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ANALYTIC TYPES OF THE SURFACE SINGULARITIES DEFINED BY SOME WEIGHTED HOMOGENEOUS POLYNOMIALS

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ABSTRACT. We classify analytically surface singularities defined by some weighted homogeneous polynomials which are topologically equivalent to the type $z_0^n + z_1^k + z_2^l = 0$.

1. Introduction

It is well known by Theorem 2.8 ([5]) that surface singularities defined by weighted homogeneous polynomials can be classified topologically by seven classes.

The aim in this paper is to classify analytically isolated surface singularities defined by some weighted homogeneous polynomials, which are topologically equivalent to the type $z_0^n + z_1^k + z_2^l = 0$.

Let ${}_{n+1}\mathcal{O}$ or $\mathbb{C}\{z_1, \dots, z_n\}$ be the ring of convergent power series at the origin in $\mathbb{C}^{n+1}$ and $f, g \in {}_{n+1}\mathcal{O}$. Then the natural question arises: What is the concrete criterion for f and g to have the same analytic type?

It is known by Theorem 2.6 ([3]) that two germs of complex analytic hypersurface singularities defined by f and g with isolated singular points at the origin in $\mathbb{C}^{n+1}$ are analytically equivalent if and only if their moduli algebra ${}_{n+1}\mathcal{O}/(f, \Delta f)$ and ${}_{n+1}\mathcal{O}/(g, \Delta g)$ are isomorphic as a $\mathbb{C}$-algebra, where $(f, \Delta f) = (f, \partial f/\partial z_0, \dots, \partial f/\partial z_n)$ is an ideal in ${}_{n+1}\mathcal{O}$ generated by $f, \partial f/\partial z_0, \dots, \partial f/\partial z_n$ and so on. In spite of the above

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theorem, it is still difficult to find a concrete criterion for analytic equivalence between two surfaces with isolated singular points at the origin.

By Theorem 2.7 ([5]), if $V = \{z \in \mathbb{C}^{n+1} : f(z) = 0\}$ and $W = \{z \in \mathbb{C}^{n+1} : g(z) = 0\}$ are surface singularities at the origin defined by weighted homogeneous polynomials f and g with the same weights, then V and W are topologically equivalent. But, for the analytic case, V and W may not be analytically equivalent, even though they have the same weights.

By the above motivation, we find a necessary and sufficient condition for given four different types of some surface singularities, which are topologically equivalent to the type $z_0^n + z_1^k + z_2^l = 0$, to be analytically equivalent.

2. Definitions and Known Preliminaries

Let ${}_{n+1}\mathcal{O}$ be the ring of germs of holomorphic functions at the origin in $\mathbb{C}^{n+1}$ and $f(z_0, \dots, z_n)$ and $g(z_0, \dots, z_n)$ are in ${}_{n+1}\mathcal{O}$ which have isolated singular points at the origin in $\mathbb{C}^{n+1}$.

Definition 2.1. Let $V = \{z \in \mathbb{C}^{n+1} : f(z) = 0\}$ and $W = \{z \in \mathbb{C}^{n+1} : g(z) = 0\}$ be germs of complex hypersurfaces with isolated singularity at the origin. f and g are said to have the same analytic type of singularity at the origin, if there is a germ at the origin of biholomorphism $\psi : (U_1, O) \to (U_2, O)$ such that $\psi(V) = W$ and $\psi(O) = O$ where U_1 and U_2 are open subsets in $\mathbb{C}^{n+1}$, that is, $f \circ \psi = ug$ where u is a unit in ${}_{n+1}\mathcal{O}$. Then we write $f \approx g$. If not, we write $f \not\approx g$.

Definition 2.2. Two germs of holomorphic functions $f, g : (\mathbb{C}^{n+1}, O) \to (\mathbb{C}, O)$ are called right equivalent if there exists a biholomorphism $\varphi : (\mathbb{C}^{n+1}, O) \to (\mathbb{C}^{n+1}, O)$ such that $f = g \circ \varphi$.

Definition 2.3. $f(z_0, \dots, z_n)$ is called a weighted homogeneous polynomial with weights $(\omega_0, \dots, \omega_n)$, where $\omega_0, \dots, \omega_n$ are fixed positive rational numbers, if it can be expressed as a linear combination of monomials $z_0^{i_0} z_1^{i_1} \cdots z_n^{i_n}$ for which $\frac{i_0}{\omega_0} + \cdots + \frac{i_n}{\omega_n} = 1$.

Definition 2.4. $f \in {}_{n+1}\mathcal{O}$ is called quasihomogeneous if $f \approx g$ for some weighted homogeneous polynomial g.

THEOREM 2.5 ([4]). *If* (V, O) *and* (W, O) *be germs of isolated hypersurface singularities at the origin in* $\mathbb{C}^{n+1}$ *defined by weighted homogeneous polynomials* f *and* g *respectively, then* (V, O) *and* (W, O) *are analytically equivalent if and only if* f *and* g *are right equivalent. That is, there exists a biholomorphism* $\varphi : (\mathbb{C}^{n+1}, O) \to (\mathbb{C}^{n+1}, O)$ *such that* $f \circ \varphi = g$.

THEOREM 2.6 ([3]). *Suppose that* $V = \{(z_0, \dots, z_n) \in \mathbb{C}^{n+1} : f(z_0, \dots, z_n) = 0\}$ *and* $W = \{(z_0, \dots, z_n) \in \mathbb{C}^{n+1} : g(z_0, \dots, z_n) = 0\}$ *have the isolated singular point at the origin. Then the following conditions are equivalent.*

(i): $f \approx g$.

(ii): $A(f)$ *is isomorphic to* $A(g)$ *as a* $\mathbb{C}$*-algebra where* $A(f) = {}_{n+1}\mathcal{O}/(f, \Delta(f))$, $A(g) = {}_{n+1}\mathcal{O}/(g, \Delta(g))$ *and* $(f, \Delta(f))$ *is the ideal in* ${}_{n+1}\mathcal{O}$ *generated by* $f, \partial f/\partial z_0, \dots, \partial f/\partial z_n$.

(iii): $B(f)$ *is isomorphic to* $B(g)$ *as a* $\mathbb{C}$*-algebra where* $B(f) = {}_{n+1}\mathcal{O}/(f, \mathbf{m}\Delta(f))$, $B(g) = {}_{n+1}\mathcal{O}/(g, \mathbf{m}\Delta(g))$ *and* $(f, \mathbf{m}\Delta(f))$ *is the ideal in* ${}_{n+1}\mathcal{O}$ *generated by* f *and* $z_i \partial f/\partial z_j$ *for all* $i, j = 0, 1, \dots, n$.

THEOREM 2.7 ([5]). *Suppose that* $f(z_0, z_1, z_2)$ *and* $g(z_0, z_1, z_2)$ *are weighted homogeneous polynomials with the same weights* $(\omega_0, \omega_1, \omega_2)$. *If* f *and* g *have isolated singularities at the origin in* $\mathbb{C}^3$, *then* f *is topologically equivalent to* g.

THEOREM 2.8 ([5]). *Let* $(V, 0)$ *and* $(W, 0)$ *be two isolated quasihomogeneous surface singularities having the same topological type. Then* $(V, 0)$ *is connected to* $(W, 0)$ *by a family of constant topological type. In fact* $(V, 0)$ *is connected to one of the followings:*

Class I. $V(a_0, a_1, a_2; 1) = \{z_0^{a_0} + z_1^{a_1} + z_2^{a_2} = 0\}$

Class II. $V(a_0, a_1, a_2; 2) = \{z_0^{a_0} + z_1^{a_1} + z_1 z_2^{a_2} = 0\}$ where $a_1 > 0$

Class III. $V(a_0, a_1, a_2; 3) = \{z_0^{a_0} + z_1^{a_1} z_2 + z_1 z_2^{a_2} = 0\}$ where $a_1 > 0, a_2 > 0$

Class IV. $V(a_0, a_1, a_2; 4) = \{z_0^{a_0} + z_1^{a_1} z_2 + z_0 z_2^{a_2} = 0\}$ where $a_0 > 0$

Class V. $V(a_0, a_1, a_2; 5) = \{z_0 z_1 + z_1^{a_1} z_2 + z_0 z_2^{a_2} = 0\}$

Class VI. $V(a_0, a_1, a_2; 6) = \{z_0^{a_0} + z_0 z_1^{a_1} + z_1^{b_1} z_2^{b_2} = 0\}$ where $(a_0 - 1)(a_1 b_2 + a_2 b_1) = a_0 a_1 a_2$

