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PERMANENTS OF DOUBLY STOCHASTIC FERRERS MATRICES

SUK-GEUN HWANG AND SUNG-SOO PYO

ABSTRACT. The minimum permanent and the set of minimizing matrices over the face of the polytope Ω_n of all doubly stochastic matrices of order n determined by any staircase matrix was determined in [4] in terms of some parameter called frame. A staircase matrix can be described very simply as a Ferrers matrix by its row sum vector. In this paper, some simple exposition of the permanent minimization problem over the faces determined by Ferrers matrices of the polytope of Ω_n are presented in terms of row sum vectors along with simple proofs.

1. Introduction

For an $n \times n$ matrix $A = [a_{ij}]$, the *permanent* of A, per A, is defined by

$$\mathrm{per}A = \sum_{\sigma \in S_n} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

where S_n stands for the symmetric group on the set $\{1, 2, \cdots, n\}$. Let Ω_n denote the polytope consisting of all $n \times n$ doubly stochastic matrices. For an $n \times n$ (0,1)-matrix $D = [d_{ij}]$ with per$D \neq 0$, let $\Omega(D) = \{X \in \Omega_n | X \leq D\}$ where $X \leq D$ denotes that every entry of X is less than or equal to the corresponding entry of D. Then $\Omega(D)$ forms a face of Ω_n and every face of Ω_n is defined in this fashion [1]. Let $\mu(D)$ denote the minimum permanent over $\Omega(D)$. A matrix $A \in \Omega(D)$ is called a *minimizing matrix* over $\Omega(D)$ if per$A = \mu(D)$. A square (0,1)-matrix D

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is called a *staircase matrix* if it is partitioned as

$$D = \begin{bmatrix} D_{11} & D_{12} & \cdots & D_{1k} \\ D_{21} & D_{22} & \cdots & D_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ D_{k1} & D_{k2} & \cdots & D_{kk} \end{bmatrix}$$

where D_{ij} is a zero matrix if $i < j$ and D_{ij} is an all 1's matrix if $i \geq j$. Suppose that D_{ij} is of size $p_i \times q_j$ for all $i, j = 1, 2, \cdots, k$. For a $(0,1)$-matrix D, the matrix Λ_D defined by

$$\Lambda_D = \frac{1}{\text{per}D} \sum_P P,$$

where the summation runs over all permutation matrices P satisfying $P \leq D$, is the barycenter of $\Omega(D)$. In [4] the problem of minimizing the permanent function over the faces $\Omega(D)$ of Ω_n for staircase matrices D is investigated, and the minimum permanent, minimizing matrices and the barycenter of $\Omega(D)$ are determined in terms of numbers n_i and π_i defined by the numbers p_i, q_j such as

$$n_i = \sum_{j=1}^{i} q_j - \sum_{j=1}^{i-1} p_j, \quad \pi = n_i - q_i \ (i = 1, 2, \cdots, k).$$

for any staircase matrix D with the above mentioned partition. The expressions of various quantities and the proofs in [4] look rather complicated and uncomfortable to apply to some other combinatorial problems. In this paper we give some simple expressions and proofs for the results in [4], mainly in terms of row sums of the staircase matrix D considered.

2. The minimum permanent

From now on in the sequel, for an $n \times n$ matrix A, and for subsets K, L of $\{1, 2, \cdots, n\}$, let $A(K|L)$ denote the matrix obtained from A by deleting rows in K and columns in L. D is called *barycentric* if Λ_D is a minimizing matrix over $\Omega(D)$. We start this section with a lemma which is essentially the same as Lemma 3.3 of [5], and we omit the proof.

LEMMA 1. *Let D be a fully indecomposable $(0,1)$-matrix of which the first row contains exactly t 1's in the first t positions and the first t columns are identical, and let $E = D(1|1)$. Then*

(a) $\mu(D) = \left(\frac{t-1}{t}\right)^{t-1} \mu(E)$.

