韓國經營科學會誌
第24卷 第3號
1999年 9月

중앙창 재고가 있는 수리가능시스템을 위한 해법

김종수* · 신규철**

An Optimal Algorithm for Repairable-Item Inventory System with Depot Spares

Jong-Soo Kim* · Kyu-Chul Shin**

■ Abstract ■

We consider the problem of determining the spare inventory level for a multiechelon repairable-item inventory system. Our model extends the previous results to the system which has an inventory at the central depot as well as at bases. We develop an optimal algorithm to find spare inventory levels, which minimize the total expected cost and simultaneously satisfy a specified minimum service rate. The algorithm is tested using problems of various sizes to verify the efficiency and accuracy.

1. Introduction

Repairable items are referred to as components which are expensive, critically important, and subject to infrequent failures such as engines of a fighter plane or a ship. They should be replaced or repaired immediately, if failed, for the system to maintain availability. For this reason the policy on the inventory or shortage levels is very important and naturally has been studied for a long time by many researchers. There are two main streams of research in this area. METRIC model, developed by Sherbrooke [13] assumes infinite repair capacity. In his model, there are many bases and a central depot. A failed item at a base is dispatched to a repair facility and a spare, if available, is plugged in. Otherwise, it is backordered. A repaired item fills the backorder or is stored at a spare inventory point if there is no backorder. Feeney

* 한양대학교 산업공학과 부교수
** 공군본부 소령

and Sherbrooke [4], Muckstadt [10, 11] and Muckstadt and Thomas [12] extended this model. However, as Albright [1] has pointed out, models assuming infinite repair capacity always underestimate the amount of congestion in the system and, consequently, result in fewer spares than are really needed to achieve a specified backorder level.

Another stream of study adopts the finite repair capacity, constant-failure-rate assumptions. The models in this stream are more realistic than the comparable METRIC models, and are certainly more difficult to solve due to the huge multidimensional state spaces involved. Gross *et al.* [7] considered a two-echelon (two levels of repair, one level of supply) system and presented an implicit enumeration algorithm to calculate the capacities of the base and depot repair facilities as well as the spares level which together guarantee a specified service rate at a minimum cost. Inevitably, the enumeration scheme of the method requires considerable computer running times even for relatively small problems. Gross *et al.* [5, 6] and Albright and Soni [2, 3] analyzed the operating characteristics of a given system with multidimensional Markov process. In another paper Albright [1] developed an approximation algorithm with a single type of item stocked and repaired by several bases and a central depot. The proposed methods in this stream concentrate on the analysis of the current status of a given system and, consequently, are impractical to apply to optimization problems.

More recently, Kim *et al.* [9] developed an algorithm to determine the optimal inventory level under finite repair capacities. They presented a method to solve a two-echelon (two levels of repair, one level of supply) system. In this paper we consider a more general system than the one analyzed in Kim *et al.* [9]. In other words, we consider the system whose central depot also has its inventory as well as the bases. Using properties of the system, we are able to develop an optimal algorithm to find the amount of spare items at each inventory which minimize the total expected holding plus shortage costs and simultaneously achieve a specified minimum service rate for large real problems.

This article is organized as follows. In Section 2 the model we consider is described and, in Section 3, we introduce the algorithm for the model and present an example to explain the algorithm. Lastly, in Section 4, we summarize the results of the study and identify some areas for future research.

2. Model Description

We consider a system with $I (I < \infty)$ bases, a central depot and a single type of repairable-item as depicted in Figure 1. The depot has its own spares inventory which enables the depot-repairable item to be replaced immediately with a spare, if available. Time intervals between failures at base i are exponentially distributed with mean, $1/\lambda_i$, $i = 1, 2, \cdots, I$. A failed item at base i is base-repairable with probability α_i and a base spare replaces it if one is available. Otherwise, replacement is delayed until a spare becomes available. A failed item, which is depot-repairable with probability $1-\alpha_i$, should be sent to the depot for repair. If the depot has spares available, then a spare is immediately sent to the base where the failed item has originated and the failed item is stored at the depot inventory after repair. On the other hand, if the depot spares are not available, then the replacement of the failed item with a spare is delayed until a spare becomes available. We refer to this case as depot-shortage with respect to a base. If there exist two or more depot-shortages outstanding when a spare becomes available, we assume that the depot-shortages are filled by FCFS(first-come, first-served) basis. Additionally, we assume that no lateral transshipment is allowed due to the computational complexities required to do optimization.

