J. Korean Math. Soc. **36** (1999), No. 5, pp. 855–890

FRACTIONAL MAXIMAL AND INTEGRAL OPERATORS ON WEIGHTED AMALGAM SPACES

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ABSTRACT. Necessary and sufficient conditions on the weight functions $u(.)$ and $v(.)$ are derived in order that the fractional maximal operator M_α, $0 \leq \alpha < 1$, is bounded from the weighted amalgam space $\ell^s(L^p(\mathbb{R}, v(x)dx))$ into $\ell^r(L^q(\mathbb{R}, u(x)dx))$ whenever $1 < s \leq r < \infty$ and $1 < p \leq q < \infty$. The boundedness problem for the fractional integral operator I_α, $0 < \alpha < 1$, is also studied.

1. Introduction

The fractional maximal operator M_α of order α, $0 \leq \alpha < 1$, is defined as

$$(M_\alpha f)(x) = \sup_{t>0}\Big\{(2t)^{\alpha-1}\int_{x-t}^{x+t}|f(y)|dy\Big\}.$$

So $M = M_0$ is the well-known Hardy-Littlewood maximal operator.

In this paper we find necessary and sufficient conditions on the weight functions $u(.)$ and $v(.)$ for which M_α is bounded from the weighted amalgam space $\ell^s(L^p_v) = \ell^s(L^p(\mathbb{R}, v(x)dx))$ into $\ell^r(L^q_u)$ whenever $1 < s \leq r < \infty$ and $1 < p \leq q < \infty$. In other words, our purpose is to determine conditions on the existence of a constant $C > 0$ such that

$$\begin{aligned}\Big\|(M_\alpha f)(.)\Big\|_{\ell^r(L^q_u)} &= \Big[\sum_{m\in\mathbb{Z}}\Big(\int_m^{m+1}(M_\alpha f)^q(x)u(x)dx\Big)^{\frac{r}{q}}\Big]^{\frac{1}{r}} \\ &\leq C\Big[\sum_{m\in\mathbb{Z}}\Big(\int_{m-1}^{m}f^p(x)v(x)dx\Big)^{\frac{s}{p}}\Big]^{\frac{1}{s}}\end{aligned} \tag{1.1}$$

Received March 18, 1998.

1991 Mathematics Subject Classification: 42B25.

Key words and phrases: weighted inequalities, fractional maximal operators, fractional integral operators, amalgam spaces.

$$= C\Big\|f(.)\Big\|_{\ell^s(L^p_v)} \quad \text{for all } f(.) \geq 0.$$

For convenience such an inequality will be denoted by $M_\alpha : \ell^s(L^p_v) \to \ell^r(L^q_u)$. The boundedness $I_\alpha : \ell^s(L^p_v) \to \ell^r(L^q_u)$ will be also studied, where I_α is the fractional integral operator defined by

$$(I_\alpha f)(x) = \int_{-\infty}^{+\infty} |x-y|^{\alpha-1} f(y)dy \quad \text{with} \quad 0 < \alpha < 1.$$

Interest on amalgam spaces (see for instance [9], [6]) comes from the fact that they naturally arose in Harmonic Analysis, since they simultaneously gave informations on local and global behaviour of the functions. Thus amalgam spaces and their weighted version are involved in the study of the Fourier Transform operator [11], [12].

It seems that the question on the boundedness of the Hardy-Littlewood maximal operator $M = M_0$ on such spaces was first considered by C. Carton-Lebrun, H. Heinig and S. Hofmann [4]. More precisely, they found a sufficient condition on $v(.)$ in order that $M : \ell^s(L^p_v) \to \ell^s(L^p_v)$, and the problem for the two-weight $u(.)$ and $v(.)$ remains unstudied until the present paper. To consider $M_\alpha : \ell^s(L^p_v) \to \ell^r(L^q_u)$ in full generality is meaningful since for some operators (like the Fourier Transform) the so-called "one-weight theory" is not interesting (because in this case necessarily the weight $u(.) = v(.)$ is up to a multiplicative constant).

The idea used by the authors in [4] to get $M : \ell^s(L^p_v) \to \ell^s(L^p_v)$ is to reduce this boundedness (via a duality argument) to the two-weight inequality

$$(1.2) \qquad \int_{\mathbb{R}} (Mf)^p(x)\overline{u}(x)dx \leq C \int_{\mathbb{R}} f^p(x)\overline{v}(x)dx \quad \text{for all } f(.) \geq 0$$

for some weights $\overline{u}(.)$ and $\overline{v}(.)$. With a well-known boundedness criterion for M on weighted Lebesgue spaces [16], this inequality (1.2) is proven by using a boundedness of a suitable discrete maximal operator on some weighted sequence space $\ell^p_{\mathcal{V}} = \{(a_k)_k | \left(\sum_k |a_k|^p \mathcal{V}_k\right)^{\frac{1}{p}} < \infty\}$. The sufficient condition found in [4] (see Theorems 4.2 and 4.5) for $M : \ell^s(L^p_v) \to \ell^s(L^p_v)$ is stronger than the usual condition $v(.) \in A_p$

for the boundedness $M : L_v^p \to L_v^p$. Such a result does not reflect the real nature of amalgams spaces. Indeed in considering the general problem $M_\alpha : \ell^s(L_v^p) \to \ell^r(L_u^q)$ it is expected to obtain a condition assuming the local property of the functions and a condition for the global behaviour.

In Theorem 2.1, we show that inequality (1.1) is equivalent both to a local $L_v^p - L_u^q$ boundedness of M_α and a boundedness of a corresponding discrete operator $\mathcal{M}_\alpha$ from a weighted sequence space $\ell_\mathcal{V}^s$ into another one $\ell_\mathcal{U}^r$. This local boundedness, denoted by $(loc)M_\alpha : L_v^p \to L_u^q$, can be derived from well-known conditions for the global boundedness $M_\alpha : L_v^p \to L_u^q$ (see [16], [13], [14], [18] and [15]). We will also see that $\mathcal{M}_\alpha : \ell_\mathcal{V}^s \to \ell_\mathcal{U}^r$ is equivalent to the boundedness of $\mathcal{M}_{\alpha,\mu}$ from $L^s(X, \mathcal{V}(n)d\mu(n))$ into $L^r(X, \mathcal{U}(n)d\mu(n))$ where $(X, d\mu)$ is a space of homogeneous type in the sense of Coifman-Weiss [5], and $\mathcal{M}_{\alpha,\mu}$ is a fractional maximal operator acting on functions of $(X, d\mu)$. Criteria for $\mathcal{M}_{\alpha,\mu} : L^s(X, \mathcal{V}(n)d\mu(n)) \to L^r(X, \mathcal{U}(n)d\mu(n))$ are now given in [3], [8] and [19]. Consequently a characterization for $M_\alpha : \ell^s(L_v^p) \to \ell^r(L_u^q)$ is obtained in Theorem 2.2. Since the characterizing condition is difficult to check, we state in Theorem 2.3 a less general result, giving a necessary condition and a sufficient condition for $M_\alpha : \ell^s(L_v^p) \to \ell^r(L_u^q)$. Problems about computabilities of the conditions introduced are discussed in Lemmas 2.4 and 2.6. As an illustration, a characterization of power weight functions for which $M_\alpha : \ell^s(L_v^p) \to \ell^r(L_u^q)$ is obtained in Corollary 2.7.

All results about the fractional maximal operator M_α are given in §2. Statements of results for the fractional integral operator I_α are given in §3. Proofs of results for M_α (resp. I_α) are presented in §4 (resp. §5).