Class VII. $V(a_0, a_1, a_2; 7) = \{z_0^{a_0} z_1 + z_0 z_1^{a_1} + z_0 z_2^{a_2} + z_1^{b_1} z_2^{b_2} = 0\}$ where $c(a_0 - 1)(a_1 b_2 + a_2 b_1) = a_2(a_0 a_1 - 1)$

THEOREM 2.9 ([1]). *Let* f *and* g *be weighted homogeneous polynomials, which are not homogeneous, with isolated singularity at the origin* *in* $\mathbb{C}^2$ *such that* $f \not\sim z_0^2+z_1^2$ *and* $g \not\sim z_0^2+z_1^2$. *Then we may assume without loss of generality that analytically,*

$$\begin{aligned} f &= z_0^{\varepsilon_1} z_1^{\varepsilon_2} f_1 \quad \text{with} \\ f_1 &= z_0^n + z_1^k + \textstyle\sum_{i=1}^{d-1} A_i z_0^{(d-i)n_1} z_1^{ik_1} \text{ and} \\ \\ g &= z_0^{\delta_1} z_1^{\delta_2} f_1 \quad \text{with} \\ g_1 &= z_0^m + z_1^l + \textstyle\sum_{j=1}^{e-1} B_j z_0^{(e-j)m_1} z_1^{jl_1} \end{aligned}$$

where

(a): $2 \le n < k, d = \gcd(n,k)$ *with* $n = dn_1$ *and* $k = dk_1$,
(b): $2 \le m < l, e = \gcd(m,l)$ *with* $m = em_1$ *and* $l = el_1$,
(c): $\varepsilon_1, \varepsilon_2, \delta_1, \delta_2$ *are either 1 or 0, respectively, and*
(d): A_i *and* B_j *are complex numbers for* $1 \le i < d-1$ *and* $1 \le j < e-1$.

Also, we need to assume without loss of generality that

$$\begin{aligned} &\text{if} \quad \gcd(n,k) = n, \quad \text{i.e.,} \quad n_1 = 1, \quad \text{then } A_1 = 0 \text{ and} \\ &\text{if} \quad \gcd(m,l) = m, \quad \text{i.e.,} \quad m_1 = 1, \quad \text{then } B_1 = 0. \end{aligned}$$

As a conclusion, we get the following:

$f \approx g$ *if and only if* $\varepsilon_i = \delta_i$ *for* $i = 1,2$ *and* $f_1 \approx g_1$ *if and only if* $\varepsilon_i = \delta_i$ *for* $i = 1,2$ *and* $n = m, k = l$ *and there is a complex number* ρ *with* $\rho^d = 1$ *such that* $A_i \rho^i = B_i$ *for* $i = 1, \ldots, d-1$.

THEOREM 2.10 ([2]). *Let* $f = z_0^n + z_1^n + \sum_{i=1}^s a_i z_0^i z_1^{n-1}$ *and* $g = z_0^n + z_1^n + \sum_{j=1}^t b_j z_0^j z_1^{n-j}$ *be homogeneous polynomials with isolated singularity at the origin in* $\mathbb{C}^2$ *where* $n \ge 2s+3, n \ge 2t+3$ *and* $n \ge 5$. *Then* $f \approx g$ *if and only if there is a complex number* ρ *with* $\rho^n = 1$ *such that* $b_i = a_i \rho^i$ *for* $i = 1, \ldots, s = t$. *Moreover, if* $f = z_0^4 + z_1^4 + az_0 z_1^3$ *and* $g = z_0^4 + z_1^4 + bz_0 z_1^3$ *have an isolated singularity at the origin, then* $f \approx g$ *if and only if* $a^4 = b^4$.

3. Main Results

We find a concrete criterion to have the same analytic type for given two surfaces singularities at the origin, which are defined by some weighted homogeneous polynomials. Consider the four different types of singularities defined by some weighted homogeneous polynomials with isolated singular points at the origin in $\mathbb{C}^3$, which are topologically equivalent to the type $z_0^n + z_1^k + z_2^l = 0$, as follows:

$$T_0(z_0, z_1, z_2) = z_0^n + z_1^k + z_2^l;$$

$$T_1(z_0, z_1, z_2) = z_0^n + z_1^k + z_2^l + \sum_{\alpha,\beta} A_{\alpha,\beta} z_0^\alpha z_1^\beta \quad \text{with some } A_{\alpha,\beta} \neq 0;$$

$$T_2(z_0, z_1, z_2) = z_0^n + z_1^k + z_2^l + \sum_{\gamma,\delta} B_{\gamma,\delta} z_1^\gamma z_2^\delta \quad \text{with some } B_{\gamma,\delta} \neq 0;$$

$$T_3(z_0, z_1, z_2) = z_0^n + z_1^k + z_2^l + \sum_{\varepsilon,\tau} C_{\varepsilon,\tau} z_0^\varepsilon z_2^\tau \quad \text{with some } C_{\varepsilon,\tau} \neq 0;$$

$$T_4(z_0, z_1, z_2) = z_0^n + z_1^k + z_2^l + \sum_{\substack{\alpha\beta\gamma \\ \alpha \leq n-2}} D_{\alpha,\beta,\gamma} z_0^\alpha z_1^\beta z_2^\gamma \text{ with some } D_{\alpha,\beta,\gamma} \neq 0.$$

DEFINITION 3.1. It is said that a weighted homogeneous polynomial f belongs to the type T_i if f can be written in the form of T_i for $i = 0, 1, 2, 3, 4$. In this case, we write $f \in T_i$. Otherwise, $f \notin T_i$.

Note that the surface singularities defined by the above four different types of weighted homogeneous polynomials are topologically equivalent to the surface singularity defined by $z_0^n + z_1^k + z_2^l = 0$. It is a consequence of Theorem 2.7 ([5]). But, for the analytic case, we find the different results. Those are followings:

First, even though f and g belong to the same type in a sense Definition 3.1, f and g may not be analytically equivalent. If f and g are analytically equivalent, then we find a necessary and sufficient condition for f and g, in an elementary way. Secondly, if f and g belong to the different types, then f and g are not analytically equivalent.

Throughout in this paper, we assume that $2 \leq n \leq k \leq l$ and all exponents of z_0, z_1 and z_2 are positive integers.

THEOREM 3.2. *Let n and k be positive integers with $2 \leq n \leq k$ and $f_1 \in T_1, g_1 \in T_1$.*

A: *Assume that $n < k$. Then, f_1 and g_1 can be written analytically as follows:*

$$\begin{aligned} f_1 &= z_0^n + A_0 z_1^k + z_2^l + \sum_{i=1}^{d-1} A_i z_0^{n_1 i} z_1^{k_1(d-i)}, \\ g_1 &= z_0^n + B_0 z_1^k + z_2^l + \sum_{i=1}^{d-1} B_i z_0^{n_1 i} z_1^{k_1(d-i)} \end{aligned} \tag{1}$$

where $d = \gcd(n, k)$ with $n = n_1 d$ and $k = k_1 d$ for some positive integers n_1, k_1, and A_i and B_i are complex numbers for $0 \leq i \leq d-1$ which satisfy the following properties:

(a): *If* $d < n$, *then* $A_0 = B_0 = 1$;

(b): *If* $d = n \geq 3$, *then* $A_{d-1} = B_{d-1} = 0$, *and* A_0 *and* B_0 *are either* 1 *or* 0, *respectively. In this case, if* $A_0 = 0$, *then* $A_1 = 1$, *and if* $B_0 = 0$, *then* $B_1 = 1$;

(c): *If* $d = n = 2$, *then* $A_0 = B_0 = 1$ *and* $A_1 = B_1 = 0$;

(d): $f_1 \approx g_1$ *if and only if there exists a complex number* ω_1 *with* $\omega_1^n = 1$ *such that* $A_i\omega_1^i = B_i$ *for* $0 \leq i \leq d-1$.