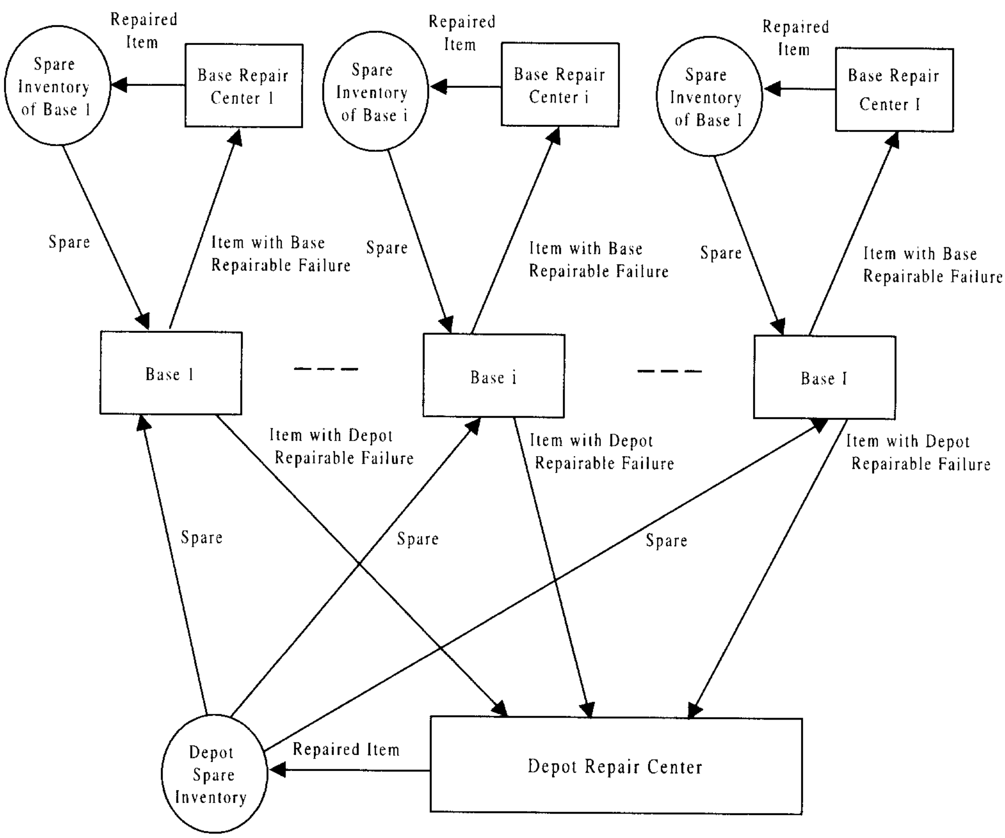
The total number of failed items of base i, in the sense that they are currently unavailable for replacement, are the sum of the items at the base repair center, items in depot-shortage with respect to the base i, and items in transit between the depot and base i.

Let us denote

$P_i(n)$ = probability distribution that there are n items at the repair center of base i,

$P_d(D)$ = probability distribution that there are D items at the depot repair center,

$P_{id}(k_i)$ = probability distribution that there are k_i items at the depot repair center which are depot-shortage with respect to base i,



[Figure 1] Schematic Representation of the Repairable-item Inventory System

$P_t(m_i)$ = probability distribution that there are m_i items in transit from or to base i,

$P(z_i)$ = probability distribution that the total number of failed items of base i is z_i

To obtain the steady-state probability distribution of total number of failed items at each base, we derive probability distributions of, $P_i(n)$, $P_{id}(k_i)$ and $P_t(m_i)$.

2.1 Probability Distribution of Items at the Base Repair Center

Let there be c_i service channels at the repair center of base i and the repair times at each channel are assumed to be i.i.d. exponential with mean $1/\mu_i$. Since we assume infinite population, the base repair center can be modeled as an M/M/ c_i queueing model, where the arrival (base-repairable failure) rate is $\alpha_i\lambda_i$ and the service rate is μ_i. So the steady-state probability distribution that there are n items at the base repair center i, $P_i(n)$, is given by the following (1) and (2).

$$P_i(n) = \begin{cases} (\alpha_i\lambda_i)^n P_i(0)/n!\mu_i^n & (1 \le n \le c_i) \\ (\alpha_i\lambda_i)^n P_i(0)/c_i^{n-c_i} c_i!\mu_i^n & (n \ge c_i) \end{cases} \quad (1)$$

where

$$P_i(0) = \left[\sum_{n=0}^{c_i-1} (\alpha_i\lambda_i)^n/n!\mu_i^n + 1/c_i!(\alpha_i\lambda_i/\mu_i)^{c_i} \times c_i\mu_i/(c_i\mu_i - \alpha_i\lambda_i) \right]^{-1} \quad (2)$$

Note that the above probability distributions do not exist unless the steady-state cindition, i.e., $\alpha_i\lambda_i/c_i\mu_i < 1$, is satisfied.

