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THE LEAST NUMBER OF COINCIDENCES WITH A COVERING MAP OF A POLYHEDRON

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ABSTRACT. We define the coincidence index of pairs of maps p, $f : \tilde{X} \to X$ where p is a covering of a polyhedron X. We use a polyhedral transversality Theorem due to T. Plavchak. When $p = identity$ we get the classical fixed point index of self map of polyhedra without using homology.

1. Introduction

The fixed point index can be defined for self maps of quite arbitrary spaces (for example, ANR's [4]). In contrast, to define the coincidence index of a pair of maps $g, f : M \to N$ one has to assume that M and N are manifolds [15, 2, 9]. Since the fixed points may be regarded as coincidences of the pair $id, f : X \to X$, the following question arises: what one has to assume on the map g to get a coincidence index on a larger class of spaces than manifolds? In this paper we consider a pair of maps $p, f : \tilde{X} \to X$ where p is a covering map of a polyhedron X and we construct the coincidence index of this pair. The most natural method, converting the formalism of [4], cannot be applied here since the domain does not coincide with the range space and the commutativity property cannot be reformulated into our situation. We bypass this difficulty applying a version of the transversality theorem (Theorem B in [13]) given below as Theorem 1.1.

In this paper by a polyhedron we will mean a polyhedron which can be imbedded as a closed subset of a Euclidean space $\mathbb{R}^n$ (unless other

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stated), i.e., finitely dimensional, locally finite and having at most countable number of simplices in a (any) triangulation (Thm. 3.2.9 in [16]). Simplex denotes open simplex.

Let $p : \tilde{X} \to X$ be a covering of a polyhedron. We fix a triangulation K of X. It induces the triangulation $\tilde{K}$ of $\tilde{X}$. Then we regard $\tilde{X} \times I$ with a triangulation $\tilde{K}'$ without adding new vertices and then we consider a subdivision $\tilde{K}''$ of $\tilde{K}'$ [14]. Let $W \subset \tilde{X} \times I$ be a compact subpolyhedron in $\tilde{K}''$. Define $\pi : \tilde{X} \times I \to X$ by $\pi(\tilde{x}, t) = p\tilde{x}$.

THEOREM 1.1. (compare Thm. B in 1.1) *Any PL-map $f : W \to X$ (with the respect to triangulations $\tilde{K}''$ and K) with $C(\pi, f) \subset \operatorname{int} W$ can be deformed rel bdW to a PL-map f' such that $C(p, f')$ is 1-dimensional PL-manifold. If moreover $C(p, f(.,0)) \cup C(p, f(.,1))$ is finite and contained inside simplices of maximal dimension then the above deformation may be also constant on $W \cap (\tilde{X} \times \{0,1\})$.*

Proof. Theorem B in [13] implies the above for $\tilde{X} = X$ a compact polyhedron and $W = X \times I$. The idea of that proof is to push the coincidences (called there fixed points) off lower dimensional skeleta of $X \times I$. This is obtained in many steps but in each step the given map is deformed inside the star of a simplex containing a coincidence. This procedure can be applied in our situation. First we choose a subdivision of $\tilde{X} \times I$ so fine that the star of any simplex, containing a concidence, is contained in W. On the other hand since this star is small, it is contained in the preimage $\pi^{-1}U$ of an open contractible subset of $U \subset X$. But this preimage splits into components each of them mapped by π as the product map onto U ($\pi : \pi^{-1}U = \bigcup U_i \times I \to U$). Now we may apply locally each step of Plavchak construction moving the coincidences into higher dimensional simplices. Since the subdivision of W is fine, the carrier of this deformation is contained inside W.