(b) *D is barycentric if and only if E is barycentric.*

Let $r_1, r_2, \cdots, r_n$ be integers with $r_1 \le r_2 \le \cdots \le r_n$. The $n \times n$ (0,1)-matrix $A = [a_{ij}]$ defined by $a_{ij} = 1$ if and only if $1 \le j \le r_i$ ($i = 1, 2, \cdots, n$) is a *Ferrers matrix* and is denoted by $F(r_1, r_2, \cdots, r_n)$. An $n \times n$ staircase matrix with row sum vector $(r_1, r_2, \cdots, r_n)$ is just the Ferrers matrix $F(r_1, r_2, \cdots, r_n)$. It is well known (see Corollary 7.2.6 of [1] for example) that

$$\mathrm{per}F(r_1, r_2, \cdots, r_n) = \prod_{i=1}^{n} \max\{r_i - i + 1, 0\}.$$

If $F(r_1, r_2, \cdots, r_n)$ is fully indecomposable, then, clearly,

$$\mathrm{per}F(r_1, r_2, \cdots, r_n) = \prod_{i=1}^{n} (r_i - i + 1).$$

In the sequel, let φ denote the function defined on $\{1, 2, \cdots\}$ by

$$\varphi(k) = \left(\frac{k-1}{k}\right)^{k-1}$$

with the conventional assumption that $0^0 = 1$. In the following theorem, the minimum permanent over $\Omega(D)$, for a Ferrers matrix D, is given in terms of row sums. It is clear from Lemma 1 that a Ferrers matrix is barycentric.

THEOREM 2. *Let $D = F(r_1, r_2, \cdots, r_n)$ be fully indecomposable. Then*

$$\mu(D) = \prod_{i=1}^{n} \varphi(r_i - i + 1).$$

Proof. We use induction on n. If $n = 2$, then $D = F(2, 2)$. We know that $\mu(F(2, 2)) = 1/2$. On the other hand,

$$\prod_{i=1}^{2} \varphi(r_i - i + 1) = \varphi(2)\varphi(1) = \left(\frac{1}{2}\right)^1 \left(\frac{0}{1}\right)^0 = \frac{1}{2},$$

and the induction starts. Let $n > 2$. Let $E = D(1|1)$. Then $E = F(r_2 - 1, r_3 - 1, \cdots, r_n - 1)$. Now, by induction, we have

$$\mu(E) = \prod_{i=1}^{n-1} \varphi(r_{i+1} - 1 - i + 1) = \prod_{i=2}^{n} \varphi(r_i - i + 1).$$

Thus, by Lemma 1, it follows that

$$\mu(D) = \varphi(r_1)\mu(E) = \prod_{i=1}^{n} \varphi((r_i - i + 1),$$

and the proof is complete. □

3. The barycenter of $\Omega(F(r_1, r_2, \cdots, r_n))$

By Lemma 1 (b) and by induction, we see that a Ferrers matrix is barycentric. In what follows we give a formula for the entries of the barycenter of $\Omega(F(r_1, r_2, \cdots, r_n))$, in terms of the row sums $r_1, r_2, \cdots, r_n$. We first prove the following

LEMMA 3. *Let D be a fully indecomposable $(0,1)$-matrix of which the first row contains exactly t 1's in the first t positions and the first t columns are identical, and let $E = D(1|1)$. Then*

$$\Lambda_E = [\Lambda_D(1|1)] \left(\frac{t}{t-1} I_{t-1} \oplus I_{n-1} \right).$$

Proof. Let $\Lambda_D = [\beta_{ij}]$, and for the consistency of positions with those of D, let the rows and columns of E and Λ_E be indexed by $2, \cdots, n$ as

$$(1) \qquad E = \begin{bmatrix} e_{22} & e_{23} & \cdots & e_{2n} \\ e_{32} & e_{33} & \cdots & e_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ e_{n2} & e_{n3} & \cdots & e_{nn} \end{bmatrix}, \ \Lambda_E = \begin{bmatrix} \gamma_{22} & \gamma_{23} & \cdots & \gamma_{2n} \\ \gamma_{32} & \gamma_{33} & \cdots & \gamma_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{n2} & \gamma_{n3} & \cdots & \gamma_{nn} \end{bmatrix}.$$

We make use of the following well known formula for β_{ij} and γ_{ij};

$$(2) \qquad \beta_{ij} = \frac{d_{ij}}{\text{per}D}\text{per}D(1|1), \ \gamma_{ij} = \frac{e_{ij}}{\text{per}E}\text{per}E(1|1).$$

Notice that

$$(3) \qquad \text{per}D = t \ \text{per}E.$$

Let (i, j) be such that $e_{ij} = 1$ or, equivalently, $d_{ij} = 1$. We show that

$$\gamma_{ij} = \begin{cases} \beta_{ij} & \text{, if } j > t, \\ \dfrac{t}{t-1}\beta_{ij}, & \text{if } j \leq t. \end{cases}$$