2.2 Probability Distribution of Depot-Shortaged Items at the Depot Repair Center

The depot-repairable failures at base i occur according to Poisson process with rate $(1-\alpha_i)\lambda_i$ which are independent of each other. This implies the superposed arrival stream at the depot repair center is Poisson with rate $\Sigma_{i=1}^{I}(1-\alpha_i)\lambda_i$. As we assume that there are c_d channels at the depot repair center, and also, the repair times are i.i.d. exponential with mean $1/\mu_d$, the probability that there are D items at the depot repair center, $P_d(D)$, is derived from the equations of an M/M/ c_d queueing model as follows:

$$P_d(D) = \begin{cases} (\lambda_d)^D P_d(0)/D!\mu_d^D & (D \le c_d) \\ (\lambda_d)^D P_d(0)/c_d^{D-c_d} c_d!\mu_d^D & (D \ge c_d) \end{cases} \quad (3)$$

where

$$P_d(0) = \left[\sum_{D=0}^{c_d-1} (\lambda_d)^D/D!\mu_d^D + 1/c_d!(\lambda_d/\mu_d)^{c_d} \times c_d\mu_d/(c_c\mu_d - \lambda_d) \right]^{-1} \quad (4)$$

and the steady-state condition $\lambda_d/c_d\mu_d < 1$ has to be satisfied.

Now let us find the probability distribution of k_i, the number of items at the depot repair center, which is supposed to be returned to base i, i.e., depot-shortaged items of base i. When we denote actual fill rate of depot as F_d, the arrival rate of total depot-shortaged items is $\Sigma_{i=1}^{I}(1-\alpha_i)\lambda_i(1-F_d)$ and the rate of depot-shortaged items of base i is $(1-\alpha_i)\lambda_i(1-F_d)$. Actual fill rate of depot, F_d, means the ratio of arriving items that are replaced immediately by depot spares. Then the ratio of total depot-shortaged items to those of base i, θ_i, is expressed as in equation (5).

$$\theta_i \equiv (1-\alpha_i)\lambda_i(1-F_d) / \sum_{i=1}^{I}(1-\alpha_i)\lambda_i(1-F_d)$$
$$= (1-\alpha_i)\lambda_i / \sum_{i=1}^{I}(1-\alpha_i)\lambda_i \qquad (5)$$

Now, assuming that there are s_d spare items at the depot, the conditional probability that there are k_i depot-shortaged items with respect to base i given that D items are depot shortaged is given by

$$P_{id}(k_i \mid D) = \begin{cases} 1 & \text{if } k_i = 0, D \leq s_d \\ 0 & \text{if } k_i \neq 0, D \leq s_d \\ \binom{D-s_d}{k_i}\theta_i^{k_i}(1-\theta_i)^{D-s_d-k_i} & \text{if } D > s_d \end{cases} \qquad (6)$$

Equation (6) means that, if there still remain spares in the depot, then depot-shortaged item to base i can not exist. On the other hand, if $D > s_d$, i.e., all spares in the depot have been sent to bases, then depot-shortage accumulates and, among them, depot-shortage with respect to base i has binomial distribution. By unconditioning on D in $P_{id}(k_i \mid D)$, we can obtain $P_{id}(k_i)$ as follows:

$$P_{id}(k_i) = \sum_{D=k_i}^{\infty} P_{id}(k_i \mid D) \cdot P_d(D) \qquad (7)$$

2.3 Probability Distribution of Items in Transit

It is well known that the probability distribution of the number of items in transit from the depot repair center to base i is equal to that of the depot repairable failures at the base in the steady state (see, for example, p.710 of Hillier and Lieberman [8]). Therefore, the probability distribution of the number of items in transit from the depot to the base is Poisson. Since the number in transit from base i to the depot repair center is also Poisson and the sum of independent Poisson's is Poisson, the total number of items in transit is also Poisson. When we denote the transit time between base i and the depot by t_i, the probability distribution of the number of items in transit is Poisson as given in equation (8).

$$P_t(m_i) = (2(1-\alpha_i)\lambda_i t_i)^{m_i} \times \exp(-2(1-\alpha_i)\lambda_i t_i)/m_i! \qquad (8)$$