If $C(p, f(.,0)) \cup C(p, f(.,1))$ is finite and contained inside maximal simplices then we may assume that $f(x,t)$ does not depend on t for $0 \le t \le \epsilon$ and for $1 - \epsilon \le t \le 1$ for an $\epsilon > 0$. Since then the coincidence set is good near $W \cap (\tilde{X} \times \{0,1\})$ the construction can be done inside $W \cap \tilde{X} \times (\epsilon', 1-\epsilon'))$ (for an $\epsilon' \in (0,\epsilon)$). □

Now suppose that $f : W \to X$ satisfies the above theorem. Let J be a component of the 1-manifold $C(\pi, f) \subset \operatorname{int} W$. Let us fix an orientation (direction) of J. Let $(x_0, t_0) \in J$ be a point lying in a maximal simplex $\tilde{\sigma} \in \tilde{K}''$. Denote $\dim \tilde{\sigma} = q + 1$ and let $\sigma \in K$ be the simplex for which

$\pi(\tilde{\sigma}) \subset \sigma$. Since $\tilde{\sigma}$ is maximal and contains a coincidence, $\dim \sigma = q$ and $f(\tilde{\sigma}) = \sigma$. Let D_0 be a q-disk transverse to $J \cap \tilde{\sigma} \subset \tilde{\sigma}$ at the point x_0. We define the coincidence index $\mathrm{ind}(\pi_|, f_|; x_0)$ of the pair $\pi_|, f_| : D_0 \to \sigma$ as the degree of $D_0 \ni (x,t) \to p(x) - f(x,t) \in \sigma$ (we fix an orientation of σ which determines the orientation of $\sigma \times I$ and we orient D_0 to get the equality orientation $D_0 +$ orientation $J =$ orientation $\sigma \times I$).

LEMMA 1.2. *The coincidence index* $ind(\pi_{D_0}, f_{D_0}; x_0)$ *of the pair* π_{D_0}, $f_{D_0} : D_0 \to \sigma$ *does not depend on the choice of a point* $(x_0, t_0) \in J \cap$ (maximal simplex of K$''$) *where* J *is a fixed component* $C(\pi, f)$.

Proof. Since all but a finite number of points from J belong to maximal simplices, we may present $J = J_1 \cup \cdots \cup J_r$ where each J_i is closed, connected, is lying in the preimage (π^{-1}) of a contractible subpolyhedron of X and $J_i \cap J_{i+1}$ contains a point in a maximal simplex. Now it is enough to prove our claim for each J_i. Thus it remains to consider the trivial covering $\tilde{X} = X$. Then $\pi(x,t) = x$.

Let us fix a PL–imbedding $X \subset \mathbb{R}^N$ $(N \in \mathbb{N})$ and a continuous retraction $r : U \to X$ of a neighbourhood $U \subset \mathbb{R}^N$. Denote $U_0 = \{(x,t) \in U \times I; (r(x),t) \in W\}$. This is an open neighbourhood of $W \subset \mathbb{R}^N \times \mathbb{I}$. We define $f' : U_0 \to X$ by $f'(x,t) = f(r(x),t)$. Notice that $C(\pi, f') = C(\pi, f)$. We fix a point $(x_0, t_0) \in J \cap \tilde{\sigma}$ where $\tilde{\sigma} \in \tilde{K}''$ is a $q+1$-dimensional maximal simplex. Let $D \subset U_0$ be N–dimensional disk transverse to $J \subset \mathbb{R}^N \times \mathbb{R}$ at the point (x_0, t_0). Then $D_0 = D \cap \tilde{\sigma}$ is q-disk. Notice that (x_0, t_0) is the isolated coincidence point of the restrictions $\pi_D, f'_D : D \to \mathbb{R}^N$. It remains to prove:

1. $\mathrm{ind}(\pi_D, f'_D; (x_0, t_0))$ does not depend on the choice of the point $(x_0, t_0) \in J$.
2. $\mathrm{ind}(\pi_D, f'_D; (x_0, t_0)) = \mathrm{ind}(\pi_{D_0}, f'_{D_0}; (x_0, t_0))$ for $(x_0, t_0) \in J \cap \{$maximal simplex$\}$.

Ad 1. Obvious, since $J \subset \mathrm{int}\, U_0$ is connected and $\mathrm{int}\, U_0$ is open subset of $\mathbb{R}^{N+1}$.

