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〈Original Paper〉

Natural Frequencies of Simply Supported Tapered Beams

단순지지된 변단면 보의 고유진동수

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ABSTRACT

Natural frequencies of non-symmetrically tapered beams with simply supported ends were determined by solving the frequency equations. In the case of symmetrically tapered beams, the finite element method was adopted for frequency computation. Computed frequencies of tapered beams were expressed as functions of taper ratio, α, and sectional properties, (m, n).

요 약

일반적으로 보의 고유진동수는 주어진 보의 진동해석 및 동적 안정해석에서 대단히 중요한 역할을 한다. 그러나 보의 단면이 부재축에 따라 연속적으로 변할 때의 고유진동수 산정은 해석적 방법으로는 불가능하거나 대단히 복잡한 것이 보통이기 때문에 수치해석법을 이용하게 된다. 여기서는 유한요소법에 의하여 고유진동수를 산정하였으며 수치해석법의 결과를 일반화하기 위하여 taper ratio를 변수로 하는 회귀식을 제안하였다. 회귀식을 이용한 고유진동수의 추정치는 해석적 방법으로 얻어진 값 또는 수치해석 결과로 부터 얻어진 값들과 비교적 잘 일치한다.

1. Introduction

The natural frequency of lateral vibration, especially the fundamental frequency of a beam, plays a very important role in the dynamic analysis of a beam and also in the vibration analysis of a beam. The fundamental natural frequency of a beam with uniform cross section can be determined with ease by an analytical method without regard to boundary conditions. For this reason many studies have concentrated on single-span or multi-span beams whose sectional properties are constant along the beam axis.

However, the determination of the natural frequency of a tapered beam, that is, when the width and/or depth of a beam is a linear function of distance along the beam length, is more difficult[(1)]. In this paper, the exact

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fundamental frequencies of simply supported non-symmetrically tapered beams with variable taper ratio, α, are determined by solving frequency equations. Also, for some of the simply supported symmetrically tapered beams, finite element method is used to determine the fundamental frequencies. Finally, frequencies are expressed by algebraic equations through a regression technique, which will help structural engineers to assess dynamic and vibration analysis results of tapered beams.

2. Non-symmetrically Tapered Beams with Simple Supports

The natural frequency, ω_o, of a tapered beam, shown in Fig.1, is governed by[5]

$$EI_o \frac{d^2}{dx^2}\left(1+\alpha\frac{x}{L}\right)^m \frac{d^2 v}{dx^2} - \rho A_o \omega_o^2 \left(1+\alpha\frac{x}{L}\right)^n \cdot v = 0 \quad (1)$$

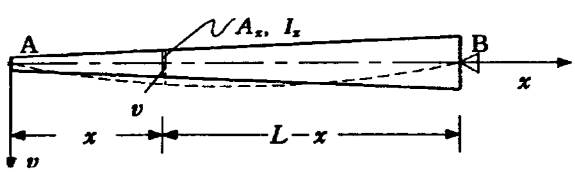
where EI_o and ρA_o are the flexural rigidity and mass per unit length at the origin, respectively.

In this paper, only 3 cases of (m, n) pairs, shown in Eq. 2, will be discussed.

$$(m, n) : (2,0),\ (3,1),\ (4,2) \quad (2)$$

The geometrical meanings of sectional property parameter, (m, n) can be found from some references. With nondimensional quality X defined by

$$X = 1+\alpha\frac{x}{L}\ , \quad x = \frac{L}{\alpha}(X-1)$$



$$A_x = A_o\left(1+\alpha\frac{x}{L}\right)^n\ ,\ I_x = I_o\left(1+\alpha\frac{x}{L}\right)^m$$

α = taper ratio (taper parameter)

Fig. 1 Nonsymmetrically tapered beam

Eq. 1 is transformed into

$$\frac{d^2}{dX^2}\left(X^m \frac{d^2 v}{dX^2}\right) - \beta^4 X^n v = 0 \quad (3)$$

where β is the nondimensional frequency defined by Eq. 4

$$\beta^4 = \frac{\rho A_o}{EI_o}\left(\frac{L}{\alpha}\right)^4 \cdot \omega_o^2 \quad (4)$$

The general solution of Eq. 3 becomes very complicated with any pair of (m, n) given by Eq. 2. For this reason, only the frequency equation of a simply supported beam with sectional properties (m, n) = (2,0) will be discussed.