2.4 Probability Distribution of Total Failed Items

In the steady state, the total failed items of a base are the sum of the depot-shortaged items, the items currently at the base repair center and in transit. So we can obtain the probability distribution of the total failed items of base i, $P(z_i)$, by convolution of the previously derived probability distributions as in equation (9).

$$P(z_i) = \sum_{m_i}\sum_{k_i} P_t(m_i) \cdot P_{id}(k_i) \cdot P_i(z_i - k_i - m_i) \qquad (9)$$

2.5 Total Expected Cost and Minimum Fill Rate

If the total failed items of base i, z_i, is greater than the spare inventory level s_i, then the shortage cost b_i is incurred for each backorder. On the other hand, the holding cost h_d or h_i is charged on the number of spares in the depot or in bases since it is reasonable to consider that the holding cost is charged depending on where it belongs rather than where it is located. When we assume linear holding and shortage costs, the total expected cost of the system, which is the sum of the expected holding cost of the depot and the shortage and holding costs of the bases, can be obtained by:

$$\begin{aligned} TC(S) &= h_d s_d + \sum_{i=1}^{I} h_i s_i + \sum_{i=1}^{I} b_i \sum_{z_i = s_i+1}^{\infty} (z_i - s_i) P(z_i) \\ &= h_d s_d + \sum_{i=1}^{I} \Big(h_i s_i + b_i \sum_{z_i = s_i+1}^{\infty} (z_i - s_i) P(z_i) \Big) \\ &\triangleq h_d s_d + \sum_{i=1}^{I} TC_i(s_i) \end{aligned} \qquad (10)$$

Let $TC_i(s_i \mid s_d)$ = conditional expected cost of base i when the depot inventory level is set to s_d.

Theorem 1. *The conditional expected cost function of base* i, $TC_i(s_i \mid s_d)$, *is unimodal on the interval* $[0, \infty]$.

Theorem 2. *When* $h_i / b_i \le 1$, *spare inventory level of base* i *minimizing* $TC_i(s_i \mid s_d)$ *satisfies equation* (11).

$$\sum_{k=1}^{\infty} P(z_i = s_i + k \mid s_d) < h_i / b_i < \sum_{k=0}^{\infty} P(z_i = s_i + k \mid s_d) \qquad (11)$$

where $P(z_i \mid s_d)$ *is the conditional probability distribution of the total failed item of base* i.

Note that the the probability distribution of the total failed item is defined in equation (9).

Lemma 1. *When* $h_i / b_i > 1$, *spare inventory level minimizing* $TC_i(s_i \mid s_d)$ *is zero.*

The problem we are going to solve is to find the spare inventory levels of the depot and base which fulfills the specified minimum required fill rate of each base with minimum total expected cost, which is defined in equation (10). Thus the actual fill rate F_i, which is the ratio of failed items that are replaced immediately by base spares, should be larger than or equal to the minimum required fill rate f_i. Here the actual fill rate means ratio actually achieved in the field compared with the minimum required fill rate, which is a target value at each base. The relationship between the two parameters can be expressed as in (12).

$$F_i = \Pr\{z_i < s_i\} = \sum_{z_i=0}^{s_i-1} P(z_i) \ge f_i \qquad (12)$$

Theorem 3. *Let* s_i^* *be the inventory level with the minimum expected cost of base* i *and* $\bar{s}_i$ *be the minimum of* s_i *values satisfying equation (12) for a given* s_d. *Then the spare inventory level to achieve the minimum fill rate at minimum cost of base* i *for the given* s_d *is* max $\{s_i^*, \bar{s}_i\}$.

The total expected cost of the system also has interesting properties. Before introducing them, we define additional notations.

$TC_i^*(s_i \mid s_d)$ = minimum of the $TC_i(s_i)$ satisfying specified minimum fill rate of base i for a given depot inventory level s_d.

$\underline{TC} = \sum TC_i^*(s_i \mid s_d \to \infty)$ = sum of the minimum expected base costs when the depot has infinite number of spares.

Theorem 4. s_d greater than $\bar{s}_d$ such that $\{\bar{s}_d \mid h_d > \sum_{i=1}^{I} TC_i^*(s_i \mid \bar{s}_d) - \underline{TC}\}$ *can not be optimal.*

When the depot has infinite number of spares, there exists no depot-shortaged item with respect to bases. In this case, the total failed items of a base are sum of the items currently at the base repair center and in transit. Thus $\underline{TC}$ in Theorem 4 can be calculated using the following equations (13) and (14).

$$\underline{TC} = \sum TC_i^*(s_i \mid s_d \to \infty) = \sum_{i=1}^{I} \min_{s_i \text{ given } s_d \to \infty} (h_i s_i + b_i \sum_{z_i = s_i + 1}^{\infty} (z_i - s_i) \underline{P}(z_i)) \quad (13)$$

$$\underline{P}(z_i) = \sum_{k_i} P_t(m_i) \cdot P_i(z_i - m_i) \quad (14)$$

The proofs of Theorems and Lemma are given in the Appendix.