Suppose first that $j > t$. Then the matrix $D(i|j)$ still has the property that the first t columns are identical and the first row has t 1's in the first t positions, so that

$$\begin{aligned} \text{(4)} \qquad \text{per}D(i|j) &= \sum_{p=1}^{t} \text{per}D(1, i|p, j) \\ &= t \; \text{per}D(1, i|p, j) = t \; \text{per}E(i|j). \end{aligned}$$

Thus $\beta_{ij} = \gamma_{ij}$ by (2) and (3). Suppose now that $j \leq t$. Since the first t columns of D are identical and the first row of D has t 1's in the first t positions, we have

$$\begin{aligned} \text{per}D(i|j) = \text{per}D(i|1) &= \sum_{p=2}^{t} \text{per}D(1, i|1, p) \\ &= (t-1)\text{per}D(1, i|1, 2) = (t-1)\text{per}E(i|2) \\ &= (t-1)\text{per}E(i|j). \end{aligned}$$

Thus by (2) and (3) again, we have $\gamma_{ij} = t\;\beta_{ij}/(t-1)$ and we are done. □

For a fully indecomposable Ferrers matrix $F = F(r_1, r_2, \cdots, r_n) = [f_{ij}]$, and for a position (p, q), let

$$\text{(5)} \qquad \epsilon_F(q) = \min\{i | f_{iq} = 1\}.$$

Then clearly $\epsilon_F(q) < p$. Let $\Delta_{p,q}(F)$ denote the matrix obtained from F by applying the following two operations consecutively;

(i) Replace row i by a zero vector if $i \notin \{\epsilon_F(q), \epsilon_F(q) + 1, \cdots, p\}$,

(ii) Replace each of the entries below the main diagonal by a 0.

With the above notations in mind, we now prove the following

THEOREM 4. *Let* $D = F(r_1, r_2, \cdots, r_n) = [d_{ij}]$, *and let* $\Lambda_D = [\beta_{ij}]$. *Then*

$$\text{(6)} \qquad \beta_{pq} = \frac{d_{pq}}{r_p - p} \prod_{i=\epsilon_F(q)}^{p} \frac{r_i - i}{r_i - i + 1},$$

for $p, q = 1, 2, \cdots, n$, where $\epsilon_D(q)$ is the number defined by (4).

Proof. Let (p, q) be given and fixed. If $d_{pq} = 0$, then certainly $\beta_{pq} = 0$. So, suppose that $d_{pq} = 1$ which is equivalent to that $q \leq r_p$. For the proof of this theorem, for a fully indecomposable Ferrers matrix $F = F(t_1, t_2, \cdots, t_n)$, let $f_{p,q}(F)$ denote the number defined by

$$f_{p,q}(F) = \prod_{i=\epsilon_F(q)}^{p} \frac{t_i - i}{t_i - i + 1}.$$

To prove (5), we may show that $\beta_{pq} = f_{p,q}(D)/(r_p - p)$, instead. Let $(s_1, s_2, \cdots, s_n)$ be the row sum vectors of $\Delta_{p,q}(D)$. Then $s_i = r_i - i + 1$ for i with $s_i \neq 0$. Hence, it suffices to show that $\beta_{pq} = f_{p,q}(D)/(s_p - 1)$. Note also that

$$f_{p,q}(D) = \prod_{i=\epsilon_D(q)}^{p} \frac{s_i - 1}{s_i}.$$

CASE (i): $p = 1$. Clearly, $\beta_{1q} = 1/r_1$, because the first row of D equals $(1, \cdots, 1, 0, \cdots, 0)$ in which the number of 1's is r_1, and each of the first r_1 columns of D is the all 1's vector. On the other hand, we have

$$\Delta_{1,q}(D) = \begin{bmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix},$$

where the first row has sum r_1. Thus it follows that $f_{p,q}(D) = (r_1 - 1)/r_1 = (s_1 - 1)/s_1$ so that $\beta_{1q} = f_{p,q}(D)/(s_1 - 1)$, and we are done for this case.