B: *Assume that* $n = k$. *Then* f_1 *and* g_1 *can be written analytically as follows:*

$$(2) \quad \begin{aligned} f_1 &= z_0^n + z_1^n + z_2^l + \sum_{i=1}^{s} C_i z_0^i z_1^{n-i}, \\ g_1 &= z_0^n + z_1^n + z_2^l + \sum_{i=1}^{t} D_i z_0^i z_1^{n-i} \end{aligned}$$

for some complex numbers C_i *and* D_i *where* $s \leq n-1, t \leq n-1$ *and* $C_s \neq 0, D_t \neq 0$. *In this case, let* $n \geq 2s+3, n \geq 2t+3$ *and* $n \geq 5$. *Then,* $f_1 \approx g_1$ *if and only if exits a complex number* ω_2 *with* $\omega_2^n = 1$ *such that* $\omega_2^i C_i = D_i$ *for each* $i = 1, 2, \ldots, s = t$.

THEOREM 3.3. *Let* k *and* l *be positive integers with* $2 \leq k \leq l$ *and* $f_2 \in T_2, g_2 \in T_2$.

A: *Assume that* $k < l$. *Then* f_1 *and* g_1 *can be written analytically as follows:*

$$(3) \quad \begin{aligned} f_2 &= z_0^n + z_1^k + A_0 z_2^l + \sum_{i=1}^{d-1} A_i z_1^{k_1 i} z_2^{l_1(d-i)}, \\ g_2 &= z_0^n + z_1^k + B_0 z_2^l + \sum_{i=1}^{d-1} B_i z_1^{k_1 i} z_2^{l_1(d-i)} \end{aligned}$$

where $d = \gcd(k, l)$ *with* $k = k_1 d$ *and* $l = l_1 d$ *for some positive integers* k_1, l_1, *and* A_i *and* B_i *are complex numbers for* $0 \leq i \leq d-1$ *which satisfy the following properties:*

(a): *If* $d < k$, *then* $A_0 = B_0 = 1$;

(b): *If* $d = k \geq 3$, *then* $A_{d-1} = B_{d-1} = 0$, *and* A_0 *and* B_0 *are either* 1 *or* 0, *respectively. In this case, if* $A_0 = 0$, *then* $A_1 = 1$, *and if* $B_0 = 0$, *then* $B_1 = 1$;

(c): *If* $d = k = 2$, *then* $A_0 = B_0 = 1$ *and* $A_1 = B_1 = 0$;

(d): $f_2 \approx g_2$ *if and only if there exists a complex number* ρ_1 *with* $\rho_1^n = 1$ *such that* $A_i\rho_1^i = B_i$ *for* $0 \leq i \leq d-1$.

B: *Assume that* $k = l$. *Then* f_2 *and* g_2 *can be written analytically as follows:*

$$(4) \quad \begin{aligned} f_2 &= z_0^n + z_1^k + z_2^k + \sum_{i=1}^{s} C_i z_1^i z_2^{k-i}, \\ g_2 &= z_0^n + z_1^k + z_2^k + \sum_{i=1}^{t} D_i z_1^i z_2^{k-i} \end{aligned}$$

for some complex numbers C_i *and* D_i *where* $s \leq k-1, t \leq k-1$ *and* $C_s \neq 0, D_t \neq 0$. *In this case, let* $k \geq 2s+3, k \geq 2t+3$ *and* $k \geq 5$. *Then,* $f_2 \approx g_2$ *if and only if there exists a complex number* ρ_2 *with* $\rho_2^k = 1$ *such that* $C_i \rho_2^i = D_i$ *for* $i = 1, 2, \ldots, s = t$.

THEOREM 3.4. *Let* n *and* l *be positive integers with* $2 \leq n \leq l$ *and* $f_3 \in T_3, g_3 \in T_3$.

A: *Assume that* $n < l$. *Then* f_1 *and* g_1 *can be written analytically as follows:*

$$\begin{aligned} f_3 &= z_0^n + z_1^k + A_0 z_2^l + \textstyle\sum_{i=1}^{d-1} A_i z_0^{n_1 i} z_2^{l_1(d-i)}, \\ g_3 &= z_0^n + z_1^k + B_0 z_2^l + \textstyle\sum_{i=1}^{d-1} B_i z_0^{n_1 i} z_2^{l_1(d-i)} \end{aligned} \tag{5}$$

where $d = \gcd(n, l)$ *with* $n = n_1 d$ *and* $l = l_1 d$ *for some positive integers* n_1, l_1, *and* A_i *and* B_i *are complex numbers for* $0 \leq i \leq d-1$ *which satisfy the following properties:*

(a): *If* $d < n$, *then* $A_0 = B_0 = 1$;

(b): *If* $d = n \geq 3$, *then* $A_{d-1} = B_{d-1} = 0$, *and* A_0 *and* B_0 *are either 1 or 0, respectively. In this case, if* $A_0 = 0$, *then* $A_1 = 1$, *and if* $B_0 = 0$, *then* $B_1 = 1$;

(c): *If* $d = n = 2$, *then* $A_0 = B_0 = 1$ *and* $A_1 = B_1 = 0$;

(d): $f_3 \approx g_3$ *if and only if there exists a complex number* η_1 *with* $\eta_1^n = 1$ *such that* $A_i \eta_1^i = B_i$ *for* $0 \leq i \leq d-1$.

B: *Assume that* $n = l$. *Then* f_3 *and* g_3 *can be written analytically as follows:*

$$\begin{aligned} f_3 &= z_0^n + z_1^n + z_2^n + \textstyle\sum_{i=1}^{s} C_i z_0^i z_2^{n-i}, \\ g_3 &= z_0^n + z_1^n + z_2^n + \textstyle\sum_{i=1}^{t} D_i z_1^i z_2^{n-i} \end{aligned} \tag{6}$$

for some complex numbers C_i *and* D_i *where* $s \leq n-1, t \leq n-1$ *and* $C_s \neq 0, D_t \neq 0$. *In this case, let* $n \geq 2s+3, n \geq 2t+3$ *and* $n \geq 5$. *Then,* $f_3 \approx g_3$ *if and only if exits a complex number* η_2 *with* $\eta_2^n = 1$ *such that* $C_i \eta_2^i = D_i$ *for each* $i = 1, 2, \ldots, s = t$.

REMARK 3.5. Theorem 3.2, Theorem 3.3 and Theorem 3.4 imply the following facts:

(i): If $f_1 \in T_1$ and $n = 2$, then $f_1 \in T_0$;
(ii): If $f_2 \in T_2$ and $n = k = 2$, then $f_2 \in T_0$;
(iii): If $f_3 \in T_3$ and $n = 2$, then $f_3 \in T_0$.

THEOREM 3.6. *Suppose that $2 < n < k < l$ and that the weighted homogeneous polynomials f_i and g_j belong to the type T_i and T_j, respectively, where $0 \leq i \leq j \leq 4$. If $i \neq j$, then $f_i \not\approx g_j$ except for $i = 2$ and $j = 4$.*

Assume that f_4 belongs to the type T_4. That is, f_4 can be written as

$$f_4 = z_0^n + z_1^k + z_2^l + \sum_{\alpha,\beta,\gamma} D_{\alpha,\beta,\gamma} z_0^\alpha z_1^\beta z_2^\gamma.$$

DEFINITION 3.7. Let $f = z_0^n + z_1^k + z_2^l + \sum_{\alpha,\beta,\gamma} D_{\alpha,\beta,\gamma} z_0^\alpha z_1^\beta z_2^\gamma$ be given. Define $\min(f) = \min\{\alpha + \beta + \gamma\}$ for all nonzero monomial $z_0^\alpha z_1^\beta z_2^\gamma$ in f and $S(f) = \{(\alpha, \beta, \gamma) : \alpha + \beta + \gamma = \min(f)\}$.

LEMMA 3.8. *If $k < l$, then there exists a unique element $(\alpha_0, \beta_0, \gamma_0) \in S(f)$ such that $\alpha_0 \leq \alpha$ for any $(\alpha, \beta, \gamma) \in S(f)$ as in* Definition 3.7.