When $m=2$ and $n=0$ are introduced into Eq. 3, one obtains

$$\frac{d^2}{dX^2}\left(X^2 \frac{d^2 v}{dX^2}\right) - \beta^4 v = 0 \quad (5)$$

The general solution of this equation is found to be[2]

$$Y = c_1 J_o\left(2\beta X^{\frac{1}{2}}\right) + c_2 Y_o\left(2\beta X^{\frac{1}{2}}\right) + c_3 I_o\left(2\beta X^{\frac{1}{2}}\right) + c_4 K_o\left(2\beta X^{\frac{1}{2}}\right) \quad (6)$$

where J_n and Y_n are Bessel function and Newman function of order n, respectively and I_n and K_n are nth order modified Bessel functions of the first and second kind. When the beam is simply supported at both ends, the integral constants c_1, c_2, c_3 and c_4 in Eq. 6 are related to the following boundary conditions.

$$\begin{aligned} X &= 1\ (x=0) & v=0\ ,\ \frac{d^2 v}{dX^2} = 0 \\ X &= 1+\alpha\ (x=L) & v=0\ ,\ \frac{d^2 v}{dX^2} = 0 \end{aligned} \quad (7)$$

Now these boundary conditions are introduced into Eq. 6 to obtain the following frequency equation.

$$\{v\}^T \cdot \begin{pmatrix} J_2(2\beta) \\ -Y_2(2\beta) \\ I_2(2\beta) \\ -K_2(2\beta) \end{pmatrix} = 0 \quad (8)$$

In Eq. 8 $\{v\}^T$ denotes the transpose of $\{v\}$ and $\{v\}$ is given by the following matrix

$$\{v\}=\begin{vmatrix} Y_2(B) & -I_2(B) & K_2(B) \\ J_2(B) & -I_2(B) & K_2(B) \\ J_2(B) & -Y_2(B) & K_2(B) \\ J_2(B) & -Y_2(B) & I_2(B) \end{vmatrix}\begin{bmatrix} l_1 & l_1 & l_2 & l_3 \\ l_2 & l_4 & l_4 & l_5 \\ l_3 & l_5 & l_5 & l_6 \end{bmatrix} \quad (9)$$

where $B=2\beta\sqrt{1+\beta}$ and $l_1,\ l_2,\ \cdots,\ l_6$ are given by the following cylindrical functions.

$$l_1=\begin{vmatrix} Y_o(A) & I_o(A) & K_o(A) & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & Y_o(A) & I_o(A) & K_o(A) \\ Y_o(B) & I_o(B) & K_o(B) & -Y_o(B) & -I_o(B) & -K_o(B) \\ Y_1(B) & -I_1(B) & K_1(B) & Y_1(B) & -I_1(B) & K_1(B) \\ Y_2(B) & I_2(B) & K_2(B) & -Y_2(B) & -I_2(B) & -K_2(B) \\ Y_3(B) & -I_3(B) & K_3(B) & Y_3(B) & -I_3(B) & K_3(B) \end{vmatrix} \quad (10,a)$$

$$l_2=\begin{vmatrix} Y_o(A) & I_o(A) & K_o(A) & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & J_o(A) & I_o(A) & K_o(A) \\ Y_o(B) & I_o(B) & K_o(B) & -J_o(B) & -I_o(B) & -K_o(B) \\ Y_1(B) & -I_1(B) & K_1(B) & J_1(B) & -I_1(B) & K_1(B) \\ Y_2(B) & I_2(B) & K_2(B) & -J_2(B) & -I_2(B) & -K_2(B) \\ Y_3(B) & -I_3(B) & K_3(B) & J_3(B) & -I_3(B) & K_3(B) \end{vmatrix} \quad (10,b)$$