CASE (ii): $p \geq 2$. Let $E = D(1|1)$ and let $\Lambda_E = [\gamma_{ij}]$. For the consistency of positions with those of D again, let E and Λ_E be indexed by $2, \cdots, n$ as in (1). Let $(s_1, s_2, \cdots, s_n)$ and $(x_2, \cdots, x_n)$ be row sum vectors of $\Delta_{p,q}(D)$ and $\Delta_{p,q}(E)$ respectively. Then clearly $(x_2, \cdots, x_n) = (s_2, \cdots, s_n)$. We divide the proof into the following two subcases.

Subcase. 1: $r_1 < q$. In this case $s_1 = 0$, and $\epsilon_D(q) = \epsilon_E(q)$, so that $f_{p,q}(D) = f_{p,q}(E)$. By induction we get $\gamma_{pq} = f_{p,q}(E)/(x_p - 1) = f_{p,q}(D)/(s_p - 1)$. Since $\beta_{pq} = \gamma_{pq}$ by Lemma 3, the assertion of Theorem 4 for this case follows.

Subcase. 2: $r_1 \geq q$. In this case $\epsilon_D(q) = 1$ and $\epsilon_D(q) = 2$, and

$$f_{p,q}(D) = \prod_{i=1}^{p} \frac{s_i - 1}{s_i} = \frac{s_1 - 1}{s_1} \prod_{i=2}^{p} \frac{x_i - 1}{x_i} = \frac{s_1 - 1}{s_1} f_{p,q}(E).$$

By Lemma 3 and by induction, we have

$$\beta_{pq} = \frac{r_1 - 1}{r_1}\gamma_{pq} = \frac{s_1 - 1}{s_1}\gamma_{pq} = \frac{s_1 - 1}{s_1}\frac{f_{p,q}(E)}{x_p - 1} = \frac{f_{p,q}(D)}{s_p - 1},$$

and we are done. □

EXAMPLE 1. Let $D = F(2, 3, 5, 6, 6, 8, 8, 8)$, *i.e.*,

$$D = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} = [d_{ij}]. \tag{7}$$

Let us find, say, the (6, 4)-entry β_{64} of the barycenter Λ_D of $\Omega(D)$. We take $p = 6, q = 4$. Looking up the column $q(= 4)$ of D, we see that the smallest number i such that $d_{i6} \neq 0$ is 3 so that $\epsilon_D(q) = 3$. Replacing the rows i in the set $\{1, 2, \cdots, 8\} - \{k | \epsilon_D(q) \leq k \leq p\} = \{1, 2, 7, 8\}$ by zero vectors, we get

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Finally replace each of the 1's below the main diagonal of this matrix by a 0, to get

$$\Delta_{6,4}(D) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & 1 & 1 & 1 & 0 & 0 & 0 \\ & & & 1 & 1 & 1 & 0 & 0 \\ & & & & 1 & 1 & 0 & 0 \\ & & & & & 1 & 1 & 1 \\ & & & & & & 0 & 0 \\ & & & & & & & 0 \end{bmatrix}.$$

We see that $(s_3, s_4, s_5, s_6) = (3, 3, 2, 3)$, and

$$\beta_{64} = \frac{1}{s_6 - 1} \prod_{i=3}^{6} \frac{s_i - 1}{s_i} = \frac{1}{2} \times \frac{2}{3}\frac{2}{3}\frac{1}{2}\frac{2}{3} = \frac{2}{27}.$$

In this way Λ_D can be calculated as

$$\Lambda_D = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/4 & 1/4 & 1/2 & 0 & 0 & 0 & 0 & 0 \\ 1/12 & 1/12 & 1/6 & 1/3 & 1/3 & 0 & 0 & 0 \\ 1/18 & 1/18 & 1/9 & 2/9 & 2/9 & 1/3 & 0 & 0 \\ 1/18 & 1/18 & 1/9 & 2/9 & 2/9 & 1/3 & 0 & 0 \\ 1/54 & 1/54 & 1/27 & 2/27 & 2/27 & 1/9 & 1/3 & 1/3 \\ 1/54 & 1/54 & 1/27 & 2/27 & 2/27 & 1/9 & 1/3 & 1/3 \\ 1/54 & 1/54 & 1/27 & 2/27 & 2/27 & 1/9 & 1/3 & 1/3 \end{bmatrix}. \tag{8}$$

We close this section with a property of the barycenter of $\Omega(F(r_1, r_2, \cdots, r_n))$. The following lemma is due to Foregger [3].