Proof. Note that if $\alpha_0 = \alpha$ and $\alpha_0 + \beta_0 + \gamma_0 = \alpha + \beta + \gamma$, then

$$\frac{\beta_0 - \beta}{k} = \frac{\gamma - \gamma_0}{l}.$$

Thus $\beta_0 = \beta$ and $\gamma_0 = \gamma$ if $k < l$. Therefore if we choose an element $(\alpha_0, \beta_0, \gamma_0) \in S$ such that $\alpha_0 \leq \alpha$ for any $(\alpha, \beta, \gamma) \in S$, then the element $(\alpha_0, \beta_0, \gamma_0)$ is unique. □

THEOREM 3.9. *Suppose that $2 < n < k < l$ and that f_4 and g_4 belong to the type of T_4. Then f_4 and g_4 can be written as follows:*

$$\begin{aligned} f_4 &= z_0^n + z_1^k + z_2^l + \textstyle\sum_{(\alpha,\beta,\gamma)\in I_4} D_{\alpha,\beta,\gamma} z_0^\alpha z_1^\beta z_2^\gamma, \\ g_4 &= z_0^n + z_1^k + z_2^l + \textstyle\sum_{(\alpha',\beta',\gamma')\in I_4'} D'_{\alpha',\beta',\gamma'} z_0^{\alpha'} z_1^{\beta'} z_2^{\gamma'}. \end{aligned}$$

for some nonzero complex numbers $D_{\alpha,\beta,\gamma}$ and $D_{\alpha',\beta',\gamma'}$. For f_4, if we can choose $(\alpha_0, \beta_0, \gamma_0) \in I_4$ with $\alpha_0 + \beta_0 + \gamma_0 \leq n + k - 2$ which satisfies Definition 3.7 *and* Lemma 3.8, *then the followings hold:*

(i): *$f_4 \not\approx f_2$ where $f_2 \in T_2$;*
(ii): *If $(\alpha_0, \beta_0, \gamma_0) \notin I_4'$, then $f_4 \not\approx g_4$.*

Furthermore, if $f_4 \approx g_4$, then $(\alpha_0, \beta_0, \gamma_0) \in I_4'$ and there exist complex numbers a, b and c such that $D_{\alpha_0,\beta_0,\gamma_0} a^{\alpha_0} b^{\beta_0} c^{\gamma_0} = D'_{\alpha_0,\beta_0,\gamma_0}$.

We prove those results by using Theorem 2.5, Theorem 2.9 and Theorem 2.10. From the fact by Theorem 2.5, which two surface singularities at the origin defined by weighted homogeneous polynomials f and g are analytically equivalent if and only if $f \circ \varphi = g$ for some biholomorphisms $\varphi : (\mathbb{C}^3, O) \to (\mathbb{C}^3, O)$ at the origin, we may apply the fact to prove those results.

We use a notation to be convenient:

$z_0^\alpha z_1^\beta z_2^\gamma \in P(z_0, z_1, z_2)$ if the monomial $z_0^\alpha z_1^\beta z_2^\gamma$ belongs to the polynomial or power series $P(z_0, z_1, z_2)$. That is, the monomial $z_0^\alpha z_1^\beta z_2^\gamma$ has nonzero coefficient in $P(z_0, z_1, z_2)$.

Before proving Theorem 3.2, Theorem 3.3 and Theorem 3.4, we remarked the followings: Let $(X_f, 0) = \{(z_0, \cdots, z_n) : f(z_0, \cdots, z_n) = 0\}$. Teissier(1977) showed that the analytic type of the hypersurface X_{f+h} defined by $f(z_0, \cdots, z_n) + h(w_0, \cdots, w_m) = 0$ depends not only on the analytic types of $(X_f, 0)$ and of $(X_h, 0)$, but also in general on the choice of the equations for f and h. However, the following theorem says that in case h is quasihomogeneous, then the analytic type of X_{f+h} indeed depends only on the analytic types of $(X_f, 0)$ and of $(X_h, 0)$. In fact, a "subtraction" theorem holds !

THEOREM 3.10. *Let $f(z_0, \cdots, z_n)$ and $g(z_0, \cdots, z_n)$ be holomorphic functions with isolated singularity at origin in $\mathbb{C}^{n+1}$ and $h(w_0, \cdots, w_m)$ be a quasihomogeneous holomorphic function with an isolated singularity at the origin. Then $(X_f, 0)$ is analytically equivalent to $(X_g, 0)$ if and only if $(X_{f+h}, 0)$ is analytically equivalent to $(X_{g+h}, 0)$.*

Proof. See [6]. □

Proof of Theorem 3.2. If $\gcd(n, k) = d < n$, then f_1 and g_1 can be written analytically as (1) which satisfy (a). Suppose that $\gcd(n, k) = n$. Then,

$$f_1 = z_0^n + z_1^k + z_2^l + \sum_{i=1}^{n-1} D_i z_0^i z_1^{k_1(n-i)}$$

for some complex numbers $D_i, 1 \le i \le n-1$. If $D_{n-1} = 0$, then f_1 can be written analytically as (1). If $D_{n-1} \neq 0$, then, by the biholomorphic change of coordinates φ with

$$\varphi(z_0, z_1, z_2) = \left(z_0 - \frac{D_{n-1}}{n} z_1^{k_1}, z_1, z_2 \right)$$

at the origin, $(f_1 \circ \varphi)(z_0, z_1, z_2)$ can be written analytically as (1) which satisfies (b) and (c). Since $f_1 \circ \varphi \approx f_1$, f_1 can be written analytically as (1) which satisfies (b) and (c). By a similar method, g_1 can be written analytically as (1) which satisfies (a), (b) and (c). Let $h(z_2) = z_2^l$. Then h is a quasihomogeneous holomorphic function. Let $f = f_1 - h$ and $g = g_1 - h$. By Theorem 3.10,

$$f_1 = f + h \approx g + h = g_1 \text{ if and only if } f \approx g.$$

This is the case of plane curve singularities. The result of Theorem 2.9 implies A. The proof of B is similar if we set $h(z_0) = z_0^n$.

This completes the proof of Theorem 3.2. □

Proofs of Theorem 3.3 *and Theorem* 3.4. By a similar argument of the proof of Theorem 3.2, we can prove Theorem 3.3 and Theorem 3.4 without any difficulty. □

We prove Theorem 3.6 by Proposition 3.11, Proposition 3.12 and Proposition 3.13.

PROPOSITION 3.11. *Assume that* $2 < n < k < l$ *and that* $f_0 \in T_0$ *and* $f_j \in T_j$ *for* $1 \leq j \leq 4$ *in a sense of* Definition 3.1. *Then* $f_0 \not\approx f_j$.

Proof. We prove that $f_0 \not\approx f_1$. Suppose that $f_0 \approx f_1$. Then f_0 and f_1 can be written analytically as

$$\begin{aligned} f_0 &= z_0^n + z_1^k + z_2^l, \\ f_1 &= z_0^n + z_1^k + z_2^l + \textstyle\sum_{\alpha,\beta} A_{\alpha,\beta} z_0^\alpha z_1^\beta, \end{aligned}$$

which are weighted homogeneous polynomials with weights (n, k, l). Let

$$\begin{aligned} &f(z_0, z_1) = z_0^n + z_1^k + \textstyle\sum_{\alpha,\beta} A_{\alpha,\beta} z_0^\alpha z_1^\beta, \\ &g(z_0, z_1) = z_0^n + z_1^k, \\ &h(z_2) = z_2^l. \end{aligned}$$

Then f and g are weighted homogeneous polynomials with weights (n, k) and h is a quasihomogeneous holomorphic function. By Theorem 3.10, $f_0 = g + h \approx f + h = f_1$ if and only if $f \approx g$. But, $f \not\approx g$ by Theorem 2.9. This leads to a contradiction. Thus $f_0 \not\approx f_1$.

The other proofs are similar to the above. □

PROPOSITION 3.12. *Suppose that* $2 < n < k < l$ *and that* $f_1 \in T_1$ *and* $f_j \in T_j$ *for* $0 \leq j \leq 4, j \neq 1$ *in a sense of* Definition 3.1. *Then* $f_1 \not\approx f_j$.

Proof. Suppose that $f_1 \approx f_j$ for some j with $j \neq 1$ and $0 \leq j \leq 4$.
Case I) $\gcd(n,k) = d_1 < n$.