2. Results for the Fractional Maximal Operator

Throughout this paper it is assumed that

$$1 < s \le r < \infty, \quad 1 < p < \infty, \quad 0 < q < \infty, \quad s' = \frac{s}{s-1}, \quad p' = \frac{p}{p-1};$$

and $u(.), v^{1-p'}(.)$ are nonnegative locally integrable functions on $\mathbb{R}$ and finite a.e. In this section it is supposed that

$$0 \le \alpha < 1.$$

First some notations are introduced. So functions defined on the set $\mathbb{Z}$ of integers are denoted by calligraphic letters as $\mathcal{F}(.)$, $\mathcal{U}(.)$, $\mathcal{V}(.)$ and $\mathcal{W}(.)$. A discrete analogue of M_α is given by

$$(\mathcal{M}_\alpha \mathcal{F})(n) = \sup_{N\in\mathbb{N}} \Big\{ (2N+1)^{\alpha-1} \sum_{k=n-N}^{n+N} |\mathcal{F}(k)| \Big\}.$$

The notation $\mathcal{M}_\alpha : \ell^s_{\mathcal{V}} \to \ell^r_{\mathcal{U}}$ means that
(2.1)

$$\Big(\sum_{n\in\mathbb{Z}} (\mathcal{M}_\alpha\mathcal{F})^r(n)\mathcal{U}(n)\Big)^{\frac{1}{r}} \le C\Big(\sum_{n\in\mathbb{Z}} \mathcal{F}^s(n)\mathcal{V}(n)\Big)^{\frac{1}{s}} \quad \text{for all } \mathcal{F}(.) \ge 0.$$

Similarly $(loc)M_\alpha : L^p_v \to L^q_u$is defined as

$$\Big(\int_n^{n+3} (M_\alpha f)^q(x)u(x)dx\Big)^{\frac{1}{q}} \le C\Big(\int_n^{n+3} f^p(x)v(x)dx\Big)^{\frac{1}{p}} \tag{2.2}$$
$$\text{for all } f(.) \ge 0$$

and for all $n \in \mathbb{Z}$. In (2.1) and (2.2), $C > 0$ is a fixed constant. In (2.2) each considered function $f(.)$ is supported by the interval $[n, n+3]$.

We are now in position to state our first main result.

THEOREM 2.1. *The boundedness* $M_\alpha : \ell^s(L^p_v) \to \ell^r(L^q_u)$ *does hold if and only if*

$$(loc)M_\alpha : L^p_v \to L^q_u$$

and

$$\mathcal{M}_\alpha : \ell^s_{\mathcal{V}} \to \ell^r_{\mathcal{U}}$$

where $\mathcal{U}(n) = \Big(\int_n^{n+1} u(y)dy\Big)^{\frac{r}{q}}$ *and* $\mathcal{V}(n) = \Big(\int_{n-1}^{n} v^{1-p'}(y)dy\Big)^{-\frac{s}{p'}}$.

This result is the high point for solving $M_\alpha : \ell^s(L^p_v) \to \ell^r(L^q_u)$ since it reduces the problem to boundednesses for maximal operators, whose characterizations are at present known in [16], [18] and [8].
Indeed for $\widetilde{u}(.) = u(.)\mathbb{1}_{[n,n+3]}(.)$ and $\widetilde{v}(.) = v(.)\mathbb{1}_{[n,n+3]}(.)$, inequality (2.2) is a consequence of the global boundedness $M_\alpha : L^p_{\widetilde{v}} \to L^q_{\widetilde{u}}$ which is also equivalent to
(2.3)

$$\Big(\int_{\mathbb{R}} (M_\alpha \widetilde{\sigma} g)^q(x)\widetilde{u}(x)dx\Big)^{\frac{1}{q}} \le C\Big(\int_{\mathbb{R}} g^p(x)\widetilde{\sigma}(x)dx\Big)^{\frac{1}{p}} \quad \text{for all } g(.) \ge 0,$$

where $\widetilde{\sigma}(.) = v^{1-p'}(.)\mathbb{1}_{[n,n+3]}(.)$. Of course $\mathbb{1}_E(.)$ denotes the characteristic function of the measurable set E. Necessary and sufficient conditions for (2.3) were obtained by E. Sawyer [16] for $p \leq q$ and by I. Verbitsky [18] for $q < p$. Therefore the characterization for $M_\alpha : L^p_{\widetilde{v}} \to L^q_{\widetilde{u}}$ would lead to a criterion for the local boundedness $(loc)M_\alpha : L^p_v \to L^q_u$. However a slight difficulty appears since the constant C in (2.3) must not depend on the integer n, though it is proved in [16] and [18] that this constant generally depends on the weights $\widetilde{u}(.)$ and $\widetilde{v}(.)$. This inconvenience will be overcome below in the proof of next Theorem 2.2.

The boundedness $\mathcal{M}_\alpha : \ell^s_{\mathcal{V}} \to \ell^r_{\mathcal{U}}$ can be reduced to a boundedness of a suitable maximal operator acting on a homogeneous space type. To be clear, the set $\mathbb{Z}$ of all integers is also denoted by X. On $X \times X$ is defined the usual distance $d(m,n) = |m-n|$. For each $n \in X$ and $t > 0$, the ball $B(n,t)$ centered at n and with the radius t is defined by

$$B(n,t) = \{m \in X;\ d(n,m) < t\} =]n-t, n+t[\cap \mathbb{Z}.$$

Observe that $B(n,t) = \{n\}$ for $0 \leq t < 1$, and $B(n,t) = \{n-N, \dots, n-1, n, n+1, \dots, n+N\}$ whenever N is the unique nonnegative integer such that $N < t \leq N+1$. The usual comptage measure on $\mathbb{Z}$ is given by

$$d\mu(n) = \sum_{l \in \mathbb{Z}} \delta_l(n) \quad \text{where } \delta_l(n) = 1 \text{ if } l = n \text{ else } \delta_l(n) = 0.$$

Obviously

$$|B(n,t)|_\mu = \int_{m \in X} \mathbb{1}_{B(n,t)}(m)\, d\mu(m) = \sum_{m=n-N}^{n+N} 1 = 2N+1$$

and

$$\text{(2.4)} \qquad |B(n,2t)|_\mu \leq 4|B(n,t)|_\mu$$

since $|B(n,2t)|_\mu \leq 2(2N+1)+1 \leq 4(2N+1) = 4|B(n,t)|_\mu$. Property (2.4) is known as a doubling condition for $d\mu$. Thus the triplet

$(X, d, d\mu)$ has the homogeneous structure in the sense of Coifman-Weiss [5]. The discrete maximal operator can be interpreted as

$$(\mathcal{M}_\alpha \mathcal{F})(n) = (\mathcal{M}_{\alpha,\mu}\mathcal{F})(n) = \sup_{t>0}\Big\{|B(n,t)|_\mu^{\alpha-1}\int_{B(n,t)}|\mathcal{F}(m)|d\mu(m)\Big\}.$$

Consequently $\mathcal{M}_\alpha : \ell^s_{\mathcal{V}} \to \ell^r_{\mathcal{U}}$ is the same as $\mathcal{M}_{\alpha,\mu} : L^s(X, \mathcal{V}(m)d\mu(m)) \to L^r(X, \mathcal{U}(m)d\mu(m))$. A characterization of $\mathcal{V}(m)$ and $\mathcal{U}(m)$ for which this boundedness holds, has recently been given by A. Gogatishvili and V. Kokilashvili [8] (Theorem 2.1, p. 425).

Therefore Theorem 2.1 and the boundedness criteria for maximal operators given in [16] and [8] lead to a complete characterization of weights $u(.)$ and $v(.)$ for which (1.1) is satisfied.

THEOREM 2.2. *Assume that $M_\alpha : \ell^s(L^p_v) \to \ell^r(L^q_u)$. Then for some constant $A > 0$ the following two conditions are satisfied*

$$\int_I (M_\alpha v^{1-p'}\mathbb{1}_I)^q(x)u(x)dx \le A^q\Big(\int_I v^{1-p'}(x)dx\Big)^{\frac{q}{p}} \tag{2.5}$$

for all intervals $I = [a, b]$ with $|I| = |b-a| \le 3$

and

(2.6)

$$\sum_{m=n-N}^{n+N}\Big[\sup_{M\in\mathbb{N}}(2M+1)^{\alpha-1}\sum_{j=m-M}^{m+M}\Big(\int_{j-1}^{j}v^{1-p'}(y)dy\Big)^{\frac{s'}{p'}}\Big]^r\Big(\int_m^{m+1}u(x)dx\Big)^{\frac{r}{q}}$$

$$\le A^r\Big(\sum_{m=n-N}^{n+N}\Big(\int_{m-1}^{m}v^{1-p'}(x)dx\Big)^{\frac{s'}{p'}}\Big)^{\frac{r}{s}} \quad \text{for all } n\in\mathbb{Z} \text{ and } N\in\mathbb{N}^*.$$

Conversely for $p \le q$, conditions (2.5) and (2.6) are sufficient to imply the boundedness $M_\alpha : \ell^s(L^p_v) \to \ell^r(L^q_u)$.

