences to the appropriate Gaussian process is reestablished particularly for severely dependent α-mixing sequences. Issue of block size is discussed from the point of validity of bootstrap method.

Keywords: Bootstrap; Empirical process; Stationary and strong mixing sequences; Weak convergence.

1. INTRODUCTION

It is well-established that Efron's bootstrap provides a very useful nonparametric technique to investigate the sampling distribution of complicated statistics. However, the assumption of independence of the observations is crucial. It is easily seen that bootstrap gives incorrect answers if dependence presents; see Künsch (1989). Recently an extension of Efron's bootstrap to general stationary dependent sequences of observations has been made by Künsch (1989), namely Moving Blocks Bootstrap (MBB). Instead of selecting single observation X_i from the sample $\{X_1, \dots, X_n\}$ with replacement, his extended method involves selecting k blocks of consecutive observations of length b. Here b is a function of n, tending to infinity with $b = o(n)$ and $n \sim kb$. Künsch (1989) showed that under some regular conditions the blockwise bootstrap correctly estimates the sampling distribution as well as the asymptotic variance of the sample mean, when n is sufficiently large.

Several recent papers have shown that the validity of the blockwise bootstrap for empirical processes under the assumption of weak dependence. For example, see Shao and Yu (1996), Radulović (1996a, 1996b, 1998), Bühlman (1994), and Naik-Nimbalkar and Rajarshi (1994). For long range dependence, Lahiri (1993)

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showed that the MBB procedure fails to provide a valid approximation to the distribution of normalized sample mean. In this paper we reestablish the bootstrap CLT for empirical processes with α-mixing coefficients $\alpha_n = n^{-\nu}$ satisfying $1 < \nu < 2$. Our result addresses the issue of block size the choice of which may be critical to excute the *MBB*. Throughout this paper we will mainly follow the approach and setting taken by Radulović (1996b, 1998).

2. PRELIMINARIES AND RESULTS

For dependence structure, α-mixing will be employed. Let $\mathcal{A}$ and $\mathcal{B}$ be two σ-fields. Then

$$\alpha(\mathcal{A}, \mathcal{B}) = \sup_{(A,B)\in \mathcal{A}x\mathcal{B}} |P(A \cap B) - P(A)P(B)|.$$

Let $\{X_i\}$ be a strictly stationary strong-mixing sequence with mixing coefficients α_n where n denotes the time lag difference between two σ-fields. Now the blockwise bootstrap procedure is summerized: Given the sample $X_1, \dots, X_n$ and $b \in N$, set $X_{n+i} = X_i$ for $i \in \{1, \dots, b\}$. Then the MBB sample with block size b is defined as follows: if $I_1, \dots, I_k, k = [n/b]$, are sampled with replacement from $\{1, \dots, n\}$ (i.e., iid uniform on $\{1, \dots, n\}$), then the MBB sample consists of all the data points X_i that belong to the sampled blocks $B_{I_1,b} \dots, B_{I_k,b}$, i.e., $X_1^* = X_{I_1}, \dots, X_b^* = X_{I_1+b-1}, X_{b+1}^* = X_{I_2}, \dots, X_l^* = X_{I_k+b-1}$ where $l = kb$. Without loss of generality we may assume $l = n$. From now on P^*, E^*, and Var^* are P, E, and Var given the sample.

For a given stationary sequence of real valued r.v.'s $\{X_i\}_{i=1}^{\infty}$, and class of functions $\mathcal{F} = \{f_t(x) = 1_{x \le t}(x) : t \in R\}$, define the empirical process indexed by the class $\mathcal{F}$ as

$$Z_n(f)_{f\in\mathcal{F}} = \sqrt{n}(P_n - P)(f) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (f(X_i) - P(f)). \tag{2.1}$$

The bootstrap version of this process is defined as

$$Z_n^*(f, \omega)_{f\in\mathcal{F}} = \sqrt{n}(\hat{P}_n(\omega) - P_n(\omega))(f) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (f(X_i^*(\omega)) - P_n(f, \omega)), \tag{2.2}$$

where $\{X_i^*\}_{i=1}^n$ is obtained by the MBB procedure with block size $b(n)$ and $\hat{P}_n(\omega) = n^{-1} \sum_{i=1}^n \delta_{X_i^*(\omega)}$. Obviously the bootstrap process (2.2) is defined given the sample (given ω). It is well known that the above defined bootstrap is sufficient for most of the applications since it allows us to construct asymptotic confidence regions for P.