$$l_3=\begin{vmatrix} J_o(A) & Y_o(A) & K_o(A) & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & Y_o(A) & I_o(A) & K_o(A) \\ J_o(B) & Y_o(B) & K_o(B) & -Y_o(B) & -I_o(B) & -K_o(B) \\ J_1(B) & Y_1(B) & K_1(B) & Y_1(B) & -I_1(B) & K_1(B) \\ J_2(B) & Y_2(B) & K_2(B) & -Y_2(B) & -I_2(B) & -K_2(B) \\ J_3(B) & Y_3(B) & K_3(B) & Y_3(B) & -I_3(B) & K_3(B) \end{vmatrix} \quad (10,c)$$

$$l_4=\begin{vmatrix} J_o(A) & Y_o(A) & I_o(A) & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & Y_o(A) & I_o(A) & K_o(A) \\ J_o(B) & Y_o(B) & I_o(B) & -Y_o(B) & -I_o(B) & -K_o(B) \\ J_1(B) & Y_1(B) & -I_1(B) & Y_1(B) & -I_1(B) & K_1(B) \\ J_2(B) & Y_2(B) & I_2(B) & -Y_2(B) & -I_2(B) & -K_2(B) \\ J_3(B) & Y_3(B) & -I_3(B) & Y_3(B) & -I_3(B) & K_3(B) \end{vmatrix} \quad (10,d)$$

$$l_5=\begin{vmatrix} J_o(A) & I_o(A) & K_o(A) & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & J_o(A) & I_o(A) & K_o(A) \\ J_o(B) & I_o(B) & K_o(B) & -J_o(B) & -I_o(B) & -K_o(B) \\ J_1(B) & -I_1(B) & K_1(B) & J_1(B) & -I_1(B) & K_1(B) \\ J_2(B) & I_2(B) & K_2(B) & -J_2(B) & -I_2(B) & -K_2(B) \\ J_3(B) & -I_3(B) & K_3(B) & J_3(B) & -I_3(B) & K_3(B) \end{vmatrix} \quad (10,e)$$

$$l_6=\begin{vmatrix} J_o(A) & Y_o(A) & K_o(A) & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & J_o(A) & I_o(A) & K_o(A) \\ J_o(B) & Y_o(B) & K_o(B) & -J_o(B) & -I_o(B) & -K_o(B) \\ J_1(B) & Y_1(B) & K_1(B) & J_1(B) & -I_1(B) & K_1(B) \\ J_2(B) & Y_2(B) & K_2(B) & -J_2(B) & -I_2(B) & -K_2(B) \\ J_3(B) & Y_3(B) & K_3(B) & J_3(B) & -I_3(B) & K_3(B) \end{vmatrix} \quad (10,f)$$

In the above expressions A denotes $A=2\beta$.

The fundamental frequency of the simply supported tapered beam with sectional properties (m, n) = (2,0) is determined by finding the least root satisfying Eq. 6. Some examples of the calculated frequency are shown in Table 1 in the column " C_{fem}". As can be seen in Table 1, β decreases with increasing taper ratio, α.

As the procedures to obtain the fundamental frequency of a tapered beam with other pairs of