A similar result can be stated for the range $q < p$ and with (2.5) replaced by a more complicated condition (like (0.13) in [18], p. 128).

Although (2.5) and (2.6) are necessary and sufficient for $M_\alpha : \ell^s(L^p_v) \to \ell^r(L^q_u)$ unfortunately their are not of great use for practical applications because of the difficulty due to the presence of the maximal operators and the integrations over arbitrary intervals. So it might be preferable to state a less general result but more manageable for explicit computations. For this purpose some growth conditions on the weights are introduced and used. So recall that $w(.) \in A_\infty$ if for some $t > 1$ and $C > 0$

$$\left(\frac{1}{|I|}\int_I w(y)dy\right)^{\frac{1}{t}}\left(\frac{1}{|I|}\int_I w^{1-t'}(y)dy\right)^{\frac{1}{t'}} \leq C \tag{2.7}$$

for all intervals $I = [a,b]$.

Precisely (2.7) is referred to $w(.) \in A_t$. Similarly for the discrete case, then $\mathcal{W}(.) \in A_\infty(d\mu)$ (or precisely $\mathcal{W}(.) \in A_t(d\mu)$) provided that

(2.8)

$$\left(\frac{1}{(2N+1)}\sum_{m=n_0-N}^{n_0+N}\mathcal{W}(m)\right)^{\frac{1}{t}}\left(\frac{1}{(2N+1)}\sum_{m=n_0-N}^{n_0+N}\mathcal{W}^{(1-t')}(m)\right)^{\frac{1}{t'}} \leq C$$

for all $n_0 \in \mathbb{Z}$ and $N \in \mathbb{N}$.

THEOREM 2.3.

Part A

Suppose that $M_\alpha : \ell^s(L_v^p) \longrightarrow \ell^r(L_u^q)$. Then, for some constant $A > 0$, the following two conditions are satisfied

$$|I|^{\alpha+\frac{1}{q}-\frac{1}{p}}\left(\frac{1}{|I|}\int_I u(y)dy\right)^{\frac{1}{q}}\left(\frac{1}{|I|}\int_I v^{1-p'}(y)dy\right)^{\frac{1}{p'}} \leq A \tag{2.9}$$

for all $I = [a,b]$ with $|I| = |b-a| \leq 3$

and

$$(2N+1)^{\alpha+\frac{1}{r}-\frac{1}{s}}\left[\frac{1}{(2N+1)}\sum_{m=n_0-N}^{n_0+N}\left(\int_m^{m+1}u(y)dy\right)^{\frac{r}{q}}\right]^{\frac{1}{r}} \tag{2.10}$$

$$\times\left[\frac{1}{(2N+1)}\sum_{m=n_0-N}^{n_0+N}\left(\int_{m-1}^{m}v^{1-p'}(y)dy\right)^{\frac{s'}{p'}}\right]^{\frac{1}{s'}} \leq A$$

for all $n_0 \in \mathbb{Z}$ and $N \in \mathbb{N}$.

Part B

Conversely for $p \leq q$, conditions (2.9) and (2.10) imply $M_\alpha : \ell^s(L_v^p) \longrightarrow \ell^r(L_u^q)$ whenever both $v^{1-p'}(.) \in A_\infty$ and $\mathcal{V}^{1-s'}(.) \in A_\infty(d\mu)$. Here $\mathcal{V}(.)$ is defined as in Theorem 2.1.

Part C

Also for $p \leq q$, then $M_\alpha : \ell^s(L^p_v) \to \ell^r(L^q_u)$ whenever for some $A > 0$ and $\varepsilon > 1$ (with $p'(\alpha - \frac{1}{p}) < \frac{1}{\varepsilon}$), the following two conditions are satisfied

$$(2.11)\quad |I|^{\alpha+\frac{1}{q}-\frac{1}{p}}\left(\frac{1}{|I|}\int_I u(y)dy\right)^{\frac{1}{q}}\left(\frac{1}{|I|}\int_I v^{(1-p')\varepsilon}(y)dy\right)^{\frac{1}{p'\varepsilon}} \leq A$$

for all $I = [a, b]$ with $|I| = |b - a| \leq 3$

and

$$(2.12)\quad (2N+1)^{\alpha+\frac{1}{r}-\frac{1}{s}}\left[\frac{1}{(2N+1)}\sum_{m=n_0-N}^{n_0+N}\left(\int_m^{m+1} u(y)dy\right)^{\frac{r}{q}}\right]^{\frac{1}{r}} \times\left[\frac{1}{(2N+1)}\sum_{m=n_0-N}^{n_0+N}\left(\int_{m-1}^{m} v^{(1-p')}(y)dy\right)^{\frac{s'\varepsilon}{p'}}\right]^{\frac{1}{s'\varepsilon}} \leq A$$

for all $n_0 \in \mathbb{Z}$ and $N \in \mathbb{N}$.

Proofs of Parts B and C highly depend on results due to C. Pérez [13], [14] about the global boundedness (2.3) when $p \leq q$. Analogue of Theorem 2.3 for the case $q < p$ can be obtained by using results in [15]. However for shortness no statement on this perspective is stated.

Next our plan is to analyze how precisely conditions (2.9) and (2.10) can be managed in practical computations.

LEMMA 2.4. *Suppose that condition* (2.9) *is satisfied. Then*

$$(2.13)\quad \alpha + \frac{1}{q} - \frac{1}{p} \geq 0$$

and for some constant $A > 0$

$$(2.14)\quad R^{\alpha+\frac{1}{q}-\frac{1}{p}}\left(\frac{1}{R}\int_{|y|<R} u(y)dy\right)^{\frac{1}{q}}\left(\frac{1}{R}\int_{|y|<R} v^{(1-p')}(y)dy\right)^{\frac{1}{p'}} \leq A$$

for all R with $0 < R \leq \frac{3}{2}$.

Conversely condition (2.9) *holds whenever* (2.14) *and*

$$(2.15)\quad R_1^{\alpha+\frac{1}{q}-\frac{1}{p}}\Big(\sup_{\frac{1}{2}R<|y|<2R} u(y)\Big)^{\frac{1}{q}}\Big(\sup_{\frac{1}{2}R<|z|<2R} v^{(1-p')}(z)\Big)^{\frac{1}{p'}} \le A$$

$$\text{for all } R, R_1 > 0 \text{ with } R_1 \le \frac{3}{2} \text{ and } 4R_1 < R$$

are both satisfied.

An analogue result can be stated for (2.11) and (2.12).

Condition (2.15) is useful to avoid the difficulty due integrations over arbitrary intervals I. As an application, the power weights for which condition (2.9) holds, can be characterized.

COROLLARY 2.5. *Let* $u(y) = |y|^{\beta-1}$ *and* $v(y) = |y|^{\delta-1}$ *with* $\beta, \delta > 0$. *Then the pair* (u, v) *satisfies condition* (2.9) *if and only if*

$$\delta < p \qquad \text{and} \qquad 0 \le \alpha + \frac{\beta}{q} - \frac{\delta}{p} \le \alpha + \frac{1}{q} - \frac{1}{p}.$$

A similar result to Lemma 2.4 for checking condition (2.10) will be also needed. This condition is more difficult to handle than (2.9) due both to integrations over intervals $[m, m+1]$ and discrete summations.

LEMMA 2.6. *Suppose that condition* (2.10) *is satisfied. Then*

$$(2.16)\qquad \alpha + \frac{1}{r} - \frac{1}{s} \ge 0$$

and for each integer $N_0 \ge 4$ *there is constant* $A > 0$ *such that*

$$(2.17)\quad (2N+1)^{\alpha+\frac{1}{r}-\frac{1}{s}}\left[\frac{1}{(2N+1)}\sum_{N_0\le|m|\le N}\left(\int_m^{m+1} u(y)dy\right)^{\frac{r}{q}}\right]^{\frac{1}{r}}$$

$$\times\left[\frac{1}{(2N+1)}\sum_{N_0\le|m|\le N}\left(\int_{m-1}^{m} v^{1-p'}(y)dy\right)^{\frac{s'}{p'}}\right]^{\frac{1}{s'}} \le A$$

$$\text{for all integers } N > N_0.$$

Conversely condition (2.10) *does hold whenever*

(2.18)

$$(2N+1)^{\alpha+\frac{1}{r}-\frac{1}{s}}\Big(\sup_{\frac{1}{4}R<|y|<4R} u(y)\Big)^{\frac{1}{q}}\Big(\sup_{\frac{1}{4}R<|z|<4R} v^{(1-p')}(z)\Big)^{\frac{1}{p'}} \leq A$$

for all $R>0$ *with* $4(2N+1)<R$

and there is $N_0 \geq 4$ *for which* (2.17) *is satisfied.*

As an illustration, these tools lead to a characterization of power weights for which $M_\alpha : \ell^s(L^p_v) \to \ell^s(L^q_u)$.