To establish validity of MBB, we follow the classical approach taken by many authors (see Bühlman (1994), Radulović (1996b, 1998)). It is clear that $Z_n(f)$ can be viewed as a random element in $l^\infty(\mathcal{F})$, and therefore we can have

$$Z_n(f)_{f\in\mathcal{F}} \xrightarrow{D} G_P(f)_{f\in\mathcal{F}} \text{ in } l^\infty(\mathcal{F}) \tag{2.3}$$

in the sense of Hoffman-Jorgenson (1984). The Gaussian process $G_P(f)_{f\in\mathcal{F}}$ from (2.3) is determined by

$$Cov(G_P(f), G_P(g)) = \lim_{n\to\infty} Cov(Z_n(f), Z_n(g)). \tag{2.4}$$

if the above limit exists. Analogously we may have weak convergence in $l^\infty(\mathcal{F})$ of the bootstrap version of the process

$$Z_n^*(f,\omega)_{f\in\mathcal{F}} \xrightarrow{D} G(f)_{f\in\mathcal{F}} \text{ in probability or a.s.} \tag{2.5}$$

for the centered Gaussian process G, independent of ω. Then it is said that the bootstrap works if the limiting process $G(f)$ coincides with $G_P(f)$ and as a result validity of MBB is established. In fact our Theorem 2.1 below states validity of MBB under some conditions.

In this paper we consider only the bootstrap in probability as defined in Gine and Zinn (1990). Namely we say that (2.5) holds in probability if

$$d_{BL}[\mathcal{L}(Z_n^*(f)|X_1,\dots,X_n), \mathcal{L}(G(f))] \to 0 \text{ in outer probability.} \tag{2.6}$$

where $\mathcal{L}(Z)$ denotes the law of Z, $d_{BL}(\mu,\nu) = \sup\{|\int f d\mu - \int f d\nu| : \| f \|_{BL} \le 1\}$, and $\| \cdot \|_{BL}$ stands for the bounded Lipschitz norm. Throughout this paper we assume that the quantity of (2.6) is measurable. For the measurability issue, refer Radulović (1996b).

Theorem 2.1. *Let $\{X_i\}_{i=1}^\infty$ be a strictly stationary α-mixing sequence of real random variables. Suppose also that $\alpha_n = O(n^{-\nu})$ for some $1 < \nu < 2$ and that $\{X_i^*\}$ is generated by the MBB procedure with block size $b_n = O(n^\mu)$ for some $0 < \mu < 1/[3(3-\nu)]$. Then*

$$Z_n^*(f)_{f\in\mathcal{F}} \to G_P(f)_{f\in\mathcal{F}} \quad \textit{in probability,} \tag{2.7}$$

where $G_P(f)$ is a centered Gaussian process defined by the covariance structure in (2.4).

Radulović (1998) established validity under $\nu > 1$ and $b_n = n^{\mu}$ for $0 < \mu < \min(1/3, \nu/4 - 1/6)$. Theorem 2.1 takes up his result further for $1 < \nu < 2$ and provide a better one. For example, when ν is close to 1, we need $\mu < 1/6$ while his result requires $\mu < 1/12$. Thus our result certainly gives more room for choice of block size when $1 < \nu < 2$. This may be meaningful because it would be safer to have a wider range for b when the optimal block size is unknown (the optimal block size is known to be $cn^{1/3}$ only for $2 < \nu$, see remark 3.3 of Künsch (1989)).

Our result seems to suggest that a narrower block size for MBB would be necessary for handling severe dependence. However such explication is misleading in the sense that a longer block would be expected to handle severe dependence. Thus what our Theorem 2.1 really tells is that the introduction of block is not efficacious in handling severe dependence. Behind this is weakness of MBB. Indeed joining independent bootstrapped block to form the bootstrapped statistic tends to completely destroy the strong dependence of underlying observations.

3. PROOF OF THEOREM

Proof of Theorem will basically follow from the steps taken by Radulović (1998). Indeed Lemmas 3.3 and 3.4 below are the Lemmas 2 and 3 of Radulović (1998) and Lemmas 3.1 and 3.2 are established to provide Lemmas 3.3 and 3.4. Thus necessary modifications and a sketch of proof will be given here while the detailed proof will be left out whenever possible. Note that we establish the second moment bounds in Lemma 3.2 and (3.11) by exerting different technique than Radulović (1998), which are key tools for verifications of Lemmas 3.3 and 3.4 here.