Table 1 Frequency of non-symmetrically tapered beams

$$\omega_o=C(\alpha)\cdot\sqrt{\frac{EI_o}{\rho A_o L^4}}$$

α	$m=2,\ n=0$		$m=3,\ n=1$		$m=4,\ n=2$	
	C_{fem}	C_{est}	C_{fem}	C_{est}	C_{fem}	C_{est}
0.0	9.8695	10.0267	9.8695	10.0836	9.8695	10.2365
0.1	10.3556	10.4802	10.3558	10.5188	10.3545	10.6330
0.2	10.8366	10.9311	10.8256	10.9496	10.8074	11.0231
0.3	11.3049	11.3795	11.2846	11.3762	11.2472	11.4067
0.4	11.7694	11.8252	11.7310	11.7984	11.6692	11.7838
0.5	12.2322	12.2684	12.1702	12.2163	12.0742	12.1544
0.6	12.6823	12.7090	15.5961	12.6300	12.4594	12.5186
0.7	13.1242	13.1470	13.0169	13.0393	12.8413	12.8763
0.8	13.5691	13.5824	13.4362	13.4443	13.2019	13.2275
0.9	14.0070	14.0153	13.8374	13.8450	13.5576	13.5723
1.0	14.4402	14.4456	14.2401	14.2413	13.9074	13.9106
1.1	14.8722	14.8733	14.6316	14.6334	14.2422	14.2424
1.2	15.3000	15.2984	15.0227	15.0212	14.5643	14.5677
1.3	15.7194	15.7209	15.4075	15.4046	14.8913	14.8866
1.4	16.1418	16.1408	15.7806	15.7838	15.1994	15.1990
1.5	16.5596	16.5582	16.1599	16.1586	15.5052	15.5049
1.6	16.9732	16.9730	16.5302	16.5292	15.8020	15.8044
1.7	17.3825	17.3852	16.8935	16.8954	16.1017	16.0974
1.8	17.7959	17.7948	17.2551	17.2573	16.3799	16.3839
1.9	18.2005	18.2018	17.6180	17.6149	16.6606	16.6639
2.0	18.6074	18.6063	17.9676	17.9682	16.9410	16.9375

(m, n) of Eq. 2 are very complicated, only the final results are presented in Table 1.

3. Symmetrically Tapered Beams

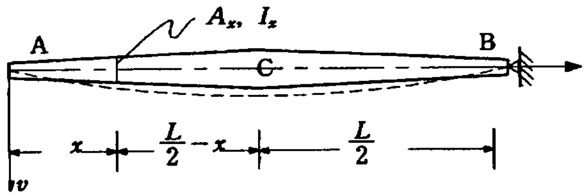
The frequency equation for a tapered beam, shown in Fig. 2, can be derived by repeating a similar method as that of non-symmetrically tapered beam.

The procedures to obtain the final frequency equation, however, become much more complicated than that of the non-symmetrically tapered beams discussed before. Thus, a finite element method is adopted for the determination of fundamental frequency.

Fig. 3 shows a commonly used beam element having 2 degrees of freedom at each node. In this case, the displacement function, $v(x)$ can be expressed by

$$v(x) = [N_1\ N_2\ N_3\ N_4]\begin{Bmatrix}\delta_1\\ \delta_2\\ \delta_3\\ \delta_4\end{Bmatrix} = [N]\{\delta\} \tag{11}$$

where $\{\delta\}$ is nodal displacement vector, and the



$A_x = A_o\left(1+2\alpha\frac{x}{L}\right)^n$, $I_x = I_o\left(1+2\alpha\frac{x}{L}\right)^m$

α = taper ratio (taper parameter)

Fig. 2 Symmetrically tapered beam

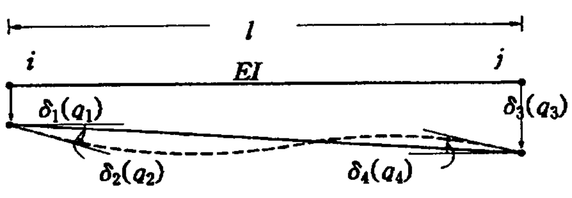


Fig. 3 Beam element

shape function component, N_i, is given by(6)

$$N_1 = 1-3\left(\frac{x}{l}\right)^2+2\left(\frac{x}{l}\right)^3$$
$$N_2 = x\left(1-\frac{x}{l}\right)^2 \tag{12}$$
$$N_3 = 3\left(\frac{x}{l}\right)^2-2\left(\frac{x}{l}\right)^3$$
$$N_4 = \frac{x^2}{l}\left(\frac{x}{l}-1\right)$$