COROLLARY 2.7. *Let* $u(y) = |y|^{\beta-1}$ *and* $v(y) = |y|^{\delta-1}$ *with* $\max(0, 1-\frac{q}{r}) < \beta$ *and* $0 < \delta < 1+\frac{p}{s'}$. *Then* $M_\alpha : \ell^s(L^p_v) \to \ell^s(L^q_u)$ *if and only if the following two conditions are satisfied*

$$\delta < p \quad \text{and} \quad 0 \leq \alpha + \frac{\beta}{q} - \frac{\delta}{p} \leq \alpha + \frac{1}{q} - \frac{1}{p}; \tag{2.19}$$

$$0 \leq \alpha + \frac{1}{r} - \frac{1}{s} \leq (\delta-1)\frac{1}{p} - (\beta-1)\frac{1}{q}. \tag{2.20}$$

Lemmas 2.4 and 2.6 can be also used to give other examples of weights non-necessarily of power type and for which $M_\alpha : \ell^s(L^p_v) \to \ell^s(L^q_v)$. But for shortness, no further examples are given.

Now we will pay attention to problem (1.1) for some weights largely used in Real Analysis. To this end, it is useful to specify when $\mathcal{V}^{1-s'}(.) \in A_\infty(d\mu)$.

LEMMA 2.8. *Let* $\mathcal{W}(m) = \Big(\int_{m-1}^m w(y)dy\Big)^{\frac{r_1}{q_1}}$. *For* $q_1 < r_1$, *then* $\mathcal{W}(.) \in A_\infty(d\mu)$ *whenever* $w(.) \in A_{2-\frac{q_1}{r_1}} \cap RH_{\frac{r_1}{q_1}}$. *And for* $r_1 \leq q_1$, *then* $\mathcal{W}(.) \in A_\infty(d\mu)$ *provided that* $w(.) \in A_{2-\frac{1}{t}} \cap RH_t$ *for some* $t > 1$.

Recall that $w(.) \in RH_t$ if $\Big(|I|^{-1}\int_I w^t(y)dy\Big)^{\frac{1}{t}} \leq c|I|^{-1}\int_I w(y)dy$ for a fixed constant $c > 0$ and for all intervalls $I = [a,b]$.

It will be seen in the proof of this result that, for $q_1 < r_1$, the assumptions on $w(.)$ can be weakened as: $w(.) \in RH_{(t'-1)'\frac{r_1}{q_1}}$ and $w^{\frac{r_1}{q_1}}(.) \in A_t$ for some $1 < t < \frac{r_1}{q_1}$.

COROLLARY 2.9. *Let $p \leq q$. For $s < p$ and $v^{1-p'}(.) \in A_{2-\frac{p'}{s'}} \cap RH_{\frac{s'}{p'}}$ then $M_\alpha : \ell^s(L^p_v) \to \ell^r(L^q_u)$ if and only conditions (2.9) and (2.10) are satisfied. This above equivalence remains also true for $p \leq s$ whenever $v^{1-p'}(.) \in A_{2-\frac{1}{t}} \cap RH_t$ for some $t > 1$.*

It is also interesting to derive (1.1) from a global condition rather than both (2.9) and (2.10).

COROLLARY 2.10. *Let $p \leq q$. The (global) condition*

(2.21)

$$|I|^{\alpha+\frac{1}{q}-\frac{1}{p}}\left(\frac{1}{|I|}\int_I u(y)dy\right)^{\frac{1}{q}}\left(\frac{1}{|I|}\int_I v^{1-p'}(y)dy\right)^{\frac{1}{p'}} \leq A \text{ for all } I = [a,b]$$

implies the boundedness $M_\alpha : \ell^s(L^p_v) \to \ell^r(L^q_u)$ provided that

(2.22)

$$v^{1-p'}(.) \in A_{2-\frac{p'}{s'}} \cap RH_{\frac{s'}{p'}} \quad \text{if } s < p, \text{ else } \; v^{1-p'}(.) \in A_{2-\frac{1}{t}} \cap RH_t, \quad t > 1$$

and

$$u(.) \in RH_{\frac{r}{q}} \quad \text{if } q < r. \tag{2.23}$$

No assumption on $u(.)$ is made when $r \leq q$.

This is an alternative of a result due to C. Carton, H. Heinig and S. Hofmann [Ca-He-St] (Theorem 4.2, p.147), which states that for $s < p$ then $M = M_0 : \ell^s(L^p_v) \to \ell^s(L^p_v)$ whenever $v(.) \in A_{2-\frac{p}{s}} \cap RH_{\frac{s}{p}} \cap A_p$.

3. Results for the Fractional Integral Operator

In this section it is always assumed that

$$0 < \alpha < 1$$

and the problem under consideration is about the inequality

$$\left\|(I_\alpha f)(.)\right\|_{\ell^r(L^q_u)} \leq C\left\|f(.)\right\|_{\ell^s(L^p_v)} \quad \text{for all } f(.) \geq 0 \tag{3.1}$$

which is also denoted by $I_\alpha : \ell^s(L^p_v) \to \ell^r(L^q_u)$.

Since $(I_\alpha f)(x) = \int_x^\infty (y-x)^{\alpha-1} f(y)dy + \int_{-\infty}^x (x-y)^{\alpha-1} f(y)dy =$ $(W_\alpha f)(x) + (R_\alpha f)(x)$ then the boundedness $I_\alpha : \ell^s(L^p_v) \to \ell^r(L^q_u)$ is equivalent to both $W_\alpha : \ell^s(L^p_v) \to \ell^r(L^q_u)$ and $R_\alpha : \ell^s(L^p_v) \to \ell^r(L^q_u)$. At this time there is no available characterization of weights $u(.)$ and $v(.)$ for which these last two boundednesses hold. However C. Carton-Lebrun, H. Heinig and S. Hofmann [4] were able to give a sufficient condition for $T : \ell^s(L^p_v) \to \ell^r(L^q_u)$, with T an integral operator including R_α and W_α. Consequently as in the previous section, it would be interesting to get a necessary and sufficient condition for $I_\alpha : \ell^s(L^p_v) \to \ell^r(L^q_u)$ by means of a local boundedness for I_α and a global boundedness on discrete spaces for a suitable operator.

For this purpose, define the local boundedness $(loc)I_\alpha : L^p_v \to L^q_u$ by the existence of a constant $C > 0$ such that

(3.2)
$$\Big(\int_n^{n+3} (I_\alpha f)^q(x)u(x)dx\Big)^{\frac{1}{q}} \le C\Big(\int_n^{n+3} f^p(x)v(x)dx\Big)^{\frac{1}{p}} \quad \text{for all } f(.) \ge 0$$

and for all $n \in \mathbb{Z}$.

Now our fourth main result can be stated.

THEOREM 3.1. *The boundedness $I_\alpha : \ell^s(L^p_v) \to \ell^r(L^q_u)$ does hold if and only if*

$$(loc)I_\alpha : L^p_v \to L^q_u$$

and for some constant $C > 0$ the following two conditions are satisfied

$$\text{(3.3)} \quad \Big(\sum_{n\in\mathbb{Z}}\Big[\sum_{m=-\infty}^{n-1} (n-m)^{\alpha-1}\mathcal{F}(m)\Big]^r \mathcal{U}(n)\Big)^{\frac{1}{r}} \le C\Big(\sum_{n\in\mathbb{Z}} \mathcal{F}^s(n)\mathcal{V}(n)\Big)^{\frac{1}{s}} \quad \text{for all } \mathcal{F}(.) \ge 0,$$

and

$$\text{(3.4)} \quad \Big(\sum_{n\in\mathbb{Z}}\Big[\sum_{m=n+3}^{\infty} (m-n)^{\alpha-1}\mathcal{F}(m)\Big]^r \mathcal{U}(n)\Big)^{\frac{1}{r}} \le C\Big(\sum_{n\in\mathbb{Z}} \mathcal{F}^s(n)\mathcal{V}(n)\Big)^{\frac{1}{s}} \quad \text{for all } \mathcal{F}(.) \ge 0,$$

where $\mathcal{U}(n) = \left(\int_n^{n+1} u(y)dy\right)^{\frac{r}{q}}$ and $\mathcal{V}(n) = \left(\int_{n-1}^{n} v^{1-p'}(y)dy\right)^{-\frac{s}{p'}}$.