Ad 2. Fix (x_0, t_0). First we make a local correction of the projection $r : U \to X$ which changes no side of the proved equality. Denote by V_0, $V_\perp$ and $V_J \subset \mathbb{R}^{N+1}$ the affine subspaces: generated by $D_0 = \tilde{\sigma} \cap D$, orthogonal to $\tilde{\sigma}$ at (x_0, t_0) and generated by $J \cap \tilde{\sigma}$ respectively. Their dimensions are q, $N - q$ and 1 respectively. Now we notice that for arbitrarily fixed $\epsilon, \delta > 0$ there is an ϵ-homotopy $\{r_t\}$ which is constant outside

$B((x_0,t_0);\delta) - X$ such that $r_0 = r$ and for a δ_1 ($\delta > \delta_1 > 0$) the restriction $r_1 : B((x_0,t_0);\delta_1) \to X$ is an affine map sending $B((x_0,t_0);\delta_1) \cap V_\perp$ to the point (x_0,t_0). Since the homotopy r_t is constant on X, $C(\pi, fr_t)$ is still the same and the indices do not vary. Thus we may prove the equality under the assumption that r in a neighbourhood of (x_0,t_0) is affine and $r(B((x_0,t_0);\delta_1) \cap V_\perp) = x_0$. Let $W_0, W_\perp \subset \mathbb{R}^N$ denote the subspaces tangent and orthogonal to σ respectively. Then f'_D is locally given by $V_0 \oplus V_\perp \ni (x,y) \to (fx, x_0) \in W_0 \oplus W_\perp$.

Since $\pi(V_0) \subset W_0$ and $\pi(V_\perp) \subset W_\perp$,

$$\begin{aligned}
&\operatorname{ind}(\pi_D, f'_D; (x_0,t_0)) = \deg(\pi_D - f'_D; (x_0,t_0)) \\
&= \deg(\pi_{V_0} - f'_{V_0}; (x_0,t_0)) \cdot \deg(\pi_{V_\perp} - f'_\perp; (x_0,t_0)) \\
&= \deg(\pi_{D_0} - f'_{D_0}; (x_0,t_0)) \cdot \deg(\pi_{V_\perp} - x_0; (x_0,t_0)) \\
&= \operatorname{ind}(\pi_{D_0}, f'_{D_0}; (x_0,t_0)) \cdot 1 = \operatorname{ind}(\pi_{D_0}, f'_{D_0}; (x_0,t_0)).
\end{aligned}$$

□

2. Index

Let X be a locally finite polyhedron. A point $x \in X$ is called *Euclidean* iff a neighbourhood of x in X is homeomorphic to $\mathbb{R}^n$ for some $n \in \mathbb{N}$. The set of all Euclidean points of X will be denoted by $\mathfrak{E}(X)$. Notice that in any locally finite polyhedron the subset $\mathfrak{E}(X)$ is dense in X: all points in maximal simplices (of a fixed triangulation) are Euclidean. Let $p : \tilde{X} \to X$ be a covering and let $U \subset \tilde{X}$ be an open subset. Let $f : U \to X$ be a continuous map whose coincidence set $C(p,f)$ is compact. We will define the coincidence index of such map.

The above map f will be called $\mathfrak{E}$ *– map* if $C(p,f)$ is a finite subset of $\mathfrak{E}(X)$. We define the index of an isolated coincidence point $x_0 \in C(p,f) \cap \mathfrak{E}(X)$ as $\operatorname{ind}(p, f : x_0) = \deg(p - f; x_0)$ where deg denotes the degree of the map $V' \ni x \to p(x) - f(x) \in V$ [12, 5]. Here V', V denote Euclidean neighbourhoods of $x_0 \in U, p(x_0) \in X$ respectively, satisfying $p(V') \cup f(V') \subset V$ and with orientations compatible via p. For an $\mathfrak{E}$-map f we define $\operatorname{ind}(p,f;U) = \sum_x \operatorname{ind}(p,f;x)$ where the summation runs the (finite) set $C(p,f)$.

For any continuous map $f : U \to X$ with $C(p,f)$ compact we define $\operatorname{ind}(p,f;U) = \operatorname{ind}(p,f';U)$ where f' is an $\mathfrak{E}$-map *C-compactly homotopic* to f i.e. there is a homotopy $F : U \times [0,1] \to X$ from f to f' with $C(p,F) = \{(x,t) \in U \times [0,1]; p(x) = F(x,t)\}$ compact.