To establish Lemmas we introduce Davydov's covariance inequality

$$|Cov(X,Y)| \leq Const.\alpha(X,Y)^r \parallel X \parallel_p \parallel Y \parallel_q, \tag{3.1}$$

where r, p, q are positive real numbers such that $r + 1/p + 1/q = 1$. As a consequence of this inequality Yokoyama (1980) has proved the following.

Lemma 3.1. *Let $\{X_i\}$ be a strictly stationary strong mixing sequence with $EX_i = 0$ and $\parallel X_i \parallel_\infty \leq C < \infty$. Suppose also that $\sum_{i=0}^{\infty}(i+1)^{r/2-1}\alpha_i < \infty$. Then there exists a constant K such that*

$$E\left|\sum_{i=1}^{n} X_i\right|^r \leq Kn^{r/2}. \tag{3.2}$$

Lemma 3.2. *Let $\{Z_i^n, i \in \{1, \dots, n\}, n \in N\}$ be a triangular array of centered r.v.'s such that for every fixed n the row $\{Z_i^n, i = 1, \dots, n\}$ is strictly stationary. Let us also assume that $\alpha(Z_1^n, Z_i^n) = a_{0 \vee (i-b_n)}$ where $\{a_i\}_{i \geq 0}$ is some decreasing sequence of positive real numbers and $\{b_n\}_{n>0}$ is some unbounded nondecreasing sequence of integers $\ll n$. Finally if $\sup_{i,n \in N} E|Z_i^n|^s < \infty$ for some $s \leq 2$, then*

$$E\left|\sum_{i=1}^{n} Z_i^n\right|^2 \leq Const.n\{b \parallel Z \parallel_{\infty}^{2-s} + \parallel Z \parallel_{\infty}^2 \sum_{i=b}^{n} a_i\}.$$

Proof of Lemma 3.2. In the following computation the constant C might change from line to line. In order to ease notation we will occasionally write Z_i and b instead of Z_i^n and b_n. It is easy to see that by the stationarity

$$E|\sum_{i=1}^{n} Z_i|^2 \leq Cn \sum_{i=0}^{2b-1} E(Z_0 Z_i) + Cn \sum_{i=2b}^{n} E(Z_0 Z_i).$$

Then the first term can be easily estimated by Hölder's inequality. Indeed it is bounded by

$$\sum_{i=0}^{2b-1} E(Z_0 Z_i) \leq bE(Z)^2 \leq Cb \parallel Z \parallel_{\infty}^{2-s}.$$

In order to estimate the second term we observe that by Davydov's inequality (3.1) with $p = \infty$, $q = \infty$, and $r = 1$ and Hölder's inequality

$$\sum_{i=2b}^{n} E(Z_0 Z_i) \leq C \parallel Z \parallel_{\infty}^2 \sum_{i=2b}^{n} \alpha_i \leq C \parallel Z \parallel_{\infty}^2 \sum_{i=b}^{n} a_i.$$

Combining these results, we have proved Lemma 3.2.

Lemma 3.3. *Let $\{X_i\}_{i=1}^{\infty}$ be a strictly stationary sequence of real r.v.'s such that X_i is uniformly distributed on $[0,1]$ and let α_n, b_n, μ and ν be the same as in Theorem 2.1. Let $\rho(n) = (\ln n)^4$ and H_n be some sequence of subsets of $[0,1]^2$ such that $Card(H_n) \leq Cn^{2/3}$. Finally, if we let $\{X_i^*\}_{i=1}^{n}$ be generated by the MBB procedure with block size b_n, then*

$$\rho(n) \sup_{(s,t) \in H_n} \left| E^*\left(b_n^{-1/2} \sum_{i=1}^{b_n} \left(1_{(s<X_i^* \leq t)} - P_n(s,t)\right)\right)^2 \right.$$
$$\left. - \ E\left(b_n^{-1/2} \textstyle\sum_{i=1}^{b_n} \left(1_{(s<X_i \leq t)} - (t-s)\right)\right)^2 \right| \to 0 \textit{ in probability.}$$