This analogue of Theorem 2.1 may be considered as a crucial point for solving $I_\alpha : \ell^s(L^p_v) \to \ell^r(L^q_u)$. Taking $\widetilde{u}(.) = u(.)1\!\!1_{[n,n+3]}(.)$ and $\widetilde{v}(.) = v(.)1\!\!1_{[n,n+3]}(.)$ then inequality (3.2) is a consequence of

(3.5)
$$\left(\int_{\mathbb{R}} (I_\alpha \widetilde{\sigma} g)^q(x)\widetilde{u}(x)dx\right)^{\frac{1}{q}} \le C\left(\int_{\mathbb{R}} g^p(x)\widetilde{\sigma}(x)dx\right)^{\frac{1}{p}} \quad \text{for all } g(.) \ge 0,$$

where $\widetilde{\sigma}(.) = v^{1-p'}(.)1\!\!1_{[n,n+3]}(.)$. A necessary and sufficient condition for (3.5) with $p \le q$ was obtained by E. Sawyer [17].

The problem of getting a characterization for (3.3) (or (3.4)) seems still open, and we will return on this question in a next paper. A sufficient condition for (3.3) was already found by K. Andersen and H. Heinig in [He-An] (see Theorem 4.1 and 4.2, p. 843).

As in the case for M_α it is interesting to connect (3.3) and (3.4) with a discrete version of I_α in homogeneous space which is given by

$$(\mathcal{I}_{\alpha,\mu}\mathcal{F})(x) = \int_{y\in X} \left|B(x, d(x,y)\right|_\mu^{\alpha-1} \mathcal{F}(y)d\mu(y).$$

Here $d(x,y)$, $d\mu(.)$ and X are defined as in the previous section. Observe that for $x = n$ and $y = m = n+k$, where k is an integer with $|k| \ge 1$, then $B(x, d(x,y)) = \{n - (|k|-1), \dots, n \dots, n + (|k|-1)\}$ and $\left|B(x,d(x,y)\right|_\mu = 2(|k|-1)+1 = 2(|m-n|-1)+1$ and $|m-n| \le \left|B(x,d(x,y)\right|_\mu \le 2|m-n|$. So

$$(\mathcal{I}_{\alpha,\mu}\mathcal{F})(x) \approx \sum_{m=-\infty}^{n-1} (n-m)^{\alpha-1}\mathcal{F}(m) + \sum_{m=n+1}^{\infty} (m-n)^{\alpha-1}\mathcal{F}(m).$$

Consequently the boundedness

(3.6) $\qquad \mathcal{I}_{\alpha,\mu} : L^s(X, \mathcal{V}(m)d\mu(m)) \to L^r(X, \mathcal{U}(m)d\mu(m))$

implies both inequalities (3.3) and (3.4), but the equivalence is not clear since moreover (3.6) requires the inequality

$$\left(\sum_{n\in\mathbb{Z}}\left[\sum_{m=n+1}^{n+2} (m-n)^{\alpha-1}\mathcal{F}(m)\right]^r \mathcal{U}(n)\right)^{\frac{1}{r}} \le C\left(\sum_{n\in\mathbb{Z}} \mathcal{F}^s(n)\mathcal{V}(n)\right)^{\frac{1}{s}}$$

for all $\mathcal{F}(.) \ge 0$.

Results for (3.6) given in [7] and [17] could be useful to study the boundedness of I_α on weighted amalgam spaces. However a difficult appears since the annulus $B(x,R) - B(x,r)$ is reduced to the empty set whenever $0 < R - r \leq 1$. Further discussions on this point are not given, since our purpose is essentially to prove Theorem 3.1 and also to treat inequality (3.1) for some restrictive cases but manageable for explicit computations.

THEOREM 3.2. *Suppose* $I_\alpha : \ell^s(L^p_v) \to \ell^r(L^q_u)$. *Then conditions* (2.9) *and* (2.10) *are satisfied. Conversely let* $p \leq q$. *Then* (2.9) *and* (2.10) *imply the boundedness* $I_\alpha : \ell^s(L^p_v) \to \ell^r(L^q_u)$ *whenever* $u(.), v^{1-p'}(.) \in A_\infty$ *and* $\mathcal{U}(.), \mathcal{V}^{1-s'}(.) \in A_\infty(d\mu)$. *Here* $\mathcal{V}(.)$ *and* $\mathcal{U}(.)$ *are defined as in* Theorem 3.1.

Similarly to Corollary 2.10, we have

COROLLARY 3.3. *Let* $p \leq q$. *The (global) condition* (2.21) *implies the boundedness* $I_\alpha : \ell^s(L^p_v) \to \ell^r(L^q_u)$ *provided that*

(3.7)

$$u(.) \in A_{2-\frac{q}{r}} \cap RH_{\frac{r}{q}} \quad \text{if } q < r, \text{ else } \quad u(.) \in A_{2-\frac{1}{t}} \cap RH_t \quad \text{for some } t > 1$$

and the condition (2.22) *is satisfied.*

4. Proofs of results for M_α, $0 \leq \alpha < 1$

Proof of Theorem 2.1.

The Necessary Part

To get $\mathcal{M}_\alpha : \ell^s_{\mathcal{V}} \to \ell^r_{\mathcal{U}}$, let $\mathcal{F}(.)$ be a nonnegative function acting on $\mathbb{Z}$. Define the nonnegative and real function

$$f(.) = v^{1-p'}(.) \sum_{k\in\mathbb{Z}} \Big(\int_{k-1}^{k} v^{1-p'}(y)dy \Big)^{-1} \mathcal{F}(k) \mathrm{1\!I}_{[k-1,k[}(.) \tag{4.1}$$

where $\mathrm{1\!I}_{[k-1,k[}(.)$ is the characteristic function of the interval $[k-1,k[$. The main keys to obtain the conclusion are

$$\int_{n-1}^{n} f^p(x)v(x)dx = \Big(\int_{n-1}^{n} v^{1-p'}(y)dy \Big)^{1-p} \mathcal{F}^p(n) \tag{4.2}$$

and

$$(4.3) \qquad (\mathcal{M}_\alpha \mathcal{F})(n) \le C(M_\alpha f)(y) \quad \text{for all } y \in [n, n+1]$$

where $C > 0$ does not depend on n and y. Indeed from $M_\alpha : \ell^s(L^p_v) \to \ell^r(L^q_u)$ we get the conclusion since

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} (\mathcal{M}_\alpha \mathcal{F})^r(n)\mathcal{U}(n) &= \sum_{n\in\mathbb{Z}} (\mathcal{M}_\alpha \mathcal{F})^r(n)\Big(\int_n^{n+1} u(y)dy\Big)^{\frac{r}{q}} \\
&\le c_1 \sum_{n\in\mathbb{Z}} \Big(\int_n^{n+1} (M_\alpha f)^q(y)u(y)dy\Big)^{\frac{r}{q}} \quad \text{by (4.3)} \\
&\le c_2 \Big[\sum_{n\in\mathbb{Z}} \Big(\int_{n-1}^{n} f^p(x)v(x)dx\Big)^{\frac{s}{p}}\Big]^{\frac{r}{s}} \quad \text{by the hypothesis} \\
&= c_2 \Big[\sum_{n\in\mathbb{Z}} \mathcal{F}^s(n)\Big(\int_{n-1}^{n} v^{1-p'}(y)dy\Big)^{-\frac{s}{p'}}\Big]^{\frac{r}{s}} \quad \text{by (4.2)} \\
&= c_2 \Big[\sum_{n\in\mathbb{Z}} \mathcal{F}^s(n)\mathcal{V}(n)\Big]^{\frac{r}{s}}.
\end{aligned}$$