Then $n = n_1 d_1$ and $k = k_1 d_1$ for some positive integers k_1 and n_1. Note that $d_1 > 1$. By Theorem 3.2, A, f_1 can be written analytically as

$$f_1 = z_0^n + z_1^k + z_2^l + \sum_{i=1}^{d} A_i z_0^{n_1 i} z_1^{k_1(d_1-i)}$$

for some complex numbers A_i where d is the largest number of i with $A_i \neq 0$ and $1 \leq i \leq d_1 - 1$. Choose a biholomorphism $\varphi : (\mathbb{C}^3, O) \to (\mathbb{C}^3, O)$ at the origin $O = (0,0,0)$ such that $f_1 \circ \varphi = f_j$. We set $\varphi(z_0, z_1, z_2) = (H, L, M)$ as follows:

(7)
$$H = a_1 z_0 + b_1 z_1 + c_1 z_2 + H_2 + \cdots + H_s + \cdots, \quad H_s = \sum_{p+q+r=s} A_{p,q,r} z_0^p z_1^q z_2^r,$$
$$L = a_2 z_0 + b_2 z_1 + c_2 z_2 + L_2 + \cdots + L_s + \cdots, \quad L_s = \sum_{p+q+r=s} B_{p,q,r} z_0^p z_1^q z_2^r,$$
$$M = a_3 z_0 + b_3 z_1 + c_3 z_2 + M_2 + \cdots + M_s + \cdots, \quad M_s = \sum_{p+q+r=s} C_{p,q,r} z_0^p z_1^q z_2^r.$$

Since φ is a biholomorphism at the origin, we have

$$(8) \qquad | J_\varphi(O) | = \det \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \neq 0.$$

Consider the expansion of $f_1 \circ \varphi = f_j$,

$$(9) \qquad H^n + L^k + M^l + \sum_{i=1}^{d} A_i H^{n_1 i} L^{k_1(d_1-i)} = f_j.$$

Note that

$$n < n_1 d + k_1(d_1 - d) < n_1(d-1) + k_1(d_1 - d + 1)$$
$$< \cdots < n_1 + k_1(d_1 - 1) < k.$$

Then, by comparison of degrees in (9), we have

$$b_1 = c_1 = c_2 = 0$$

and

$$H_2 = \cdots = H_{(k_1 - n_1)(d_1 - d)} = 0.$$

Therefore

$$| J_\varphi(O) | = a_1 b_2 c_3 \neq 0.$$

In the expansion

$$\begin{aligned} &A_d H^{n_1 d} L^{k_1(d_1-d)} \\ &= A_d \left(a_1 z_0 + H_{(k_1-n_1)(d_1-d)+1} + \cdots\right)^{n_1 d} \left(a_2 z_0 + b_2 z_1 + L_2 + \cdots\right)^{k_1(d_1-d)}, \end{aligned}$$

every monomial with degree $n_1 d + k_1(d_1 - d)$ is contained in the expansion of $A_d (a_1 z_0)^{n_1 d} (a_2 z_0 + b_2 z_1)^{k_1(d_1-d)}$. In particular, the monomial $z_0^{n_1 d} z_1^{k_1(d_1-d)}$ has nonzero coefficient $A_d a_1^{n_1 d} b_2^{k_1(d_1-d)}$ in the expansion of $A_d H^{n_1 d} L^{k_1(d_1-d)}$. It is clear that the monomial $z_0^{n_1 d} z_1^{k_1(d_1-d)}$ does not belong to L^k and M^l by the inequalities

$$n_1 d + k_1(d_1 - d) < k < l.$$

We claim that the monomial $z_0^{n_1 d} z_1^{k_1(d_1-d)}$ does not belong to the expansion of H^n. If $z_0^{n_1 d} z_1^{k_1(d_1-d)}$ belongs to H^n, then the monomial is contained in the expansion of $(a_1 z_0)^\eta (H_{(k_1-n_1)(d_1-d)+1} + \cdots)^{n-\eta}$ for some η where $0 \le \eta \le n_1 d$. Since $n \ge n_1 d + 2$ and $n_1 d + k_1(d_1 - d) > n$, the inequalities

$$\begin{aligned} &\eta + (n - \eta)\left\{(k_1 - n_1)(d_1 - d) + 1\right\} \\ &\ge n_1 d + n_1(d_1 - d)\left\{(d_1 - d)(k_1 - n_1) + 1\right\} \\ &> n_1 d + k_1(d_1 - d) \end{aligned} \tag{10}$$

hold for all η where $0 \le \eta \le n_1 d$. That is to say, every monomial in the expansion of $(a_1 z_0)^\eta (H_{(k_1-n_1)(d_1-d)+1} + \cdots)^{n-\eta}$ has degree greater than $n_1 d + k_1(d_1 - d)$ if $0 \le \eta \le n_1 d$. This leads to a contradiction. Consequently, the monomial $z_0^{n_1 d} z_1^{k_1(d_1-d)}$ has nonzero coefficient $A_d a_1^{n_1 d}$ $b_2^{k_1(d_1-d)}$ in the left expansion of (9) and it must belong to f_j for $0 \le j \le 4, j \ne 1$. This also leads to a contradiction. Thus $f_1 \not\approx f_j$ if $j \ne 1$.

Case II) $\gcd(n, k) = n < k$.

Then $k = n k_1$ for some positive integer k_1. By Theorem 3.2, A, f_1 has analytically two different representations as follows:

$$\begin{aligned} f_{11} &= z_0^n + z_1^k + z_2^l + \textstyle\sum_{i=1}^{\alpha} A_{1,i} z_0^i z_1^{k_1(n-i)}, \\ f_{12} &= z_0^n + z_2^l + \textstyle\sum_{r=1}^{\beta} A_{2,r} z_0^r z_1^{k_1(n-r)} \end{aligned} \tag{11}$$

for some complex numbers $A_{1,i}, A_{2,r}$ with $1 \le i \le \alpha \le n-2, 1 \le r \le \beta \le n-2, A_{2,1} = 1$, where α and β are the largest numbers of i and r such that $A_{1,i} \ne 0$ and $A_{2,r} \ne 0$, respectively. Therefore, if $f_1 \approx f_j$, then either $f_{11} \approx f_j$ or $f_{12} \approx f_j$.

Suppose that $f_{11} \approx f_j$. Choose a biholomorpism $\varphi : (\mathbb{C}^3, O) \to (\mathbb{C}^3, O)$ at the origin as (7) such that $f_{11} \circ \varphi = f_j$. By a similar method in the proof of Case I, the monomial $z_0^\alpha z_1^{k_1(n-\alpha)}$ has nonzero coefficient $A_{1,\alpha}a_1^\alpha b_2^{k_1(n-\alpha)}$ in the expansion of $f_{11}\circ\varphi$. Thus the monomial $z_0^\alpha z_1^{k_1(n-\alpha)}$ must belong to f_j. This leads to a contradiction. Thus $f_{11}\not\approx f_j$.

Similarly, we have $f_{12}\not\approx f_j$.

Consequently, $f_1\not\approx f_j$.

This completes the proof. □

PROPOSITION 3.13. *Suppose that* $2<n<k<l$ *and that* $f_3\in T_3$ *and* $f_j\in T_j$ *for* $j=0,2,4$ *in a sense of* Definition 3.1. *Then* $f_3\not\approx f_j$.

Proof. Suppose that $f_3\approx f_j$ for some $j=0,2,4$.

Case I) $\gcd(n,l)=d_3<n$.

Then $n=n_3d_3$ and $l=l_3d_3$ for some positive integers n_3 and l_3. Note that $d_3>1$. By Theorem 3.4, A, f_3 can be written analytically as follows:

$$f_3=z_0^n+z_1^k+z_2^l+\sum_{i=1}^{d}E_iz_0^{n_3i}z_2^{l_3(d_3-i)} \tag{12}$$

for some complex numbers E_i, where d is the largest number of i with $E_i\neq 0$ and $1\le i\le d_3-1$. Take a biholomorphism $\varphi:(\mathbb{C}^3,O)\to(\mathbb{C}^3,O)$ at the origin $O=(0,0,0)$ such that $f_3\circ\varphi=f_j$ as (7). Note that

$$n<n_3d+l_3(d_3-d)<n_3(d-1)+l_3(d_3-d+1)<\cdots<n_3+l_3(d_3-d)<l.$$

By a similar method in the proof of Proposition 3.12, we have $b_1=c_1=0$ in (7). Therefore, we may write H as

$$H=a_1z_0+H_{\min\{k-n+1,n_3d+l_3(d_3-d)-n+1\}}+\cdots.$$

Note that

$$\begin{aligned}&(n-n_3d)(n_3d+l_3(d_3-d)-n+1)>l_3(d_3-d),\\&n(n_3d+l_3(d_3-d)-n+1)>l.\end{aligned} \tag{13}$$

Using the above inequalities (13), we have

$$\begin{aligned}&n\cdot\min\{k-n+1,n_3d+l_3(d_3-d)-n+1\}>k,\\&n_3d\cdot\min\{k-n+1,n_3d+l_3(d_3-d)-n+1\}>l.\end{aligned} \tag{14}$$

So, we have $c_2=0$ and $L=a_2z_0+b_2z_1+L_2+\cdots$ in (7). Since φ is a biholomorphism at the origin,

$$|\,J_\varphi(O)\,|=a_1b_2c_3\neq 0.$$

We consider two subcases (Ia) and (Ib) of Case I.