Proof: (Proof of Lemma 3.3.) Let $A_b^* := b^{-1/2}\sum_{i=1}^b 1_{(s<X_i^*\le t)}$ and $A_b := b^{-1/2}\sum_{i=1}^b 1_{(s<X_i\le t)}$. Then the above expression is bounded by

$$\rho(n)\sup_{(s,t)\in H_n}\left|E^*\left(A_b^*-\sqrt{b}(t-s)\right)^2-E\left(A_b-\sqrt{b}(t-s)\right)^2\right| \tag{3.3}$$

$$+\rho(n)b\sup_{(s,t)\in H_n}|P_n(s,t)-(t-s)|^2$$

$$+2\rho(n)\sqrt{b}\sup_{(s,t)\in H_n}|P_n(s,t)-(t-s)|\left|E^*\left(A_b^*-\sqrt{b}(t-s)\right)\right| = I_n+II_n+III_n.$$

According to Radulović (1998), Lemma 3.3 will follow if one shows that I_n and II_n converges to 0 in probability. Verification of III_n follows from verification of II_n. For every $\epsilon>0$

$$P(II_n>\epsilon)\le Const.Card(H_n)\rho(n)b\sup_{(s,t)\in H_n}E\left|\frac{1}{n}\sum_{i=1}^n 1_{(s<X_i\le t)}-(t-s)\right|^2$$

$$\le C(\ln n)^4\frac{n^{2/3}b}{n^2}\sup_{(s,t)\in H_n}E\left|\sum_{i=1}^n 1_{(s<X_i\le t)}-(t-s)\right|^2.$$

If $\nu>1$, then by Lemma 3.1 the above is bounded by

$$\le C(\ln n)^4\frac{b}{n^{1/3}} = C(\ln n)^4 n^{\mu-1/3}.$$

By the choice of b_n, the above expression converges to 0 as n tends to infinity. Now it remains to show that I_n converges to 0 in probability. Let $Y_i^n = (b^{-1/2}\sum_{j=i}^{i+b}(1_{(s<X_j\le t)}-(t-s)))^2$. Then for every $\epsilon>0$

$$P(I_n>\epsilon) = P\left(\sup_{(s,t)\in H_n}\rho(n)\left|n^{-1}\sum_{i=1}^n Y_i^n - EY_i^n\right|>\epsilon\right). \tag{3.4}$$

Without loss of generality we can assume stationarity of $\{Y_i^n\}_{i=1}^n$. Now (3.4) does not exceed

$$C\ Card(H_n)\sup_{(s,t)\in H_n}P\left(\rho(n)\left|\frac{1}{n}\sum_{i=1}^n(Y_i^n-EY_i^n)\right|>\epsilon\right)$$

$$\le Cn^{2/3}\rho(n)^2\sup_{(s,t)\in H_n}E\left|\frac{1}{n}\sum_{i=1}^n(Y_i^n-EY_i^n)\right|^2. \tag{3.5}$$

Letting $Z_i^n := Y_i^n - EY_i^n$ and observing that by Lemma 3.1 and the assumptions of Lemma 3.3, $\sup_{i,n\in N} E|Z_i^n|^s < \infty$ holds for any $1 \le s < \nu$, we can apply Lemma 3.2 and bound expression (3.5) with

$$C(\ln n)^8 n^{2/3}\left(\frac{b}{n} \parallel Z \parallel_\infty^{2-s} + \frac{1}{n} \parallel Z \parallel_\infty^2 \sum_{i=b}^{n} a_i\right) \tag{3.6}$$

$$\le C(\ln n)^8 n^{-1/3}\left(b^{3-s} + b^2 \sum_{i=b}^{n} a_i\right)$$

by definition of Z_i^n. If $1 < \nu < 2$ the above expression is bounded by

$$C(\ln n)^8 n^{-1/3}\left(b^{3-s} + b^{3-\nu}\right),$$

which tends to zero as n tends to ∞ by the choice of b_n. This proves Lemma 3.3.