Inequality (4.2) appears immediately from $p(1-p')+1 = 1-p'$ and the definition of $f(.)$ since

$$\begin{aligned}
&\int_{n-1}^{n} f^p(x)v(x)dx \\
&= \int_{n-1}^{n} v^{1-p'}(x)\Big[\sum_{k\in\mathbb{Z}} \Big(\int_{k-1}^{k} v^{1-p'}(y)dy\Big)^{-p} \mathcal{F}^p(k)\mathbb{1}_{[k-1,k]}(x)\Big]dx \\
&= \Big(\int_{n-1}^{n} v^{1-p'}(y)dy\Big)^{1-p} \mathcal{F}^p(n).
\end{aligned}$$

To see (4.3), note that
(4.4)

$$\begin{aligned}
\mathcal{F}(k) &= \int_{k-1}^{k} v^{1-p'}(x)\Big[\Big(\int_{k-1}^{k} v^{1-p'}(y)dy\Big)^{-1}\mathcal{F}(k)\Big]dx \\
&= \int_{k-1}^{k} v^{1-p'}(x)\Big[\sum_{l\in\mathbb{Z}} \Big(\int_{l-1}^{l} v^{1-p'}(y)dy\Big)^{-1}\mathcal{F}(l)\mathbb{1}_{[l-1,l[}(x)\Big]dx
\end{aligned}$$

$$= \int_{k-1}^{k} f(x)dx = \mathcal{G}(k).$$

So for each $y \in [n, n+1]$, it appears that

$$\begin{aligned}
(\mathcal{M}_\alpha \mathcal{F})(n) &= \sup_{N\in\mathbb{N}} \Big\{ (2N+1)^{\alpha-1} \sum_{k=n-N}^{n+N} \mathcal{F}(k) \Big\} \\
&\leq \sup_{N\in\mathbb{N}} \Big\{ (2N+1)^{\alpha-1} \int_{n-(N+1)}^{n+N} f(x)dx \Big\} \\
&\leq \sup_{N\in\mathbb{N}} \Big\{ (2N+1)^{\alpha-1} \int_{y-(N+2)}^{y+N} f(x)dx \Big\} \\
&\leq \sup_{N\in\mathbb{N}} \Big\{ (2N+1)^{\alpha-1} \int_{y-(N+2)}^{y+(N+2)} f(x)dx \Big\} \leq \times 5^{1-\alpha} (M_\alpha f)(y).
\end{aligned}$$

To get $(loc) M_\alpha : L^p_v \to L^q_u$ put in (1.1) a nonnegative function $f(.)$ whose support is $[n_0 - 1, n_0 + 2]$, for some $n_0 \in \mathbb{Z}$. Then

$$\begin{aligned}
\Big\| f(.) \Big\|^s_{\ell^s(L^p_v)} &= \Big(\int_{n_0-1}^{n_0} f^p(y)v(y)dy \Big)^{\frac{s}{p}} \\
&\quad + \Big(\int_{n_0}^{n_0+1} f^p(y)v(y)dy \Big)^{\frac{s}{p}} + \Big(\int_{n_0+1}^{n_0+2} f^p(y)v(y)dy \Big)^{\frac{s}{p}} \\
&\leq c_1(s,p) \Big(\int_{n_0-1}^{n_0+2} f^p(y)v(y)dy \Big)^{\frac{s}{p}},
\end{aligned}$$

and on the other hand

$$\begin{aligned}
&\Big\| (M_\alpha f)(.) \Big\|^r_{\ell^r(L^q_u)} \\
&\geq \Big(\int_{n_0-1}^{n_0} (M_\alpha f)^q(y)u(y)dy \Big)^{\frac{r}{q}} \\
&\quad + \Big(\int_{n_0}^{n_0+1} (M_\alpha f)^q(y)u(y)dy \Big)^{\frac{r}{q}} + \Big(\int_{n_0+1}^{n_0+2} (M_\alpha f)^q(y)u(y)dy \Big)^{\frac{r}{q}} \\
&\geq c_2(r,q) \Big(\int_{n_0-1}^{n_0+2} (M_\alpha f)^q(y)u(y)dy \Big)^{\frac{r}{q}}.
\end{aligned}$$

And the conclusion follows from the above two inequalities.

The Sufficient Part

Observe that

$$\Big\|(M_\alpha f)(.)\Big\|^r_{\ell^r(L^q_u)} \leq c(r,q)\Big\{\mathcal{S}_1+\mathcal{S}_2+\mathcal{S}_3\Big\} \tag{4.5}$$

where

$$\mathcal{S}_1=\sum_{n\in\mathbb{Z}}\left(\int_n^{n+1}(M_\alpha f \mathbb{1}_{]-\infty,n-1[})^q(x)u(x)dx\right)^{\frac{r}{q}},$$

$$\mathcal{S}_2=\sum_{n\in\mathbb{Z}}\left(\int_n^{n+1}(M_\alpha f \mathbb{1}_{[n-1,n+2]})^q(x)u(x)dx\right)^{\frac{r}{q}},$$

$$\mathcal{S}_3=\sum_{n\in\mathbb{Z}}\left(\int_n^{n+1}(M_\alpha f \mathbb{1}_{]n+2,\infty[})^q(x)u(x)dx\right)^{\frac{r}{q}}.$$

So it remains to estimate each of $\mathcal{S}_i$, $i=1,2,3$ by $c\Big\|f(.)\Big\|^r_{\ell^s(L^p_v)}$ where $c>0$ is a fixed constant.

The main points for controlling $\mathcal{S}_2$ are the local boundedness $(loc)M_\alpha : L^p_v \to L^q_u$ and the fact that $s \leq r$. Indeed

$$\begin{aligned}
\mathcal{S}_2 &\leq \sum_{n\in\mathbb{Z}}\left[\int_{n-1}^{n+2}(M_\alpha f \mathbb{1}_{[n-1,n+2]})^q(x)u(x)dx\right]^{\frac{r}{q}}\\
&\leq c_1\sum_{n\in\mathbb{Z}}\left[\int_{n-1}^{n+2}f^p(x)v(x)dx\right]^{\frac{s}{p}\times\frac{r}{s}}\\
&\leq c_1\left[\sum_{n\in\mathbb{Z}}\left(\int_{n-1}^{n+2}f^p(x)v(x)dx\right)^{\frac{s}{p}}\right]^{\frac{r}{s}} \quad \text{since } \frac{r}{s}\geq 1\\
&= c_1\left[\sum_{n\in\mathbb{Z}}\left(\int_{n-1}^{n}+\int_n^{n+1}+\int_{n+1}^{n+2}f^p(x)v(x)dx\right)^{\frac{s}{p}}\right]^{\frac{r}{s}}\\
&\leq c_2\left[\sum_{n\in\mathbb{Z}}\left(\int_{n-1}^{n}f^p(x)v(x)dx\right)^{\frac{s}{p}}\right]^{\frac{r}{s}} = c_2\Big\|f(.)\Big\|^r_{\ell^s(L^p_v)}.
\end{aligned}$$

Expressions $\mathcal{S}_1$ and $\mathcal{S}_3$ will be estimated by using the discrete boundedness $\mathcal{M}_\alpha : \ell^s_{\mathcal{V}} \to \ell^r_{\mathcal{U}}$ once for some constant $c > 0$

$$\left[(M_\alpha f \mathrm{1\!I}_{]-\infty,n-1[})(x) + (M_\alpha f \mathrm{1\!I}_{]n+2,\infty[})(x)\right] \leq c(\mathcal{M}_\alpha \mathcal{G})(n) \tag{4.6}$$

for all $x \in [n, n+1]$, and where $\mathcal{G}(n) = \int_{n-1}^{n} f(y)dy$ as in (4.4). Indeed for $\mathcal{S}_1$ then

$$\begin{aligned}
\mathcal{S}_1 &\leq c_3 \sum_{n\in\mathbb{Z}} (\mathcal{M}_\alpha \mathcal{G})^r(n) \left(\int_n^{n+1} u(y)dy\right)^{\frac{r}{q}} = c_3 \sum_{n\in\mathbb{Z}} (\mathcal{M}_\alpha \mathcal{G})^r(n) \mathcal{U}(n) \\
&\leq c_4 \left[\sum_{n\in\mathbb{Z}} \mathcal{G}^s(n) \mathcal{V}(n)\right]^{\frac{r}{s}} = c_4 \left[\sum_{n\in\mathbb{Z}} \left(\int_{n-1}^{n} f(y)dy\right)^s \mathcal{V}(n)\right]^{\frac{r}{s}} \\
&\leq c_4 \left[\sum_{n\in\mathbb{Z}} \left(\int_{n-1}^{n} f^p(y)v(y)dy\right)^{\frac{s}{p}} \left(\int_{n-1}^{n} v^{1-p'}(y)dy\right)^{\frac{s}{p'}} \mathcal{V}(n)\right]^{\frac{r}{s}} \\
&= c_4 \left[\sum_{n\in\mathbb{Z}} \left(\int_{n-1}^{n} f^p(y)v(y)dy\right)^{\frac{s}{p}}\right]^{\frac{r}{s}} = c_4 \left\| f(.) \right\|^r_{\ell^s(L^p_v)}.
\end{aligned}$$

Expression $\mathcal{S}_3$ can be majorized similarly.