(Ia) $n < n_3d + l_3(d_3 - d) \le k$.

Then we may write H as

$$H = a_1 z_0 + H_{n_3d+l_3(d_3-d)-n+1} + \cdots .$$

In the expansion of

$$\begin{aligned}&E_d H^{n_3d} M^{l_3(d_3-d)}\\&= E_d(a_1z_0 + H_{n_3d+l_3(d_3-d)-n+1} + \cdots)^{n_3d}(a_3z_0 + b_3z_1 + c_3z_2 + M_2 + \cdots)^{l_3(d_3-d)}\end{aligned}$$

in $f_3 \circ \varphi$, every monomial with degree $n_3d + l_3(d_3 - d)$ is contained in the expansion of $E_d(a_1z_0)^{n_3d}(a_3z_0 + b_3z_1 + c_3z_2)^{l_3(d_3-d)}$ only. In particular, the monomial $z_0^{n_3d} z_2^{l_3(d_3-d)}$ has nonzero coefficient $E_d a_1^{n_3d} c_3^{l_3(d_3-d)}$ in the expansion of $E_d H^{n_3d} M^{l_3(d_3-d)}$.

We claim that the monomial $z_0^{n_3d} z_2^{l_3(d_3-d)}$ does not belong to H^n. If $z_0^{n_3d} z_2^{l_3(d_3-d)} \in H^n$, then the monomial is contained in the expansion of $(a_1z_0)^\eta (H_{n_3d+l_3(d_3-d)-n+1} + \cdots)^{n-\eta}$ for some η where $0 \le \eta \le n_3d$. By the inequalities

$$\begin{aligned}\eta + (n-\eta)\ &(n_3d + l_3(d_3-d) - n + 1)\\&\ge n_3d + (n - n_3d)(n_3d + l_3(d_3-d) - n + 1)\\&> n_3d + l_3(d_3-d),\end{aligned} \tag{15}$$

every monomial in the expansion of $(a_1z_0)^\eta (H_{n_3d+l_3(d_3-d)-n+1} + \cdots)^{n-\eta}$ has degree greater than $n_3d + l_3(d_3 - d)$ if $0 \le \eta \le n_3d$. It is impossible if $z_0^{n_3d} z_2^{l_3(d_3-d)}$ belongs to H^n.

Since $L = a_2z_0 + b_2z_1 + L_2 + \cdots$ and $n_3d + l_3(d_3 - d) \le k < l$, the monomial $z_0^{n_3d} z_2^{l_3(d_3-d)}$ does not belong to L^k and M^l. That is to say, the monomial $z_0^{n_3d} z_2^{l_3(d_3-d)}$ has nonzero coefficient $E_d a_1^{n_3d} c_3^{l_3(d_3-d)}$ in the expansion of $f_3 \circ \varphi$ and it must belong to f_j for some $j = 0, 2, 4$. This leads to a contradiction.

(Ib) $k < n_3d + l_3(d_3 - d)$.

In this case, H can be written as

$$H = a_1 z_0 + H_{k-n+1} + \cdots .$$

In the expansion of $f_3 \circ \varphi$, every monomial with degree k is contained in the expansions of $(a_2z_0 + b_2z_1)^k$ in L^k and $(a_1z_0)^{n-1} H_{k-n+1}$ in H^n only. Since $n \ge n_3d + 2 \ge 3$, the monomial $z_0 z_1^{k-1}$ does not belong to $(a_1z_0)^{n-1} H_{k-n+1}$. This says that $a_2 = 0$. Note that the following assertion.

ASSERTION. If $s < \min\{n-1, n_3 d + l_3(d_3 - d) - k + 1\} = e$, then $L_s = 0$ and $H_{k-n+s} = 0$. That is to say, H and L can be written as follows:

$$\begin{aligned} H &= a_1 z_0 + H_{k-n+e} + \cdots , \\ L &= b_2 z_1 + L_e + \cdots . \end{aligned} \tag{16}$$

Proof of Assertion. It is obvious that $H_{k-n+1} = 0$. In the expansion of $f_3 \circ \varphi$, every monomial with degree $k+1$ is contained in the expansions of $z_0^{n-1} H_{k-n+2}$ in H^n and $z_1^{k-1} L_2$ in L^k only. Thus $H_{k-n+2} = L_2 = 0$ if $2 < \min\{n-1, n_3 d + l_3(d_3 - d) - k + 1\}$. If $\min\{n-1, n_3 d + l_3(d_3-d) - k + 1\} = 3$, we are done. If not, i.e., $\min\{n-1, n_3 d + l_3(d_3 - d) - k + 1\} > 3$, then every monomial with degree $k+2$ is contained in the expansions of $z_0^{n-1} H_{k-n+3}$ in H^n and $z_1^{k-1} L_3$ in L^k only in $f_3 \circ \varphi$. Thus we have $H_{k-n+3} = L_3 = 0$ if $\min\{n-1, n_3 d + l_3(d_3 - d) - k + 1\} > 3$. Continuing this process, we have the above assertion.

We claim that the monomial $z_0^{n_3 d} z_2^{l_3(d_3-d)}$ does not belong to H^n and L^k by using the assertion. Consider two subcases of the case (Ib).

(Ib-I) $n_3 d + l_3(d_3 - d) - k + 1 \le n - 1$.

In this case, H and L can be written as follows:

$$\begin{aligned} H &= a_1 z_0 + H_{n_3 d + l_3(d_3 - d) - n + 1} + \cdots , \\ L &= b_2 z_1 + L_{n_3 d + l_3(d_3-d) - k + 1} + \cdots . \end{aligned} \tag{17}$$

If the monomial $z_0^{n_3 d} z_2^{l_3(d_3-d)}$ belongs to H^n, then the monomial $z_0^{n_3 d}$ $z_2^{l_3(d_3-d)}$ belongs to $(a_1 z_0)^\eta (H_{n_3 d + l_3(d_3-d) - n + 1} + \cdots)^{n-\eta}$ for some η where $0 \le \eta \le n_3 d$. The inequality (15) implies that every monomial in the expansion of $(a_1 z_0)^\eta (H_{n_3 d + l_3(d_3 - d) - n + 1} + \cdots)^{n-\eta}$ has degree greater than $n_3 d + l_3(d_3 - d)$ if $0 \le \eta \le n_3 d$. This leads to a contradiction.

Similarly, we have $z_0^{n_3 d} z_2^{l_3(d_3-d)} \notin L^k$ by using the inequality

$$k(n_3 d + l_3(d_3 - d) - k + 1) > n_3 d + l_3(d_3 - d).$$

(Ib-II) $n - 1 < n_3 d + l_3(d_3 - d) - k + 1$.