LEMMA 5. *Let $D = [d_{ij}]$ be a fully indecomposable square $(0,1)$-matrix and let $A = [a_{ij}]$ be a minimizing matrix over $\Omega(D)$. Then A is fully indecomposable and, for (i, j) such that $d_{ij} = 1$, $\mathrm{per}A(i|j) \geq \mathrm{per}A$ with equality if $a_{ij} > 0$.*

THEOREM 6. *Let $D = F(r_1, r_2, \cdots, r_n) = [d_{ij}]$. Suppose that $r_i = i + 1$ for some i and let t be the smallest of such i's so that Λ_D has the form*

$$\Lambda_D = \begin{bmatrix} U & O \\ X & G \end{bmatrix} \tag{9}$$

with U being of size $t \times (t+1)$. Let U_j be the matrix obtained from U by deleting the column j for each $j = 1, 2, \cdots, t+1$, then the matrices $U_1, U_2, \cdots, U_{t+1}$ have the same permanent.

Proof. We use induction on n. If $n = 1$, the theorem clearly holds. If the zero submatrix in the upper right corner of (8) is vacuous, then U is of size $(n-1) \times n$ and

$$\Lambda_D = \begin{bmatrix} U \\ \mathbf{x} \end{bmatrix}.$$

Since $\mathbf{x} > \mathbf{0}$ and since $U_j = \Lambda_D(n|j)$ for $j = 1, 2, \cdots, n$, the assertion of the theorem for this case follows by Lemma 5 because a Ferrers is barycentric. Suppose that the zero submatrix in (8) is not vacuous. Let s be the number of 1's in the last column of D and let $E = D(n|n)$. Then, by Lemma 3, we have

$$\Lambda_E = \left(I_{n-s} \oplus \frac{s}{s-1} I_s \right) [\Lambda_D(n|n)].$$

Since the row t of E still has sum $r_t = t + 1$, and since Λ_E has the form

$$\Lambda_E = \begin{bmatrix} U & O \\ * & * \end{bmatrix},$$

the assertion of the theorem for this case follows by induction. □

4. The structure of minimizing matrices over $\Omega(F(r_1, r_2, \cdots, r_n))$

In this section, we give a description of the structure of minimizing matrices over $\Omega(D)$ for fully indecomposable Ferrers matrices D in terms of row sums. The following lemma is due to Minc [6].

LEMMA 7. *Let D be a $(0,1)$-matrix of which the first t columns are identical, and let A be a minimizing matrix over $\Omega(D)$. Then the matrix obtained from A by replacing each of the first t columns by the average of those is also a minimizing matrix over $\Omega(D)$. A similar statement holds for rows.*

In the sequel, for a fully indecomposable Ferrers matrix $F = F(s_1, s_2, \cdots, s_n)$, let $\mathfrak{R}(F)$ denote the set of all $G = [g_{ij}] \in \Omega(F)$ with the property that g_{ij} equals the (i,j)-entry of Λ_F for every $(i,j) \notin \bigcup_{k \in T}(\{k+1, k+2, \cdots, n\} \times \{1, 2, \cdots, k+1\})$ where $T = \{k | s_k = k+1 < n\}$, and let $\mathbf{Min}(D)$ denote the set of all minimizing matrices over $\Omega(D)$.

THEOREM 8. *Let $D = F(r_1, r_2, \cdots, r_n)$ be fully indecomposable. Then $\mathbf{Min}(D) = \mathfrak{R}(D)$.*

Proof. Let $E = D(1|1)$ and let $A \in \Omega(D)$ be given. Let A_1 be the matrix obtained from A by replacing each of the first r_1 columns by the average of those columns. Then

$$\text{per}A_1 = \text{per}A_1(1|1), \tag{10}$$

because the first row of A_1 equals $(1/r_1, \cdots, 1/r_1, 0, \cdots, 0)$ and the first r_1 columns of A_1 are identical. Let B be the matrix obtained from $A_1(1|1)$ by multiplying each of the first $r_1 - 1$ columns by $r_1/(r_1 - 1)$. Then we get, by (9),

$$\text{per}B = \left(\frac{r_1}{r_1 - 1}\right)^{r_1 - 1} \text{per}A_1. \tag{11}$$

We also have, by Lemma 3, that

$$A \in \mathfrak{R}(D) \text{ if and only if } B \in \mathfrak{R}(E). \tag{12}$$