□

Before stating the next Lemma several definitions are necessary. Let $\mathcal{F}$ be a class of functions and d be a pseudo metric on $\mathcal{F}$. Then for every $\epsilon > 0$, the covering number $N(\epsilon, \mathcal{F}, d)$ is defined by

$$N(\epsilon, \mathcal{F}, d) = \min\{m : \text{ there are } f_1, \dots, f_m \in \mathcal{F} \text{ such that } \sup_{f\in\mathcal{F}} \min_{1\le j\le m} d(f, f_j) \le \epsilon\}.$$

The collection $\mathcal{F}_\epsilon = \{f_1, \dots, f_m\}$ is called an ϵ-net in $\mathcal{F}$. Finally we set

$$\sigma_b^2 = Var(b^{-1/2}\sum_{i=1}^{b} f(X_i)) \text{ and } \hat{\sigma}_b^2 = Var^*(b^{-1/2}\sum_{i=1}^{b} f(X_i^*)).$$

Now we have the following Lemma which is Lemma 3 of Radulović (1998).

Lemma 3.4. *Let $\{X_i\}_{i>0}$, $\{X_i^*\}_{i=1}^n$, b_n and α_n be the same as in Lemma 3.3. Let $\mathcal{F} = \{f_t(x) = 1_{0<x\le t}, t \in [0,1]\}$, and $\mathcal{F}_n$ be an $n^{-1/3}$-net in $\mathcal{F}$ under the pseudo distance $d(f_t, f_s) = |t - s|$ such that $Card(\mathcal{F}_n) \le const.n^{1/3}$. Finally, let $\mathcal{F}_n' := \{f - g, f, g \in \mathcal{F}_n\}$. Then*

$$A_n = (\ln n)^4 \sup_{h\in\mathcal{F}_n'} |\hat{\sigma}_b^2(h) - \sigma_b^2(h)| \to 0 \textit{ in probability} \tag{3.7}$$

$$B_n = (\ln n)^2 \sup_{f,g\in\mathcal{F};d(f,g)\le n^{-1/3}} \hat{\sigma}_b^2(f-g) \to 0 \textit{ in probability.} \tag{3.8}$$

Proof of Lemma 3.4. For (3.7), observe that $Card.\mathcal{F}_n' \leq Const.n^{2/3}$. Then by an application of Lemma 3.3 and by the definition of $\hat{\sigma}_b^2(h)$ and $\sigma_b^2(h)$ we have $A_n \to 0$ in probability.

In order to show the second part of Lemma 3.4, following the argument of proof of Lemma 3 of Radulović (1998), it is sufficient to consider the following

$$P\left(\sup_{s<t\in[0,1],|s-t|\leq n^{-1/3}}(\ln n)^2|n^{-1}\sum_{i=1}^{n}[\hat{f}_i^b(s,t)-Ef_i^b(s,t)]|>\epsilon\right). \tag{3.9}$$

Furthurmore note that we are able to replace $P_n(s,t)$ with $(t-s)$ in the above expression, since

$$\sup_{s<t\in[0,1],|s-t|\leq n^{-1/3}} b(\ln n)^2|P_n(s,t)-(t-s)|\to 0 \text{ in probability }. \tag{3.10}$$

Indeed (3.10) follows if one shows that

$$\max_{0\leq p\leq r\leq n^{1/3},|p-r|\leq 2} b(\ln n)^2|P_n(\frac{p}{n^{1/3}},\frac{r}{n^{1/3}})-(\frac{r}{n^{1/3}}-\frac{p}{n^{1/3}})|=I_n,$$

converges to 0 in probability. For every $\epsilon>0$

$$P(I_n>\epsilon)\leq C(\ln n)^4n^{1/3}b^2\sup_{s,t\in[0,1]}E(n^{-1}\sum_{i=1}^{n}[1_{s<X_i\leq t}-(t-s)])^2$$

(by Lemma 3.1 with $r=2$) if $\nu>1$

$$\leq C(\ln n)^4n^{1/3}b^2n^{-1}.$$

Then the above expression is dominated by

$$C(\ln n)^4n^{-2/3}b^2,$$

which converges to zero by the choice of b_n.