In this case, we claim that H and L can be written as follows:

$$\begin{aligned} H &= z_0 P_1 + z_1 Q_1 + H_{n_3 d + l_3(d_3-d) - n + 1} + \cdots , \\ L &= z_0 P_2 + z_1 Q_2 + L_{n_3 d + l_3(d_3 - d) - k + 1} + \cdots , \end{aligned} \tag{18}$$

for some polynomials P_1, Q_1 with degrees less than $n_3d+l_3(d_3-d)-n+1$ and P_2, Q_2 with degrees less than $n_3d+l_3(d_3-d)-k+1$. Note that

$$\begin{aligned} H &= a_1z_0 + H_{k-1} + \cdots, \\ L &= b_2z_1 + L_{n-1} + \cdots. \end{aligned} \tag{19}$$

Since $n-1+k-1 < n_3d+l_3(d_3-d)$, every monomial with degree $k+n-2$ is contained in the expansions of $(a_1z_0)^{n-1}H_{k-1}$ in H^n and $(b_2z_1)^{k-1}L_{n-1}$ in L^k in the expansion of $f_3 \circ \varphi$. Since the monomial $z_0^{n-1}z_2^{k-1}$ with coefficient $A_{0,0,k-1}a_1^{n-1}$ is not contained in $(b_2z_1)^{k-1}L_{n-1}$ and the monomial $z_1^{k-1}z_2^{n-1}$ with coefficient $B_{0,0,n-1}b_2^{k-1}$ is not contained in $(a_1z_0)^{n-1}H_{k-1}$, we have $A_{0,0,k-1} = B_{0,0,n-1} = 0$. That is to say, $z_2^{k-1} \notin H_{k-1}$ and $z_2^{n-1} \notin L_{n-1}$. If $n_3d+l_3(d_3-d) = k+n-1$, we are done. If not, i.e., $n_3d+l_3(d_3-d) > k+n-1$, continue this process. In H^n, we claim that the monomial $z_0^{n-1}z_2^k$ with degree $k+n-1$ is contained in $(a_1z_0)^{n-1}H_k$ only. In the expansion

$$\begin{aligned} H^n &= (a_1z_0 + H_{k-1} + H_k + \cdots)^n \\ &= \textstyle\sum_{\eta=0}^n {}_nC_\eta (a_1z_0 + H_{k-1})^{n-\eta}(H_k + \cdots)^\eta, \end{aligned}$$

we have $z_0^{n-1}z_2^k \notin (a_1z_0 + H_{k-1})^{n-\eta}(H_k + \cdots)^\eta$ if $\eta = 0$ or $\eta > 1$. Thus

$$z_0^{n-1}z_2^k \in (a_1z_0 + H_{k-1})^{n-1}(H_k + \cdots)$$

if $z_0^{n-1}z_2^k \in H^n$. In particular, $z_0^{n-1}z_2^k \in (a_1z_0)^{n-1}H_k$ only. Since the monomial $z_0^{n-1}z_2^k$ does not belong to L^k, M^l and $H^{n_3d}M^{l_3(d_3-d)}$ if $n_3d + l_3(d_3-d) > k+n-1$, the monomial $z_0^{n-1}z_2^k$ has coefficient $nA_{0,0,k}a_1^{n-1}$ in the expansion of $f_3 \circ \varphi$. Thus $A_{0,0,k} = 0$.

Similarly, we can show that the monomial $z_1^{k-1}z_2^n$ is contained in the expansion of L^k only and has coefficient $kB_{0,0,n}b_2^{k-1}$ in the expansion of $f_3 \circ \varphi$. Thus $B_{0,0,n} = 0$. That is, $z_2^k \notin H_k$ and $z_2^n \notin L_n$. Continue this process. Then we have the following facts:

$$\begin{aligned} A_{0,0,k-1} &= A_{0,0,k} = \cdots = A_{0,0,n_3d+l_3(d_3-d)-n} = 0, \\ B_{0,0,n-1} &= B_{0,0,n} = \cdots = B_{0,0,n_3d+l_3(d_3-d)-k} = 0. \end{aligned} \tag{20}$$

This says that (18) holds. Using the facts (20), we prove that $z_0^{n_3d}z_2^{l_3(d_3-d)}$ does not belong to H^n and L^k. If $z_0^{n_3d}z_2^{l_3(d_3-d)} \in H^n$, then

$$z_0^{n_3d}z_2^{l_3(d_3-d)} \in (z_0P_1 + z_1Q_1)^{n-\eta}(H_{n_3d+l_3(d_3-d)-n+1} + \cdots)^\eta$$

for some η where $0 \le \eta \le n$. If either $\eta = 0$ or $\eta > 1$, then

$$z_0^{n_3d}z_2^{l_3(d_3-d)} \notin (z_0P_1 + z_1Q_1)^{n-\eta}(H_{n_3d+l_3(d_3-d)-n+1} + \cdots)^\eta,$$

since $z_0^{n_3d} z_2^{l_3(d_3-d)} \notin (z_0P_1 + z_1Q_1)^n$ and

$$n - \eta + \eta(n_3d + l_3(d_3 - d) - n + 1) > n_3d + l_3(d_3 - d),$$

if $\eta > 1$. Thus, if $z_0^{n_3d} z_2^{l_3(d_3-d)} \in H^n$, then

$$z_0^{n_3d} z_2^{l_3(d_3-d)} \in (z_0P_1 + z_1Q_1)^{n-1}(H_{n_3d+l_3(d_3-d)-n+1} + \cdots).$$

But it is impossible, since $n_3d < n - 1$.

Similarly, we can prove that if $z_0^{n_3d} z_2^{l_3(d_3-d)} \in L^k$, then

$$z_0^{n_3d} z_2^{l_3(d_3-d)} \in (z_0P_2 + z_1Q_2)^{k-1}(L_{n_3d+l_3(d_3-d)-k+1} + \cdots).$$

But it is also impossible, since $n_3d < k - 1$. Thus $z_0^{n_3d} z_2^{l_3(d_3-d)}$ does not belong to the expansions of H^n and L^k in this case.

Consequently, $z_0^{n_3d} z_2^{l_3(d_3-d)}$ does not belong to H^n and L^k at any case. Furthermore, the monomial $z_0^{n_3d} z_2^{l_3(d_3-d)}$ does not belong to M^l, since $n_3d + l_3(d_3 - d) < l$.

These show that the monomial $z_0^{n_3d} z_2^{l_3(d_3-d)}$ has nonzero coefficient $E_d a_1^{n_3d} c_3^{l_3(d_3-d)}$ in the expansion of $f_3 \circ \varphi$. But the monomial $z_0^{n_3d} z_2^{l_3(d_3-d)}$ does not belong to f_j for $j = 0, 2, 4$. This leads to a contradiction.

Case II) $\gcd(n, l) = n < l$.

Then $l = nl_3$ for some positive integer l_3. By Theorem 3.4, A, f_3 has analytically two different representations as follows:

$$\begin{aligned} f_{31} &= z_0^n + z_1^k + z_2^l + \textstyle\sum_{i=1}^{\alpha} E_{1,i} z_0^i z_2^{l_3(n-i)}, \\ f_{32} &= z_0^n + z_1^k + \textstyle\sum_{r=1}^{\beta} E_{2,r} z_0^r z_2^{l_3(n-r)} \end{aligned}$$

for some complex numbers $E_{1,i}$ and $E_{2,r}$, where α is the largest number of i with $E_{1,i} \neq 0$ and β is the largest number of r with $E_{2,r} \neq 0$ for $1 \leq i, r \leq n-2$. Therefore, if $f_3 \approx f_j$, then either $f_{31} \approx f_j$ or $f_{32} \approx f_j$.

Suppose that $f_{31} \approx f_j$. Then there is a biholomorphism $\varphi : (\mathbb{C}^3, O) \to (\mathbb{C}^3, O)$ at the origin as (7) such that $f_{31} \circ \varphi = f_j$. Note that

$$n < \alpha + l_3(n - \alpha) < \alpha - 1 + l_3(n - \alpha + 1) < \cdots < 1 + l_3(n - 1) < l.$$

As in the proof of Case I, the monomial $z_0^{\alpha} z_2^{l_3(n-\alpha)}$ has nonzero coefficient $E_{1,\alpha} a_1^{\alpha} c_3^{l_3(n-\alpha)}$ in the expansion of $f_{31} \circ \varphi$. But the monomial $z_0^{\alpha} z_2^{l_3(n-\alpha)}$ does not belong to f_j for $j = 0, 2.4$. This leads to a contradiction.

Similarly, suppose that $f_{32} \approx f_j$ and $\varphi : (\mathbb{C}^3, O) \to (\mathbb{C}^3, O)$ are chosen as (7) so that $f_{32} \circ \varphi = f_j$. Then the monomial $z_0^{\beta} z_2^{l_3(n-\beta)}$ has nonzero coefficient $E_{2,\beta} z_1^{\beta} c_3^{l_3(n-\beta)}$ in the expansion of $f_{32} \circ \varphi$. But the monomial $z_0^{\beta} z_2^{l_3(n-\beta)}$ does not belong to f_j. This also leads to a contradiction.

Consequently, we show that neither $f_{31} \approx f_j$ nor $f_{32} \approx f_j$. Thus $f_3 \not\approx f_j$.

By Case I and Case II, $f_3 \not\approx f_j$ at any case.