Suppose that $A \in \mathbf{Min}(D)$. Then it follows that $A_1 \in \mathbf{Min}(D)$ by Lemma 7, and hence that $B \in \mathbf{Min}(E)$ by Lemma 1 and (9). Now, by induction, we have that $B \in \mathfrak{R}(E)$ so that $A \in \mathfrak{R}(D)$ by (11). Conversely, suppose that $A \in \mathfrak{R}(D)$. If $\bigcup_{k \in T}(\{k+1, t+2, \cdots, n\} \times \{1, 2, \cdots, k+1\}) = \{1, 2, \cdots, n\} \times \{1, 2, \cdots, n\}$, then $A = \Lambda_D$, and $A \in \mathbf{Min}(D)$. If not, let t be the smallest number in T. We have that $B \in \mathfrak{R}(E)$ by (11) and hence that $B \in \mathbf{Min}(E)$ by induction. Hence by (10) and Lemma 1, it follows that $A_1 \in \mathbf{Min}(D)$. The matrix Λ_D has the form

$$\Lambda_D = \begin{bmatrix} U & O \\ * & * \end{bmatrix},$$

with U being of size $t \times (t+1)$. In accordance with this, the matrices A and A_1 have the form

$$A = \begin{bmatrix} U & O \\ X & G \end{bmatrix}, \; A_1 = \begin{bmatrix} U & O \\ Y & G \end{bmatrix}.$$

For $j = 1, 2, \cdots, t+1$, let $\mathbf{x}_j$ and $\mathbf{y}_j$ be the j th columns of X and Y respectively, and let U_j be the matrix obtained from U by deleting the jth column. Then both $\mathbf{x}_1 + \mathbf{x}_2 + \cdots + \mathbf{x}_{t+1}$ and $\mathbf{y}_1 + \mathbf{y}_2 + \cdots + \mathbf{y}_{t+1}$ are equal to a same vector, say $\mathbf{z}$. We have

$$\begin{aligned}\mathrm{per}A &= \sum_{j=1}^{t+1}(\mathrm{per}U_j)(\mathrm{per}[\mathbf{x}_j, G]) = (\mathrm{per}U_1)\sum_{j=1}^{t+1}\mathrm{per}[\mathbf{x}_j, G] \\ &= (\mathrm{per}U_1)(\mathrm{per}[\mathbf{z}, G]),\end{aligned}$$

where the second equality is due to Theorem 6. Similarly, $\mathrm{per}A_1 = (\mathrm{per}U_1)(\mathrm{per}[\mathbf{z}, G])$ so that $\mathrm{per}A = \mathrm{per}A_1$. Therefore we have $A \in \mathbf{Min}(D)$, and the proof is complete. □

EXAMPLE 2. Let D be the matrix in (6). Since $(r_1, r_2, \cdots, r_8) = (2, 3, 5, 6, 6, 8, 8, 8)$ we see that $T = \{1, 2, 5\}$, and the positions in $\bigcup_{k\in T}(\{k+1, k+2, L, n\} \times \{1, 2, L, k+1\})$ are those astrisked in the following matrix,

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & 1 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & 1 & 1 & 0 & 0 & 0 \\ * & * & * & 1 & 1 & 1 & 0 & 0 \\ * & * & * & 1 & 1 & 1 & 0 & 0 \\ * & * & * & * & * & * & 1 & 1 \\ * & * & * & * & * & * & 1 & 1 \\ * & * & * & * & * & * & 1 & 1 \end{bmatrix}.$$

In view of (7) and Theorem 8, we conclude that a minimizing matrix over $\Omega(D)$ is a doubly stochastic matrix of the form

$$\begin{bmatrix} 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & 1/2 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & 1/3 & 1/3 & 0 & 0 & 0 \\ * & * & * & 2/9 & 2/9 & 1/3 & 0 & 0 \\ * & * & * & 2/9 & 2/9 & 1/3 & 0 & 0 \\ * & * & * & * & * & * & 1/3 & 1/3 \\ * & * & * & * & * & * & 1/3 & 1/3 \\ * & * & * & * & * & * & 1/3 & 1/3 \end{bmatrix}.$$

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Department of Mathematics Education
Kyungpook National University
Taegu 702-701, Korea
E-mail: sghwang@kyungpook.ac.kr
E-mail: sspyo@gauss.kyungpook.ac.kr