LEMMA 2.1. *The above definition is correct*

Proof. We have to prove that:

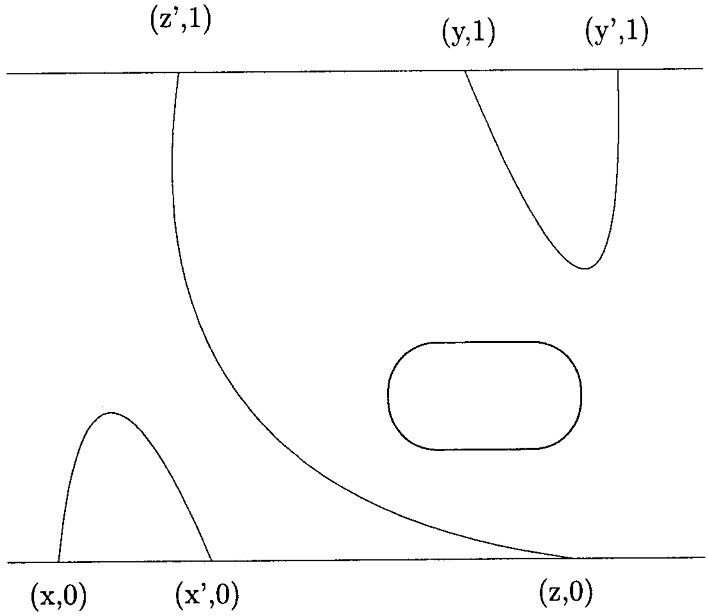
1. Any continuous map $f : U \to X$ with $C(p, f)$ compact is C-compactly homotopic to an $\mathfrak{C}$-map f_0.
2. If $f_0, f_1 : U \to X$ are C-compactly homotopic $\mathfrak{C}$-maps then $\operatorname{ind}(p, f_0; U) = \operatorname{ind}(p, f_1; U)$.

Ad 1. Let K be a triangulation of the polyhedron X and let $\tilde{K}$ be the induced triangulation of $\tilde{X}$. If K is chosen fine enough then there is a subpolyhedron $W \subset \tilde{K}$ that the star of any simplex in W containing a coincidence is contained inside W. We may apply the Hopf construction to the restriction $f : W \to X$ [1]. Recall that this construction deforms f to a map with finite number of fixed points each in a maximal simplex by pushing the fixed point set off the lower dimensional skeleta. It is done in many steps but in each step the given map is deformed only in the star of a simplex containing a fixed point. Since the restriction of p to any star is a homeomorphism hence we may adapt the Hopf construction to our situation. By the above, the final deformation is constant outside the interior of W hence it can be extended by the constant homotopy onto the whole U.

Ad 2. Fix such triangulation K of X that all the points of the (finite) set $p(C(p, f_0) \cup C(p, f_1)) \subset \mathfrak{C}(X)$ lie inside maximal simplices $\sigma_1, \cdots, \sigma_r$ of K. There is a homotopy, constant outside $\bigcup_{i=0}^{r} \sigma_i$, from f_0 to a $PL-$map f_0' with fixed point set finite and contained in $\bigcup_{i=0}^{r} \sigma_i$ (first we replace f_0 by PL-map with no coincidences outside $\bigcup_{i=0}^{r} \sigma_i$ and then we apply Hopf construction). Notice that $\operatorname{ind}(p, f_0) = \operatorname{ind}(p, f_0')$.

We replace f_1 by a PL-map f_1' with similar properties. It remains to show that $\operatorname{ind}(p, f_0') = \operatorname{ind}(p, f_1')$.

Since f_0', f_1' are $C-$compactly homotopic PL-maps, there exists a $C-$compact PL-homotopy F. Now the compact set $C(p, F)$ is contained inside a compact subpolyhedron $W \subset U$. We apply Theorem 1.1 to make $C(p, F)$ a 1-manifold. Figure 1 illustrates the possible positions of the components of $C(p, F)$ in $X \times I$.



Then by lemma 1.2:

$$\operatorname{ind}(p, f_0'; (x,0)) = -\operatorname{ind}(p, f_0'; (x',0)),$$

$$\operatorname{ind}(p, f_1'; (y,1)) = -\operatorname{ind}(p, f_1'; (y',1)),$$

$$\operatorname{ind}(p, f_0'; (z,0)) = \operatorname{ind}(p, f_1'; (z',1)).$$

Summing over all components we get the desired equality $\operatorname{ind}(p, f_0'; U) = \operatorname{ind}(p, f_1'; U)$. □

REMARK 2.2. If X is a manifold then the above definition gives the coincidence index (see [3] oriented and [7] nonoriented case) of the pair p, f.