Now to prove (4.6) we only estimate $(M_\alpha f \mathrm{1\!I}_{]n+2,\infty[})(x)$, for each $x \in [n, n+1]$ since the arguments are the same. And by the definition of M_α, the question is reduced to evaluate

$$(2t)^{\alpha-1} \int_{[x-t,x+t]\cap]n+2,\infty[} f(y)dy \quad \text{where } t > 0 \text{ and } x \in [n, n+1].$$

If $[x-t, x+t] \cap]n+2, \infty[\neq \emptyset$ then necessarily there is an integer $j > 1$ such that $j \leq t$. So consider the unique integer $j_0 = j_0(t, x)$ satisfying: $1 < j_0 \leq t < j_0 + 1$. Then

$$\begin{aligned}
\int_{[x-t,x+t]\cap]n+2,\infty[} f(y)dy &\leq \int_{(n+2)}^{(n+2)+j_0} f(y)dy \\
&= \sum_{l=1}^{j_0} \int_{(n+2)+l-1}^{(n+2)+l} f(y)dy = \sum_{l=1}^{j_0} \mathcal{G}((n+2)+l) \\
&= \sum_{j=n+3}^{n+(2+j_0)} \mathcal{G}(j) \leq \sum_{j=n-(2+j_0)}^{n+(2+j_0)} \mathcal{G}(j).
\end{aligned}$$

This last inequality implies one part of (4.6) since

$$(M_\alpha f \mathbb{I}_{]n+2,\infty[})(x) \le 6^{1-\alpha} \sup_{j_0 \ge 1} \Big\{ [2(2+j_0)+1]^{\alpha-1} \sum_{j=n-(2+j_0)}^{n+(2+j_0)} \mathcal{G}(j) \Big\}$$
$$\le 6^{1-\alpha} (\mathcal{M}_\alpha \mathcal{G})(n).$$

□

Proof of Theorem 2.2.

The necessary Part

As in the proof of Theorem 2.1, the boundedness $M_\alpha : \ell^s(L^p_v) \to \ell^r(L^q_u)$ implies that for some $A > 0$: $\Big(\int_n^{n+4} (M_\alpha f)^q(x) u(x) dx\Big)^{\frac{1}{q}} \le A\Big(\int_n^{n+4} f^p(x) v(x) dx\Big)^{\frac{1}{p}}$ for all $n \in \mathbb{Z}$ and for each function $f(.) \ge 0$ whose support is the interval $[n, n+4]$. It is equivalent to write

$$\Big(\int_{\mathbb{R}} (M_\alpha g v^{1-p'} \mathbb{I}_{[n,n+4]})^q(x) u(x) \mathbb{I}_{[n,n+4]}(x) dx\Big)^{\frac{1}{q}}$$
$$\le A\Big(\int_{\mathbb{R}} g^p(x) v^{1-p'}(x) \mathbb{I}_{[n,n+4]}(x) dx\Big)^{\frac{1}{p}} \qquad \text{for all } g(.) \ge 0.$$

Now consider an arbitrary intervall $I = [a, b]$ with $|I| = |b - a| \le 3$. Then for some $n \in \mathbb{Z}$: $I \subset [n, n+4]$. Putting $g(.) = \mathbb{I}_I(.)$ in this last inequality then

$$\Big(\int_I (M_\alpha v^{1-p'} \mathbb{I}_I)^q(x) u(x) dx\Big)^{\frac{1}{q}} \le A\Big(\int_I v^{1-p'}(x) dx\Big)^{\frac{1}{p}}. \tag{4.7}$$

And this is condition (2.5).

By Theorem 2.1, $M_\alpha : \ell^s(L^p_v) \to \ell^r(L^q_u)$ implies $\mathcal{M}_{\alpha,\mu} : L^s(X, \mathcal{V}(m) d\mu(m)) \to L^r(X, \mathcal{U}(m) d\mu(m))$, where $\mathcal{V}(.)$, $\mathcal{U}(.)$, $d\mu(.)$ and $\mathcal{M}_{\alpha,\mu}$ are defined as in §2. Therefore using the last boundedness, then for some constant $A > 0$

(4.8)

$$\Big(\int_B (\mathcal{M}_{\alpha,\mu} \mathcal{V}^{1-s'} \mathbb{I}_B)^r(x) \mathcal{U}(x) d\mu(x)\Big)^{\frac{1}{r}} \le A\Big(\int_B \mathcal{V}^{1-s'}(x) d\mu(x)\Big)^{\frac{1}{s}}$$

for all balls $B = \{n-N, \dots, n, \dots, n+N\}$ where $n \in \mathbb{Z}$ and $N \in \mathbb{N}$. By the definitions of $\mathcal{V}(.)$, $d\mu(.)$, $\mathcal{M}_{\alpha,\mu}$ and $\mathcal{U}(.)$:

$$\int_B \mathcal{V}^{1-s'}(x)d\mu(x) = \sum_{m=n-N}^{n+N} \Big(\int_{m-1}^{m} v^{1-p'}(y)dy\Big)^{\frac{s'}{p'}} \tag{4.9}$$

and

$$\begin{aligned} &\int_B (\mathcal{M}_{\alpha,\mu}\mathcal{V}^{1-s'}\mathbb{I}_B)^r(x)\mathcal{U}(x)d\mu(x) \\ &= \sum_{m=n-N}^{n+N} (\mathcal{M}_{\alpha,\mu}\mathcal{V}^{1-s'}\mathbb{I}_B)^r(m)\Big(\int_m^{m+1} u(x)dx\Big)^{\frac{r}{q}} \\ &= \sum_{m=n-N}^{n+N} \Big[\sup_{M\in\mathbb{N}}(2M+1)^{\alpha-1} \sum_{j=m-M}^{m+M} \Big(\int_{j-1}^{j} v^{1-p'}(z)dz\Big)^{\frac{s'}{p'}}\Big]^r \\ &\quad \Big(\int_m^{m+1} u(x)dx\Big)^{\frac{r}{q}}. \end{aligned} \tag{4.10}$$

Hence condition (2.6) follows from (4.9), (4.10) and inequality (4.8).

The sufficient Part

To prove $M_\alpha : \ell^s(L^p_v) \to \ell^r(L^q_u)$, by Theorem 2.1, it is sufficient to check $(loc)M_\alpha : L^p_v \to L^q_u$ and $\mathcal{M}_{\alpha,\mu} : L^s(X, \mathcal{V}(m)d\mu(m)) \to L^r(X, \mathcal{U}(m)d\mu(m))$.

As is alluded in §2, the boundedness $(loc)M_\alpha : L^p_v \to L^q_u$ holds whenever inequality (2.3) is satisfied for some constant $C > 0$ which does not depend on n. By the E. Sawyer's characterization [16] for (2.3), it is required that

(4.11)
$$\Big(\int_J (M_\alpha\widetilde{\sigma}\mathbb{I}_J)^q(x)\widetilde{u}(x)dx\Big)^{\frac{1}{q}} \le A\Big(\int_J \widetilde{\sigma}(x)dx\Big)^{\frac{1}{p}} \quad \text{for each interval } J$$

where $\widetilde{\sigma}(.) = v^{1-p'}(.)\mathbb{I}_{[n,n+3[}(.)$ and $\widetilde{u}(.) = u(.)\mathbb{I}_{[n,n+3[}(.)$. In fact in [16], it is proved that (4.11) implies (2.3) with the constant $C = cA$ where c depends only on α, p and q. Therefore it remains to see that (4.11) is satisfied for some constant A which does not depend on n.