The element force vector, $\{q\}$ and nodal displacement vector, $\{\delta\}$, are related by

$$\{q\} = \{[k_b] - \omega_o^2[m_c]\}\{\delta\} \tag{13}$$

where ω_o is the natural frequency(ies) of the beam. And flexural stiffness matrix, $[k_b]$, and the consistent mass matrix, $[m_c]$, of the beam segment are given by[3]

$$[k_b] = \int_0^l \left[\frac{d^2N}{dx^2}\right]^T \cdot [EI(x)] \cdot \left[\frac{d^2N}{dx^2}\right] dx$$

$$= \frac{EI(e)}{l^3}\begin{vmatrix} 12 & & & symm. \\ -6l & 4l^2 & & \\ -12 & 6l & 12 & \\ -6l & 2l^2 & 6l & 4l^2 \end{vmatrix} \tag{14}$$

$$[m_c] = \int_0^l [N]^T \cdot (\rho A(x)) \cdot [N]\ dx$$

$$= \frac{\rho A(e) l}{420}\begin{vmatrix} 156 & & & symm. \\ 22l & 4l^2 & & \\ 54 & 13l & 156 & \\ -13l & -3l^2 & -22l & 4l^2 \end{vmatrix} \tag{15}$$

In the derivation of Eq. 14 and 15, cross-sectional area $A(e)$ and cross-sectional moment of inertia $I(e)$ of beam segment are assumed constant. In this paper, they are the values that are calculated at the middle point of element length, which coincides well with the exact values that are obtained by the integrals with variable $I(x)$ and $A(x)$.

The assembled beam matrix from the element

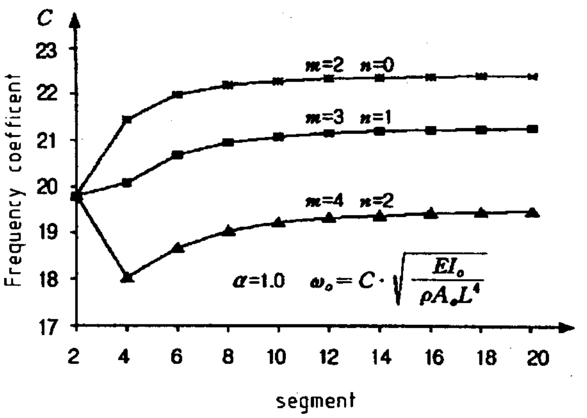


Fig. 4 Convergence of fundamental frequency

matrices is to be proceeded in the conventional manner, which is expressed by[7,8]

$$\{[K_b]-\omega_o^2[M_c]\}\{\Delta\}=\{0\} \tag{16}$$

Table 2 Frequency of symmetrically tapered beams

$$\omega_o = C(\alpha)\cdot\sqrt{\frac{EI_o}{\rho A_o L^4}}$$

α	$m=2,\ n=0$		$m=3,\ n=1$		$m=4,\ n=2$	
	C_{fem}	C_{est}	C_{fem}	C_{est}	C_{fem}	C_{est}
0.0	9.8695	10.0536	9.8695	10.2264	9.8695	10.5040
0.1	10.5527	10.6972	10.5492	10.8293	10.5421	11.0383
0.2	11.2245	11.3380	11.2093	11.4268	11.1850	11.5637
0.3	11.8925	11.9759	11.8598	12.0189	11.7855	12.0800
0.4	12.5543	12.6110	12.4893	12.6056	12.3744	12.5874
0.5	13.2041	13.2433	13.1003	13.1870	12.9336	13.0857
0.6	13.8492	13.8727	13.7003	13.7630	13.4656	13.5751
0.7	14.4783	14.4993	14.2895	14.3336	13.9847	14.0556
0.8	15.1143	15.1230	14.8767	14.8988	14.4729	14.5270
0.9	15.7433	15.7438	15.4429	15.4587	14.9544	14.9895
1.0	16.3564	16.3619	16.0110	16.0132	15.4286	15.4430
1.1	16.9804	16.9771	16.5620	16.5623	15.8861	15.8875
1.2	17.5872	17.5894	17.1031	17.1060	16.3261	16.3230
1.3	18.1987	18.1989	17.6450	17.6443	16.7476	16.7496
1.4	18.8086	18.8055	18.1812	18.1773	17.1625	17.1672
1.5	19.3955	19.4093	18.7081	18.7049	17.5799	17.5757
1.6	20.0130	20.0103	19.2263	19.2271	17.9816	17.9754
1.7	20.6122	20.6084	19.7404	19.7439	18.3713	18.3660
1.8	21.2126	21.2037	20.2547	20.2554	18.7280	18.7477
1.9	21.7953	21.7961	20.7572	20.7614	19.1277	19.1203
2.0	22.3808	22.3857	21.2666	21.2621	19.4856	19.4840