REMARK 2.3. If X is a polyhedron and $p = id$ then the above definition gives the ordinary fixed point index [4]. The above may be also regarded as the alternative definition which does not use homology. Recall the scheme: we base on the definition of the degree of a map $\phi : U \to \mathbb{R}^n$ ($U \subset \mathbb{R}^n$ open subset) [12], [5]. This allows to define the fixed point index of $\mathfrak{C}$-selfmaps of a polyhedron. Now Theorem 1.1, Hopf construction and Lemma 1.2 allow to extend the definition on all selfmaps.

Properties

ADDITIVITY. $p : \tilde{X} \to X$ a covering of polyhedron, $U \subset \tilde{X}$ open subset, $f : U \to X$ a map with $C(p, f)$ compact. Suppose that $U = U_1 \cup \cdots \cup U_k$and $U_i \cap C(p, f)$ are compact mutually disjoint sets. Then $\mathrm{ind}(p, f; U) = \sum_i \mathrm{ind}(p, f; U_i)$.

UNITS. Let $f_0 : U \to X$ denote the constant map into $x_0 \in X$. Then $\mathrm{ind}(p, f_0; U) = \#p^{-1}(x_0) \cap U$. In particular if $U = \tilde{X}$ then $\mathrm{ind}(p, f_0; U) =$ multiplicity of the covering p.

FIXED POINTS. $\mathrm{ind}(p, f : U) \neq 0$ implies $C(p, f) \neq \emptyset$.

HOMOTOPY INVARIANCE. Let $U \subset \tilde{X} \times [0,1]$ be an open subset and $F : U \to X$ a continuous map with $C(p, F)$ compact. Denote $U_t = \{x \in \tilde{X}; (x,t) \in U\}$, $f_t : U_t \to X$, $f_t(x) = F(x,t)$ $(0 \leq t \leq 1)$. Then $\mathrm{ind}(p, f_0; U_0) = \mathrm{ind}(p, f_1; U_1)$.

Proof. As in the proof of lemma 1.2 we may assume that $C(p, F)$ is 1-manifold. □

3. Lefschetz formula

We will show that the global index of the pair $p, f : \tilde{X} \to X$ is equal to the trace of an induced homomorphism on homology. Here we will consider homology groups with rational coefficients. We use the method from [8]. Let start by recalling:

LEMMA 3.1. *Let K denote a triangulation of a compact polyhedron W and let K' be its subdivision. Then the homomorphism $\rho : C_r(K) \to C_r(K')$ given by $\rho(\sigma) = \sum \sigma_i$ induces the isomorphism of homology groups (the right hand side denotes the sum of all simplices of K' contained in σ with the orientation inherited after σ).*

Let $p : \tilde{X} \to X$ be a finite covering of a polyhedron. Let K denote a fixed triangulation of X, $\tilde{K}$ the induced triangulation of $\tilde{X}$, $\tilde{K}'$ a subdivision of $\tilde{K}$ and $f : \tilde{X} \to X$ a PL map (with the respect to $\tilde{K}'$ and K) for which $C(p, f)$ is compact. By the Hopf construction we may assume that $C(p, f)$ is finite and contained inside maximal simplices. Taking the subdivision $\tilde{K}'$ fine enough we may assume that if $\tilde{\sigma}' \subset \tilde{\sigma}$ and f maps $\tilde{\sigma}'$ onto $p\tilde{\sigma}$ then f is expanding there [10].