3. The Algorithm

We now formally present an algorithm to determine the optimal spare inventory levels of the depot and bases to meet a specified minimum fill rate of each base at minimum total expected cost. The algorithm could be very effective by limiting the search region using the properties described above.

Step 0. Verify that the following steady-state conditions are satisfied.

$\sum_{i=1}^{I}(1-\alpha_i)\lambda_i / c_d\mu_d < 1$ and $\alpha_i\lambda_i / c_i\mu_i < 1$ for $i = 1, 2, \cdots, I$.

If the conditions are met, go to Step 1. Otherwise, stop since the system can not reach steady state.

Step 1. Calculate $P_d(D), P_i(n), P_t(m_i)$ until the probability becomes less than $\varepsilon = 10^{-4}$. Let $s_d \leftarrow 0$.

Step 2. For $i = 1, 2, \cdots, I$, perform the following 2.1-2.3 steps to calculate $\underline{TC}$.

Step 2.1. Calculate $\underline{P}(z_i)$ until the probability becomes less than $\varepsilon = 10^{-4}$.

Step 2.2. If $h_i / b_i > 1$, set $s_i^* \leftarrow 0$.

Otherwise set $s_i^* \leftarrow$smallest integer s_i satisfying $\sum_{k=1}^{\infty} \underline{P}(z_i = s_i + k) < h_i / b_i < \sum_{k=0}^{\infty} \underline{P}(z_i = s_i + k)$.

Step 2.3. Calculate the minimum inventory level satisfying the minimum fill rate, $\bar{s}_i$.

Step 2.4. Let $\underline{s}_i = \max\{s_i^*, \bar{s}_i\}$ and calculate $\underline{TC}_i = h_i \underline{s}_i + b_i \sum_{z_i = \underline{s}_i + 1}^{\infty} (z_i - \underline{s}_i) \underline{P}(z_i)$.

Step 3. $\underline{TC} = \sum_{i=1}^{I} \underline{TC}_i$

Step 4. For $i = 1, 2, \cdots, I$, perform the following 4.1-4.4 Steps to calculate for each base the base inventory level satisfying minimum fill rate at minimum cost.

Step 4.1. Calculateand $P_{id}(k_i \mid s_d)$ and $P(z_i \mid s_d)$ until the probability becomes less than $\varepsilon = 10^{-4}$.

Step 4.2. If $h_i / b_i > 1$, set $s_i^* \leftarrow 0$.

Otherwise set $s_i^* \leftarrow$ smallest integer s_i satisfying $\sum_{k=1}^{\infty} P(z_i = s_i + k \mid s_d) < h_i / b_i < \sum_{k=0}^{\infty} P(z_i = s_i + k \mid s_d)$.

Step 4.3. Calculate the minimum inventory level satisfying the minimum fill rate $\bar{s}_i$.

Step 4.4. Let $s_i^* = \max\{s_i^*, \bar{s}_i\}$. s_i^* denote the desired spare inventory level of base i when the depot spare is given as s_d.

Step 5. $TC_i^*(s_i \mid s_d) = TC_i(s_i^* \mid s_d) = \sum_{i=1}^{I}(h_i s_i^* + b_i \sum_{z_i = s_i^* + 1}^{\infty} (z_i - s_i^*) P(z_i \mid s_d))$.

If the current s_d satisfies $h_d > \sum_{i=1}^{I} TC_i^*(s_i \mid s_d) - \underline{TC}$, then go to Step 6. Otherwise let $s_d \leftarrow s_d + 1$ and go to Step 4.

Step 6. The s_d and s_i^*'s corresponding to the minimum total expected cost found so far are solution of the algorithm.

In Steps 2 and 3, we calculate a low bound of the total expected cost. In Step 4, we find the inventory level for each base satisfying the specified minimum fill rate at minimum base cost under the current depot spare level. Using the low bound and the current total expected cost, we check, in Step 5, if we have completed searching depot spare levels inside the region where optimal solution can reside and, if so, we stop. Otherwise the current depot spare level is incremented by one and we repeat searching.