where $\{\Delta\}$ denotes the beam displacement vector and the right side $\{0\}$ indicates that Eq. 16 is free vibration problems.

The natural frequency of a tapered beam can be determined from the following eigenvalue equation.

$$det\ \{[K_b]-\omega_o^2[M_c]\}=0 \tag{17}$$

To obtain the lowest eigenvalue or the fundamental natural frequency from Eq. 17 by the computer-aided iteration method, this expression is transformed into[4]

$$det\left\{[K_b]^{-1}[M_c]-\frac{1}{\omega_o^2}[I]\right\}=0 \tag{18}$$

in which $[I]$ is the identity matrix.

As can be seen in Fig. 4, the computed frequencies converge to certain values when the numbers of the elements are increased. In this paper, a symmetrically tapered beam is subdivided into 20 equal elements. Table 2 shows some computed results of the fundamental frequency.

4. Approximate Fundamental Frequency

As one would see in Eq. 8 and Table 1, the exact determination of the fundamental frequency of the lateral vibration of a tapered beam requires a solution of complicated cylindrical

Table 3 Regression results (range of α, 0.0~2.0)

	(m, n)	a_1	a_2	a_3	r
Fig. 1	$m=2$, $n=0$	10.0266	4.5480	-0.1291	0.9999545
	$m=3$, $n=1$	10.0836	4.3731	-0.2154	0.9998060
	$m=4$, $n=2$	10.2365	3.9977	-0.3236	0.9997433
Fig. 2	$m=2$, $n=0$	10.0536	6.4505	-0.1422	0.9999806
	$m=3$, $n=1$	10.2264	6.0557	-0.2689	0.9999740
	$m=4$, $n=2$	10.5040	5.3879	-0.4489	0.9999364

functions and time-consuming calculations.

For the convenience of structural engineers who are engaged in dynamic analysis or in vibration problems of tapered beams, the following expression is proposed :

$$C(\alpha) = a_1 + a_2\alpha + a_3\alpha^2 \qquad (19)$$

The constants a_1, a_2 and a_3 in Eq. 19 are determined by regression technique and the results are listed in Table 3

In Table 1 and 2, the columns with the name "Cest" are the frequencies, estimated with Eq. 19 and Table 3. The last column of Table 3 indicates the correlation coefficients given by

$$r = \frac{\sum(C_{fem} - \overline{C}_{fem}) \cdot (C_{est} - \overline{C}_{est})}{\sqrt{[\sum(C_{fem} - \overline{C}_{fem})^2] \cdot [\sum(C_{est} - \overline{C}_{est})^2]}} \qquad (20)$$

where $\overline{C}_{fem}$ and $\overline{C}_{est}$ denotes the mean values of C_{fem} and C_{est}, respectively. It is observed that the correlation coefficients in Table 3 are nearly the same in all the cases and so one can see that the derived regression equations estimate the frequency data well.

5. Conclusion

The fundamental frequencies of lateral vibration of non-symmetrically or symmetrically tapered beams with simply supported ends can be determined by an analytical method. The difficulties encountered in the frequency determination were overcome by choosing finite element method. In the tapered beams, the key parameters were taper parameter, α and sectional property parameter, (m, n). For easy hand calculation of natural frequencies, simple algebraic equations are proposed. The proposed equation can be easily approached by structural engineers resulting in less error in actual application.

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