Therefore we can proceed with estimating (3.8) using $f_i^b(s,t)$ instead of $\hat{f}_i^b(s,t)$. Then (3.8) will be dominated by

$$\max_{0\leq p\leq r\leq n^{1/2},|p-r|\leq 2n^{1/6}}(\ln n)^2|n^{-1}\sum_{i=1}^{n}[f_i^b(pn^{-1/2},rn^{-1/2})-Ef_i^b(pn^{-1/2},rn^{-1/2})]|$$

$$+(\ln n)^2\max_{1\leq p\leq\sqrt{n}}|n^{-1}\sum_{i=1}^{n}\sum_{j=i}^{i+b}(1_{(p-1)n^{-1/2}\leq X_j\leq pn^{-1/2}}-n^{-1/2})|$$

$$+(\ln n)^2 \max_{0\le p\le r\le\sqrt{n},|p-r|\le 2n^{1/6}} \left| n^{-1}\sum_{i=1}^{n}\left[\left(b^{-1/2}\sum_{j=i}^{i+b}(1_{(p-1)n^{-1/2}\le X_j\le rn^{-1/2}}\right)^2\right.\right.$$

$$\left.\left. - \left(b^{-1/2}\sum_{j=i}^{i+b}(1_{pn^{-1/2}\le X_j\le (r-1)n^{-1/2}}\right)^2\right]\right| + o(1)$$

$$= II_n + III_n + IV_n + o(1).$$

For every $\epsilon > 0$

$$P(III_n > \epsilon) \le Const.(\ln n)^4\sqrt{n}b^2 n^{-2}$$
$$\max_{1\le p\le\sqrt{n}} E[\textstyle\sum_{i=1}^{n}(1_{(p-1)n^{-1/2}\le X_i\le pn^{-1/2}} - n^{-1/2})]^2.$$

When using $\nu > 1$, the covariance inequality, and

$$E1_{(p-1)n^{-1/2}\le X_j\le pn^{-1/2}} \le n^{-1/2}$$

the above is easily shown to be bounded by

$$Const.(\ln n)^4\sqrt{n}b^2 n^{-2}(nPn^{-1/2} + nP^{-\nu+1})$$

for $1 \le P \le n$. (For verification of this one may employ the argument used in the proof of Lemma 3.2.) Then P could be chosen as $P = n^{1/(2\nu)}$, and the last expression is

$$\le Const.(\ln n)^4\sqrt{n}b^2 n^{-2}n^{1/2+1/(2\nu)} = Const.(\ln n)^4 b^2 n^{-1+1/(2\nu)} \tag{3.11}$$

which converges to zero by the assumption on the size of b_n (i.e., we need $\mu < 1/2 - 1/(4\nu) < 1/2$). Notice that an application of Lemma 3.3 (formulae (3.3) yields $II_n \to 0$ in probability. In order to handle IV_n we proceed as Radulović (1998) did. Indeed according to Radulović (1998), it suffices to show the following:

$$\|b^{-1/2}\sum_{i=1}^{b}\left[\left(1_{(r-1)n^{-1/2}\le X_i\le rn^{-1/2}} - n^{-1/2}\right)+\left(1_{(p-1)n^{-1/2}\le X_i\le pn^{-1/2}} - n^{-1/2}\right)\right]\|_2$$

$$\le cn^{-\epsilon} \text{ for some } \epsilon$$

provided $\nu > 1$. The last expression holds by moment bounds derived in (3.11). Therefore $IV_n \to 0$ in probability. This proves Lemma 3.4.

Proof: (Proof of the main result) Here we just brief the proof. Detailed may be found in Radulović (1998). First it will be assumed that the sequence X_i is uniformly distributed on $[0,1]$, which can be relaxed in a routine manner as in Billingsley (1968) pg. 197.

It is well known that proof of Theorem 2.1 will follow if one establishes finite dimensional convergence and stochastic equicontinuity of the process (2.2), conditionally in probability (see, e.g., Billingsley (1968) pg. 123.) To establish this, *in a view of definition (2.6) it is sufficient to show that for every subsequence* n_k there exists a furthur subsequence n_k and a set $C \subset \Omega$, $P(C) = 1$, such that:
a)for every $\omega \in C$ and for every finite collection $t_{j1}, \ldots, t_{jd} \in [0,1]$

$$(Z^*_{n_{k_l}}(T_{j1}), \ldots, Z^*_{n_{k_l}}(T_{jd})) \overset{D}{\to} (G_P(t_{j1}), \ldots, G_P(t_{jd})) \tag{3.12}$$

b)for every $\omega \in C$ and every $\tau > 0$

$$\lim_{\delta \to 0} \limsup_{l \to \infty} P^*(\sup_{|t-s|<\delta} |Z^*_{n_{kl}}(t,s)| > \tau) = 0. \tag{3.13}$$

Before going furthur, it is worthwhile to mention that the bootstrap CLT here reduces to an relaxed uniform law of large numbers (ULL). Indeed it is sufficient for the ULL to hold along $n^{-\epsilon}$−nets (whose cardinality is a bounded function of n) and for all $h = f - g$ such that $\| f - g \|_p \leq n^{-\delta}$ for some $p, \delta > 0$ (here, the cardinality is infinite but we have the control over the size h). This relaxation of the ULL is a key point in Radulović (1998) and Shao and Yu (1996) because it enables us to prove our result by using some standard techniques from empirical process theory.