3.1 Example

We test the algorithm using a program coded in C on a Pentium (166MHz CPU) based IBM compatible PC system. Consider a multiechelon inventory system with two bases and a depot. The holding costs for each base and a depot are 20 respectively, and the shortage costs for each base are set to 100. Other relevant data is described in <Table 1>. <Table 2> shows the results considering the total cost only. For base 1, when we neglect the minimum fill rate, the desirable inventory level is 24 at a minimum base expected cost of 541.115. For base 2, it is 12 items at a cost of 285.82. For the depot, it is 1 item at a cost of 20. The solution of the example, inventory levels satisfying the minimum fill rate at minimum cost, is summarized in <Table 3>. The spare levels and the cost are decreased as the minimum fill rate is decreased until the minimum point is reached. As soon as the inventory level arrives at the minimum point, it remains there despite a further decrease in the minimum fill rate.

〈Table 1〉 Data for the Example

Para meters	λ_i	α_i	c_i	μ_i	t_i	h_i	b_i
Base 1	20.0	0.623	2	18.0	1.130	20	100
Base 2	10.0	0.743	1	15.0	1.502	20	100
Depot	-	-	5	3.0		20	-

〈Table 2〉 Minimum Cost Inventory Level

Base/Depot	Inventory level	Minimum cost
Base 1	24	541.115
Base 2	12	285.820
Depot	1	20
Total cost		846.935

〈Table 3〉 Output of the Algorithm for the Example

Minimum fill rate	Actual fill rate	Spare level		Minimun cost		
		Base	Dept	Dase	Depot	Total
0.99	0.991(0.990)*	34(19)	0	681.494(381.111)	0	1062.605
0.95	0.957(0.970)	30(17)	0	608.636(343.784)	0	952.420
0.90	0.926(0.937)	27(15)	1	560.982(310.003)	20	890.985
0.85	0.860(0.867)	26(14)	1	550.494(297.135)	20	867.629
0.80	0.828(0.806)	25(13)	1	543.562(288.483)	20	852.045
0.75	0.782(0.811)	24(12)	2	526.908(280.524)	40	847.432
0.70	0.782(0.722)	24(12)	1	541.115(285.820)	20	846.935
0.65	0.782(0.722)	24(12)	1	541.115(285.820)	20	846.935
0.60	0.782(0.722)	24(12)	1	541.115(285.820)	20	846.935

* Entry in parenthesis is for the base 2.

To test the accuracy of the algorithm, we perform extensive computational experiments. For the test, we generate 10 different problems for 5, 10 and 15 base cases, i.e., 30 problems in total. We also develop a simulation model programmed in AWESIM (Version 1.2) simulation language and the interface program coded in C. For each problem, we run a simulation 10 times. Costs and spare levels obtained by the algorithm and averages of those generated by the 10 simulation runs are compared with each other. The main objective of the comparison is to verify that the algorithm is able to produce an accurate expected cost of the system. Another concern is to make certain that the algorithm detects the optimal spare inventory levels correctly. From the comparisons, we notice that the algorithm is able to generate cost values within 1.0 percent in percent difference unit and the spare inventory levels coincident with the inventory levels with a lowest simulated cost value.

In addition, to get an idea of how fast the algorithm is for real problems, we solve problems of extremely large sizes. It is observed that it takes 245, 670 and 1,310 seconds in CPU time to solve 50, 100 and 150 base problems respectively. Since the problem with 150 bases could be considered as the biggest problem we may encounter in real application, we conclude that the algorithm is efficient enough to be applied to real world problems. Reducing search region by taking advantage of the properties we found not only guarantees finding optimal solution but also drastically reduce running time of the algorithm.

4. Concluding Remarks

In this paper we develop an optimal algorithm to calculate the spare inventory level which satisfies a predetermined minimum service. With this approach we are able to solve large problems quickly and accurately. For further study, one can relax the assumption of infinite number of items operating at each base or can consider a more general case where the spares in a base can be transferred to another if it has no spare to replace the failed item.

APPENDIX

1. Proof of Theorem 1

Since the shortage and holding costs are linear with $P(z_i \mid s_d) \geq 0, TC_i(s_i \mid s_d)$ is unimodal.