We consider the composition

$$C_*(\tilde{K}) \xrightarrow{\rho} C_*(\tilde{K}') \xrightarrow{f_*} C_*(K) \xrightarrow{t} C_*(\tilde{K})$$

where $t : C_*(K) \to C_*(\tilde{K})$ denotes the transfer map $t(\sigma) = \sum \tilde{\sigma}_i$ (sum of all lifts of σ) and f_* denotes also the chain map induced by the PL-map f. Let us fix simplices $\tilde{\sigma} \in \tilde{K}$, $\tilde{\sigma}' \in \tilde{K}'$ such that $\tilde{\sigma}' \subset \tilde{\sigma}$. We notice that if f does not map $\tilde{\sigma}'$ onto $p\tilde{\sigma}$ then $\tilde{c}$ does not occur in $tf_*\tilde{\sigma} \in C_*(\tilde{K})$. If f maps $\tilde{\sigma}'$ onto $p\tilde{\sigma}$ then coefficient of $\tilde{\sigma}$ in $tf_*\tilde{\sigma}$ is $c = +1(-1)$ if f preserves (reverses) the orientation. Then $f_|; \tilde{\sigma}' \to p\tilde{\sigma}'$ is a (linear) homeomorphism hence by the Brouwer theorem $f_|$ has a coincidence with the projection p. Moreover this coincidence must lie inside $\tilde{\sigma}' \subset \tilde{\sigma}$ hence these simplices are maximal. Since $f_|$ is expanding on $\tilde{\sigma}'$, there can be no other coincidence.

On the other hand since the map $f_| : \tilde{\sigma}_i' \to p(\tilde{\sigma}_i)$ is expanding, for any coincidence $\tilde{x}_i \in \sigma_i'$

$$\mathrm{ind}(p, f; \tilde{x}_i) = \deg(p - f; \tilde{x}_i) = \deg(p(\tilde{x}_i) - f; \tilde{x}_i) = (-1)^{\dim \tilde{\sigma}_i} c_i.$$

Thus

$$\sum_{k=0}^{\infty} (-1)^k \mathrm{trace}(tf_*\rho_k) = \sum_{\tilde{\sigma}_i'} (-1)^{\dim \tilde{\sigma}_i} c_i = \sum_{\tilde{\sigma}_i'} \mathrm{ind}(f, g; \tilde{x}_i) = \mathrm{ind}(p, f).$$

On the other hand: the Hopf Trace Theorem (comp Thm. 1D2 in [1]) implies.

LEMMA 3.2. *The alternating sums of traces of the chain maps*

$$C_*(\tilde{K}) \xrightarrow{\rho} C_*(\tilde{K}') \xrightarrow{f_*} C_*(K) \xrightarrow{t} C_*(\tilde{K})$$

and of the induced homology homomorphisms

$$H_*(\tilde{X}) \overset{\rho}{=} H_*(\tilde{X}) \xrightarrow{f_*} H_*(X) \xrightarrow{t_*} H_*(\tilde{X})$$

are equal : $\sum_{k=0}^{\infty} (-1)^k tr(tf_*\rho) = \sum_{k=0}^{\infty} (-1)^k tr(t_* f_*)$.

We define the *Lefschetz number* of the pair p, f as the above alternating sum of traces and we denote it by $L(p, f)$. The above implies

THEOREM 3.3. (Normalization) *For any pair of maps $p, f : \tilde{X} \to X$, where X is a compact polyhedron and p is a finite covering, the equality $L(p, f) = ind(p, f)$ holds.*

REMARK 3.4. Since $tr(\alpha\beta) = tr(\beta\alpha)$, the Lefschetz number can be defined by $L(p, f) = \sum_{k=0}^{\infty} (-1)^k tr(f_*' t_*)$.

4. Minimizing the number of coincidences

In this section we define the Nielsen number of the pair p, f and we show that this number can realized in the homotopy class of f (Wecken-type theorem) provided X satisfies the conditions of [11].

Let $p, f : U \to X$ ($U \subset \tilde{X}$) be as before. We say that two points $x, x' \in C(p, f)$ are *Nielsen equivalent* if there exists a path $\omega : I \to U$ from x to x' such that $f\omega$ and ω are fixed end point homotopic in X. This relation divides $C(p, f)$ into classes called *Nielsen classes.* A Nielsen class is called *essential* if its coincidence index is nonzero. The number of essential classes is called the *Nielsen number* and is denoted by $N(p, f)$. If $p = id$ then $N(id, f)$ equals to the ordinary fixed point Nielsen number $N(f)$ [1], [10]. As in the case of fixed points $N(p, f)$ has two fundamental properties:

PROPERTY 4.1. *If $f, f' : U \to X$ are compactly homotopic then $N(p, f) = N(p, f')$.*

PROPERTY 4.2. $\#C(p, f) \geq N(p, f)$.