First observe that (3.12) is established because it reduces to the MBB CLT for the mean (see also Radulović (1996a)). To prove (3.13), we need to prove claims 2 and 3 of Radulović(*1996b*)*which basically follow from our Lemma 3.4* with some necessary modifications. Indeed by considering the untruncated f_s instead $f_s 1_{F<n^\gamma}$, $\| \cdot \|_\phi$ instead of $\| \cdot \|_p$, letting $\alpha_n = n^{-1/3\phi}$ instead of $\alpha_n = (\ln n)^{-3/2}$ in claims 2 and 3 of Radulović (1996b), we can obtain the following. For every subsequence n_k there exists a furthur subsequence n_{kl} (we will call it n') such that

$$\begin{aligned} P^*(\sup_{|t-s|<\delta} |Z_{n'}(t,s)| > \tau) &= P^*(\sup_{\|f_t - f_s\|_\phi < \delta^{1/\phi}} |Z_{n'}(f_t, f_s)| > \tau). \\ &\leq 2\Phi(\delta, n') + 2P^*(\sup_{f_t \in \mathcal{F}} |Z_{n'}(f_t, f_{\alpha'_n}| > \tau/3) = I'_n + II'_n, \end{aligned}$$

where $\Phi(\delta, n)$ is a real function such that $\lim_{\delta\to 0} \lim_{n\to\infty} \Phi(\delta, n) = 0$ and f_{α_n} is such that $\| f_t - f_{\alpha_n} \|_\phi \leq \alpha_n = n^{-1/3\phi}$. Now observing that the class of functions

$\mathcal{F}$ is obviously Vapnik-Chervenkis (see Dudley (1978)), we have II'_n converging to 0 as n' tends to infinity. See Radulović (1996b) and Radulović (1998) for its detailed arguments. This proves Theorem 2.1. □

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REFERENCES

Billingsley, P. (1968). *Convergence of Probability Measures,* Wiley.

Bühlmann (1994). "Blockwise bootstrapped empirical process for stationary sequences," *The Annals of Statistics,* **22** 995-1012.

Dudley, R.M. (1978). "Central limit theorem for empirical processes," *The Annals of Probability,* 899-929.

Giné, E. and Zinn, J. (1990). "Bootstrapping general empirical measures," *The Annals of Probability,* **18** 851-869

Hoffman-Jorgenson, J. (1984). "Convergence of stochastic processes on Polish spaces," *Unpublished Manuscript.*

Künsch, H. R. (1989). "The jackknife and the bootstrap for general stationary observations," *The Annals of Statistics,* **17** 1217-1241.

Lahiri, S.N. (1993). "On the moving block bootstrap under long range dependence," *Statistics and Probability Letters,* **18** 405-413.

Naik-Nimbalkar, U.V. and Rajarshi, M. B. (1994). "Validity of blockwise bootstrap for empirical processes with stationary observations," *The Annals of Statistics,* **22** 980-994.

Radulović, D. (1996a). "The bootstrap of the mean for strong mixing sequences under minimal conditions," *Statistics and Probability Letters,* **28** 65-72.

Radulović, D. (1996b). "The bootstrap for empirical processes based on stationary observations," *Stochastic Processes and their Applications,* **65** 259-279.

Radulović, D. (1998). "The bootstrap of empirical processes for α-mixing sequences," Lecture Notes. Progress in Probability. vol.3 *Birkhausen Verlag Basel*, Switzerland

Shao, Q. and Yu, B. (1996). "Bootstrapping empirical process for stationary ρ-mixing sequences," *Preprint*

Yokoyama, R. (1980). "Moment bounds for stationary mixing sequences," *Zeitschrift für Wahrscheinlichkeitstheorie Verwandte Gebiete*, **52** 45 -57.

Yu, B. (1994). "Rates of Convergence for Empirical Processes of Stationary Mixing Sequences," *The Annals of Probability*, **22** 94-116