2. Proof of Theorem 2

Since $TC_i(s_i \mid s_d)$ is unimodal function, unless $s_i = 0$ is minimum point, there should exist a point s_i satisfying $TC_i(s_i + l_1 \mid s_d) > TC_i(s_i \mid s_d)$ for $l_1 \geq 0$ and $TC_i(s_i - l_2 \mid s_d) > TC_i(s_i \mid s_d)$ for $l_2 \geq 0$.

$$TC_i(s_i \mid s_d) = h_i s_i + b_i \sum_{z_i = s_i + 1}^{\infty} (z_i - s_i) P(z_i \mid s_d).$$

$$TC_i(s_i + 1 \mid s_d)$$
$$= h_i(s_i + 1) + b_i \sum_{z_i = s_i + 2}^{\infty} (z_i - s_i - 1) P(z_i \mid s_d).$$

$$TC_i(s_i + 1 \mid s_d) - TC_i(s_i \mid s_d)$$
$$= h_i(s_i + 1) + b_i(P(s_i + 2 \mid s_d) + 2P(s_i + 3 \mid s_d) + \cdots) - h_i s_i - b_i(P(s_i + 1 \mid s_d) + 2P(s_i + 2 \mid s_d) + \cdots)$$
$$= h_i - b_i(P(s_i + 1 \mid s_d) + P(s_i + 2 \mid s_d) + P(s_i + 3 \mid s_d) + \cdots)$$
$$= h_i - b_i\left[\sum_{k=1}^{\infty} P(s_i + k \mid s_d) \right]$$

Thus, if $TC_i(s_i + 1 \mid s_d) > TC_i(s_i \mid s_d)$ is to be satisfied,

$$h_i - b_i\left[\sum_{k=1}^{\infty} P(s_i + k \mid s_d) \right] > 0 \Rightarrow$$
$$\sum_{k=1}^{\infty} P(s_i + k \mid s_d) < h_i / b_i \qquad \text{(A1)}$$

In addition, when (A1) is satisfied by s_i and $s_i + 1$,

$$TC_i(s_i + 2 \mid s_d) - TC_i(s_i \mid s_d)$$
$$= 2h_i - b_i[P(s_i + 1 \mid s_d) + 2P(s_i + 2 \mid s_d)$$

$$+2P(s_i+3 \mid s_d)+2P(s_i+4 \mid s_d)+\cdots]$$

$$=2h_i-b_i\left[\sum_{k=1}^{\infty}P(s_i+k \mid s_d)+\sum_{k=2}^{\infty}P(s_i+k \mid s_d)\right]$$

$$> 2h_i-b_i\left(\frac{h_i}{b_i}+\frac{h_i}{b_i}\right)=0$$

$$\left(\sum_{k=1}^{\infty}P(s_i+k \mid s_d) < h_i/b_i \text{ and } \sum_{k=2}^{\infty}P(s_i+k \mid s_d) < \sum_{k=1}^{\infty}P(s_i+k \mid s_d) < h_i/b_i\right)$$

Thus, $TC_i(s_i+2 \mid s_d) > TC_i(s_i \mid s_d)$ is also satisfied.

This applies to $s_i+3, s_i+4, s_i+5, \cdots$.

$$TC_i(s_i-1 \mid s_d)$$
$$=h_i(s_i-1)+b_i\sum_{z_i=s_i}^{\infty}(z_i-s_i+1)P(z_i \mid s_d)$$
$$=h_i(s_i-1)+b_i[P(s_i \mid s_d)+2P(s_i+1 \mid s_d)+3P(s_i+2 \mid s_d)+\cdots].$$

$$TC_i(s_i-1 \mid s_d)-TC_i(s_i \mid s_d)$$
$$=h_is_i-h_i+b_i[P(s_i \mid s_d)+2P(s_i+1 \mid s_d)+3P(s_i+2 \mid s_d)+\cdots]-h_is_i-b_i[P(s_i+1 \mid s_d)+2P(s_i+2 \mid s_d)+3P(s_i+3 \mid s_d)+\cdots]$$
$$=-h_i+b_i[P(s_i \mid s_d)+P(s_i+1 \mid s_d)+P(s_i+2 \mid s_d)+\cdots]$$
$$=-h_i+b_i\sum_{k=0}^{\infty}P(s_i+k \mid s_d)$$

Thus if $TC_i(s_i-1 \mid s_d) > TC_i(s_i \mid s_d)$ is to be satisfied,

$$b_i\sum_{k=0}^{\infty}P(s_i+k \mid s_d) > h_i$$
$$\Rightarrow \sum_{k=0}^{\infty}P(s_i+k \mid s_d) > h_i/b_i \qquad \text{(A2)}$$

When (A2) is satisfied by s_i and s_i-1, $TC_i(s_i-2) > TC_i(s_i)$ is also satisfied. This applies to $s_i-3, s_i-4, s_i-5, \cdots$

Thus s_i minimizing $TC_i(s_i \mid s_d)$ should satisfy $\Sigma_{k=1}^{\infty}P(s_i+k \mid s_d) < h_i/b_i < \Sigma_{k=1}^{\infty}P(s_i+k \mid s_d)$.