This completes the proof. □

Proof of Theorem 3.6. Proposition 3.11, Proposition 3.12 and Proposition 3.13 imply Theorem 3.6. □

Proof of Theorem 3.9. Suppose that $f_4 \approx f_2$. Choose a biholomorphism $\varphi : (\mathbb{C}^3, O) \to (\mathbb{C}^3, O)$ at the origin $O = (0,0,0)$ as (7) so that $f_4 \circ \varphi = f_2$.

Claim that the monomial $z_0^{\alpha_0} z_1^{\beta_0} z_2^{\gamma_0}$ has nonzero coefficient $D_{\alpha_0,\beta_0,\gamma_0} a_1^{\alpha_0} b_2^{\beta_0} c_3^{\gamma_0}$ for some complex numbers a_1, b_2 and c_3 in the expansion of $f_4 \circ \varphi$. To prove the claim, it is enough to consider two cases: $\alpha_0 + \beta_0 + \gamma_0 \leq k$ and $k < \alpha_0 + \beta_0 + \gamma_0 \leq k + n - 2$.

Case I) $\alpha_0 + \beta_0 + \gamma_0 \leq k$

In this case, we may write H as follows:

$$H = a_1 z_0 + H_{\alpha_0+\beta_0+\gamma_0-n+1} + \cdots$$

by the similar method as in the proof of Proposition 3.13. Consider the expansion of $f_4 \circ \varphi$ as follows:

(21)

$$\begin{aligned} & H^n + L^k + M^l + \textstyle\sum_{(\alpha,\beta,\gamma)\in I_4} D_{\alpha,\beta,\gamma} H^{\alpha} L^{\beta} M^{\gamma} \\ & = (a_1 z_0 + H_{\alpha_0+\beta_0+\gamma_0-n+1} + \cdots)^n + (a_2 z_0 + b_2 z_1 + c_2 z_1 + L_2 + \cdots)^k \\ & \ +(a_3 z_0 + b_3 z_1 + c_3 z_2 + M_2 + \cdots)^l + \textstyle\sum_{(\alpha,\beta,\gamma)\in I_4} D_{\alpha,\beta,\gamma} (a_1 z_0 + \\ & \ H_{\alpha_0+\beta_0+\gamma_0-n+1} + \cdots)^{\alpha} (a_2 z_1 + b_2 z_1 + c_2 z_2 + L_2 + \cdots)^{\beta} (a_3 z_0 + \\ & \ b_3 z_1 + c_3 z_2 + M_2 + \cdots)^{\gamma}. \end{aligned}$$

Note that the monomial z_2^k does not belong to the expansions of H^n, M^l and $D_{\alpha_0,\beta_0,\gamma_0} H^{\alpha_0} L^{\beta_0} M^{\gamma_0}$ by the inequalities $n(\alpha_0 + \beta_0 + \gamma_0 - n + 1) > k, \alpha_0(\alpha_0 + \beta_0 + \gamma_0 - n + 1) + \beta_0 + \gamma_0 > k$ and $l > k$. These show that $z_2^k \in L^k$ only in the expansion of (21) and the monomial z_2^k has coefficient c_2^k. Since the monomial z_2^k does not belong to $f_2(z, z_1, z_2)$, we have $c_2 = 0$. Thus $| J_{\varphi}(O) | = a_1 b_2 c_3 \neq 0$.

We claim that $z_0^{\alpha_0} z_1^{\beta_0} z_2^{\gamma_0}$ has nonzero coefficient $D_{\alpha_0,\beta_0,\gamma_0} a_1^{\alpha_0} b_2^{\beta_0} c_3^{\gamma_0}$ in the expansion of (21). By the inequalities $\alpha_0 + (n - \alpha_0)(\alpha_0 + \beta_0 + \gamma_0 - n + 1) >$

$\alpha_0+\beta_0+\gamma_0$ and $l > \alpha_0+\beta_0+\gamma_0$, the monomial $z_0^{\alpha_0}z_1^{\beta_0}z_2^{\gamma_0}$ does not belong to the expansions of H^n and M^l. If $k > \alpha_0+\beta_0+\gamma_0$, then $z_0^{\alpha_0}z_1^{\beta_0}z_2^{\gamma_0} \notin L^k$ is clear. If $k = \alpha_0+\beta_0+\gamma_0$, then $z_0^{\alpha_0}z_1^{\beta_0}z_2^{\gamma_0} \notin L^k$, since every monomial with degree $k = \alpha_0+\beta_0+\gamma_0$ is contained in the expansion of $(a_2z_0+b_2z_1)^k$ in L^k, and the monomial $z_0^{\alpha_0}z_1^{\beta_0}z_2^{\gamma_0}$ does not belong to the expansion of $(a_2z_0+b_2z_1)^k$.

Consequently, the monomial $z_0^{\alpha_0}z_1^{\beta_0}z_2^{\gamma_0}$ belongs to $D_{\alpha_0,\beta_0,\gamma_0}H^{\alpha_0}L^{\beta_0}M^{\gamma_0}$ only in the expansion of (21). Thus the monomial $z_0^{\alpha_0}z_1^{\beta_0}z_2^{\gamma_0}$ has nonzero coefficient $D_{\alpha_0,\beta_0,\gamma_0}a_1^{\alpha_0}b_2^{\beta_0}c_3^{\gamma_0}$ in the expansion of $f_4 \circ \varphi$.

Case II) $k < \alpha_0+\beta_0+\gamma_0 \leq k+n-2$

In this case, we may write H as follows:

$$H = a_1z_0 + H_{k-n+1} + \cdots .$$

In the expansion of $f_4 \circ \varphi$, every monomial with degree k is contained in the expansions of $n(a_1z_0)^{n-1}H_{k-n+1}$ in H^n and $(a_2z_0+b_2z_1+c_2z_2)^k$ in L^k only. Note that $n \geq \alpha+2 \geq 3$. Since the monomials $z_0z_1^{k-1}$ and z_2^k do not belong to the expansion of $n(a_1z_0)^{n-1}H_{k-n+1}$ and $f_2(z_0,z_1,z_2)$, we have $a_2 = c_2 = 0$. This follows

$$L = b_2z_1 + L_2 + \cdots .$$

By the Assertion in the proof of Proposition 3.13, H and L can be written as follows:

$$\begin{aligned} H &= a_1z_0 + H_{\alpha_0+\beta_0+\gamma_0-n+1} + \cdots , \\ L &= b_2z_1 + L_{\alpha_0+\beta_0+\gamma_0-k+1} + \cdots . \end{aligned}$$

Thus the monomial $z_0^{\alpha_0}z_1^{\beta_0}z_2^{\gamma_0}$ does not belong to the expansions of H^n and L^k by the inequalities $\alpha_0+(n-\alpha_0)(\alpha_0+\beta_0+\gamma_0-n+1) > \alpha_0+\beta_0+\gamma_0$ and $\beta_0+(k-\beta_0)(\alpha_0+\beta_0+\gamma_0-k+1) > \alpha_0+\beta_0+\gamma_0$, since $n \geq \alpha_0+2$ and $k \geq \beta_0+2$. It is clear that the monomial $z_0^{\alpha_0}z_1^{\beta_0}z_2^{\gamma_0}$ does not belong to M^l, since $l > \alpha_0+\beta_0+\gamma_0$.

Consequently, the monomial $z_0^{\alpha_0}z_1^{\beta_0}z_2^{\gamma_0}$ is contained in the expansion of $D_{\alpha_0,\beta_0,\gamma_0}\,H^{\alpha_0}L^{\beta_0}M^{\gamma_0}$ only in the expansion of $f_4 \circ \varphi$. Thus the monomial $z_0^{\alpha_0}z_1^{\beta_0}z_2^{\gamma_0}$ has nonzero coefficient $D_{\alpha_0,\beta_0,\gamma_0}a_1^{\alpha_0}b_2^{\beta_0}c_3^{\gamma_0}$ and must belong to $f_2(z_0,z_1,z_2)$. This says that $f_4 \not\approx f_2$ if $(\alpha,\beta,\gamma) \neq (\alpha',\beta',\gamma')$. In particular, if $f_4 \approx g_4$, then $(\alpha_0,\beta_0,\gamma_0)$ belongs to I_4' and there exist complex numbers a_1, b_2 and c_3 such that $D_{\alpha_0,\beta_0,\gamma_0}a_1^{\alpha_0}b_2^{\gamma_0}c_3^{\gamma_0} = D'_{\alpha_0,\beta_0,\gamma_0}$.

This completes the proof of Theorem 3.9. □

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