We will show that $N(p, f)$ is the largest number satisfying the above two properties if X satisfies the following two conditions from [11].

DEFINITION 4.3. We say that X is a WJ-polyhedron if

1. X has no locally cut-points (i.e. if an open subset $V \subset X$ is connected then so is $V - x$ (for any $x \in X$)).
2. In a (hence any) triangulation there is a 1-simplex being the (proper) face of at least three simplices.

THEOREM 4.4. (Wecken-Jiang) *If $p : \tilde{X} \to X$ is a finite cover of a compact WJ-polyhedron then any map $f : \tilde{X} \to X$ is homotopic to a map with exactly $N(p, f)$ coincidences.*

Proof. The construction given below allows to replace two coincidences in the same Nielsen class by one coincidence. Thus we may reduce each Nielsen class to a point. Since p is a local homeomorphism, we may remove any inessential class, as in the case of fixed points [1]. □

Now we show how to replace two coincidences in Nielsen relation by one point. We will sketch how one may adapt the results of [11] to this situation (in [11] the theorem is proved for fixed points, i.e.

$p = identity$). We may assume that there is only a finite number of coincidences each lying inside a maximal simplex. Moreover, using a local homeomorphism, if necessary, we may also assume that $a \neq b \in C(p,f)$ implies $pa \neq pb$. Let $a, b \in C(p,f)$ be Nielsen related and let a path $\tilde{\omega}$ establish this relation. Put $\omega = p\tilde{\omega}$. Since $\omega(0) = pa \neq pb = \omega(1)$, ω is homotopic to a normal $PL-$arc in X (see [11] for definition). Since this homotopy can be lifted, we may assume that ω is such arc. Moreover we may assume that $\omega(s) \notin p(C(p,f))$ for $0 < s < 1$ and the conditions α, β of lemma 3.4 from [11] are satisfied (for σ being the face of at least three simplices). Put $\omega_\epsilon(s) = \omega(s(1 - \epsilon\sin(\pi s)))$(for a small $\epsilon > 0$). Then the paths $\omega, \omega_\epsilon, f\tilde{\omega}$ have the same ends (pa and pb), are homotopic and $\omega_\epsilon(s) \neq \omega(s) \neq f\tilde{\omega}(s)$ for $0 < s < 1$. Now we may apply lemma 5.1 [11] for ($q = \omega, p_0 = \omega_\epsilon, p_1 = f\tilde{\omega}$) and we get a homotopy $h_t : I \to X$ satisfying $h_0(s) = f\tilde{\omega}(s)$, $h_1(s) = \omega_\epsilon(s)$, $h_t(s) \neq \omega(s)$. Then we define $h'_t : \tilde{\omega} \to X$ putting $h'_t(\tilde{\omega}(s)) = h_t(s)$. Now

$$h'_0(\tilde{\omega}(s)) = h_0(s) = f\tilde{\omega}(s)$$
$$h'_1(\tilde{\omega}(s)) = h_1(s) = \omega_\epsilon(s)$$
$$h'_t(\tilde{\omega}(s)) = h_t(s) \neq \omega(s).$$

Since (by the last inequality) $C(p_|, h'_t) = \{\tilde{\omega}(0), \tilde{\omega}(0)\}$ does not depend on t, the map $\phi : \tilde{X} \times 0 \cup (\tilde{\omega}[0,1] \times I) \to X$ given by $\phi(x, 0) = f(x)$, $\phi(\tilde{\omega}(s), t) = h'_t(\tilde{\omega}(s))$ extends onto the whole $\tilde{X} \times I$ to a homotopy F that $C(p, F(.,t))$ does not depend on $t \in [0,1]$ (compare Lemma 2.1 in [11]). Thus f is homotopic to a map f' that $f'(\tilde{\omega}(s)) = \omega_\epsilon(s)$ is close to $\omega(s) = p(\tilde{\omega}(s))$ and $C(p,f') = C(p,f)$. On the other hand p is a homeomorphism between some neighbourhoods $\tilde{U}$ (of $\tilde{\omega}[0,1] \subset \tilde{X}$) and U (of $\omega[0,1] \subset X$). This homeomorphism allows to apply lemmas VIII C3 and VIIIC2 from [1] and move the coincidence point b to a.

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