3. Proof of Lemma 1

We note that in proof of Theorem 2 that $TC_i(s_i-l_2 \mid s_d) > TC_i(s_i \mid s_d)$ for $l_2 > 0$ can not be satisfied when $h_i/b_i > 1$. Thus the point with the minimum $TC_i(s_i \mid s_d)$ value is zero.

4. Proof of Theorem 3

When, $s_i^* \geq \bar{s}_i$, inventory level with minimum cost simultaneously satisfies the minimum required fill rate. Thus the desired inventory level is s_i^*. On the other hand, if the reverse is true, inventory level with minimum cost can not satisfy the minimum required fill rate and the level should be increased up to the level $\bar{s}_i$, where required fill rate starts to be satisfied. Therefore the spare inventory level to achieve the minimum fill rate at minimum cost is $\max\{s_i^*, \bar{s}_i\}$.

5. Proof of Theorem 4

A lower bound of $\Sigma_{i=1}^{I}TC_i^*(s_i \mid s_d)$ is the value when $s_d \to \infty$. Thus the maximum reduction in $TC(S)$ by incrementing s_d by one is at most the difference between current value and the lower bound, i.e., $\Sigma_{i=1}^{I}TC_i^*(s_i)-\underline{TC}$. Since the additional holding cost for increasing depot stock by one is h_d, if $h_d > [\Sigma_{i=1}^{I}TC_i^*(s_i \mid s_d)-\underline{TC}]$, i.e., a cost increase due to the increase of depot stock by one unit is larger than a largest possible cost reduction by incrementing depot stock, it is not worthwhile to increase depot stock s_d by one unit. Similarly, it is not worthwhile to increase depot stock s_d by more than one unit.

REFERENCES

[1] Albright, S. C., "An Approximation to the Stationary Distribution of a Multiechelon Repairable-Item Inventory System with Finite Sources and Repair Channels," *Naval Research Logistics* Quarterly, Vol.36(1989), pp.179-195.

[2] Albright, S. C. and A. Soni, "An Approximation to the Stationary Distribution of a Multidimensional Markov Process," *IIE Transactions*, Vol.20(1989), pp.111-118.

[3] Albright, S. C. and A. Soni, "Markovian Multiechelon Repairable Inventory System," *Naval Research Logistics Quarterly*, Vol.35 (1988), pp.49-61.

[4] Feeney, G. J. and C. C. Sherbrooke, "The (*s*-1, *s*) Inventory Policy under Compound Poisson Demand," *Management Science*, Vol.12(1966), pp.391-411.

[5] Gross, D., L. C. Kioussis, and D. R. Miller, "A Network Decomposition Approach for Approximating the Steady-State Behavior of Markovian Multi-Echelon Repairable Item Inventory Systems," *Management Science*, Vol.33(1987), pp.1453-1468.

[6] Gross, D. and D. R. Miller, "Multiechelon Repairable-Item Provisioning in a Time-Varying Environment Using the Random ization Technique," *Naval Research Logistics Quarterly*, Vol.31(1984), pp.347-361.

[7] Gross, D., D. R. Miller, and R. M. Soland, "A Closed Queueing Network Model for Multi-Echelon Repairable Item Provisioning," *IIE Transactions*, Vol.15(1983), pp.344-352.

[8] Hillier, F. S. and G. J. Lieberman, *Introduction to Operations Research* (6th eds.), McGraw-Hill, New York, 1995.

[9] Kim, J. S., K. C. Shin, and H. K. Yu, "Optimal Algorithm to Determine the Spare Inventory Level for a Repairable-Item inventory," *Computers and Operations Research*, Vol.23(1996), pp.289-297.

[10] Muckstadt, J. A., "Some Approximation in Multi-Item, Multi-Echelon Inventory System for Recoverable Items," *Naval Research Logistics Quarterly*, Vol.25(1978), pp.377-394.

[11] Muckstadt, J. A., "A Model for a Multi-Item, Multi-Echelon, Multi-Indenture Inventory System," *Management Science*, Vol.20(1973), pp.472-481.

[12] Muckstadt, J. A. and L. J. Thomas, "Are Multi-Echelon Inventory Methods Worth Implementing in Systems with Low-Demand-Rate Items?," *Management Science*, Vol.26 1980), pp.483-494.

[13] Sherbrooke, C. C., "METRIC : A Multi-Ehelon Technique for Recoverable Item Control," *Operations Research*, Vol.16(1968), pp.122-141.