From the definitions of $\widetilde{\sigma}(.)$ and $\widetilde{u}(.)$, (4.11) becomes

$$\Big(\int_{J\cap[n,n+3]}(M_\alpha v^{1-p'}\mathrm{1\!I}_{J\cap[n,n+3]})^q(x)u(x)dx\Big)^{\frac{1}{q}} \le A\Big(\int_{J\cap[n,n+3]}v^{1-p'}(x)dx\Big)^{\frac{1}{p}}.$$

This inequality is trivially satisfied whenever $J\cap[n,n+3]=\emptyset$. For $J\cap[n,n+3]=I\neq\emptyset$ then I is an interval with the lenght $|I|\le 3$. So (4.11) is reduced to the inequality (4.7). This last is a consequence of condition (2.5) with a constant $A>0$ which does not depend on the integer n.

On the other hand, since $s\le r$ and following a result due A. Gogatishvili and V. Kokilashvili (see [8], Theorem 2.1, p. 425), the boundedness $\mathcal{M}_{\alpha,\mu}: L^s(X,\mathcal{V}(m)d\mu(m))\to L^r(X,\mathcal{U}(m)d\mu(m))$ holds whenever (4.8) is satisfied for all balls B. This requirement is fulfilled after using condition (2.6) and computations (4.9) and (4.10). □

Proof of Theorem 2.3.

Part A

As seen in the proof of necessary part for Theorem 2.2, the boundedness $M_\alpha:\ell^s(L^p_v)\to\ell^r(L^q_u)$ implies inequality (4.7). And this last one yields condition (2.9) since

$$\Big(M_\alpha v^{1-p'}\mathrm{1\!I}_I\Big)(x)\ge |I|^{\alpha-1}\int_I v^{1-p'}(y)dy \qquad \text{for each } x\in I.$$

Also as proved in [3], the boundedness $\mathcal{M}_{\alpha,\mu}: L^s(X,\mathcal{V}(m)d\mu(m))\to L^r(X,\mathcal{U}(m)d\mu(m))$ implies that for a fixed constant $A>0$

(4.12)

$$|B|_\mu^{\alpha+\frac{1}{r}-\frac{1}{s}}\Big(|B|_\mu^{-1}\int_B\mathcal{U}(y)d\mu(y)\Big)^{\frac{1}{r}}\Big(|B|_\mu^{-1}\int_B\mathcal{V}^{1-s'}(y)d\mu(y)\Big)^{\frac{1}{s'}}\le A$$

for all balls B. Taking $B=\{n_0-N,\dots,n_0,\dots,n_0+N\}$, ($n_0\in\mathbb{Z}$, $N\in\mathbb{N}$) and doing computations as in (4.9), then inequality (4.12) yields condition (2.10).

Part B

Conversely, by Theorem 2.1, to get $M_\alpha : \ell^s(L^p_v) \to \ell^r(L^q_u)$ it is sufficient to prove $(loc)M_\alpha : L^p_v \to L^q_u$ and $\mathcal{M}_{\alpha,\mu} : L^s(X, \mathcal{V}(m)d\mu(m)) \to L^r(X, \mathcal{U}(m)d\mu(m))$.

As seen in §2, the boundedness $(loc)M_\alpha : L^p_v \to L^q_u$ holds if inequality (2.3) is satisfied for some constant $C > 0$ independent on the integer n. But for each arbitrary interval J

(4.13)

$$\begin{aligned} &|J|^{\alpha-1}\left(\int_J \widetilde{u}(y)dy\right)^{\frac{1}{q}}\left(\int_J \widetilde{\sigma}(y)dy\right)^{\frac{1}{p'}} \\ &= |J|^{\alpha-1}\left(\int_{J\cap[n,n+3]} u(y)dy\right)^{\frac{1}{q}}\left(\int_{J\cap[n,n+3]} v^{1-p'}(y)dy\right)^{\frac{1}{p'}} \\ &\leq |I|^{\alpha-1}\left(\int_I u(y)dy\right)^{\frac{1}{q}}\left(\int_I v^{1-p'}(y)dy\right)^{\frac{1}{p'}} \leq A \end{aligned}$$

where $I = [n, n+3]$ if $|J| > 3$, else $I = J$. The last inequality appears due to condition (2.9). Here the constant $A > 0$ does not depend on n and J. Similarly using $v^{1-p'}(.) \in A_t$, for some $t > 1$, then

$$\begin{aligned} &|J|^{-1}\left(\int_J \widetilde{\sigma}(y)dy\right)^{\frac{1}{t}}\left(\int_J \widetilde{\sigma}^{1-t'}(y)dy\right)^{\frac{1}{t'}} \\ &\leq |I|^{-1}\left(\int_I v^{1-p'}(y)dy\right)^{\frac{1}{t}}\left(\int_I v^{(1-p')(1-t')}(y)dy\right)^{\frac{1}{t'}} \leq B. \end{aligned} \tag{4.14}$$

The constant $B > 0$ does not depend on the integer n. According to a result of C. Pérez [13], both (4.13) and $\widetilde{\sigma}(.) \in A_\infty$ (i.e., (4.14)) imply inequality (2.3) with the constant $C = cA$, where c depends only on α, p, q and B. Therefore C does not depend on n as it is required.

On the other hand with $\mathcal{V}^{1-s'}(.) \in A_\infty$ and condition (4.12), A. Bernardis and O. Salinas [3] proved that $\mathcal{M}_{\alpha,\mu} : L^s(X, \mathcal{V}(m)d\mu(m)) \to L^r(X, \mathcal{U}(m)d\mu(m))$. So the conclusion appears since (4.12) is equivalent to condition (2.10).

Part C

As in Part B, to prove $M_\alpha : \ell^s(L^p_v) \to \ell^r(L^q_u)$ it is sufficient to check inequality (2.3) (with a constant $C > 0$ independent on n) and $\mathcal{M}_{\alpha,\mu} : L^s(X, \mathcal{V}(m)d\mu(m)) \to L^r(X, \mathcal{U}(m)d\mu(m))$.

Since $\alpha - \frac{1}{p} - \frac{1}{\varepsilon p'} < 0$, then for each arbitrary interval J

(4.15)

$$|J|^{\alpha+\frac{1}{q}-\frac{1}{p}}\left(\frac{1}{|J|}\int_J \widetilde{u}(y)dy\right)^{\frac{1}{q}}\left(\frac{1}{|J|}\int_J \widetilde{\sigma}^{\varepsilon}(y)dy\right)^{\frac{1}{\varepsilon p'}}$$
$$= |J|^{\alpha-\frac{1}{p}-\frac{1}{\varepsilon p'}}\left(\int_{J\cap[n,n+3]} u(y)dy\right)^{\frac{1}{q}}\left(\int_{J\cap[n,n+3]} v^{(1-p')\varepsilon}(y)dy\right)^{\frac{1}{\varepsilon p'}}$$
$$\leq |I|^{\alpha+\frac{1}{q}-\frac{1}{p}}\left(\frac{1}{|I|}\int_I u(y)dy\right)^{\frac{1}{q}}\left(\frac{1}{|I|}\int_I v^{(1-p')\varepsilon}(y)dy\right)^{\frac{1}{\varepsilon p'}} \leq A$$

where $I = [n, n+3]$ if $|J| > 3$, else $I = J$. The last inequality appears due to condition (2.11). Moreover the constant $A > 0$ depends only on $u(.)$ and $v(.)$ but not on the integer n and the interval J. By a result of C. Pérez [13] (Theorem 2.11, p. 670), condition (4.15) implies inequality (2.3) with the constant $C = cA$ where c depends only on α, p and q. Consequently, the local boundedness $(loc)M_\alpha : L^p_v \to L^q_u$ follows.

Condition (2.12) implies that for fixed constants $A > 0$ and $\varepsilon > 1$

(4.16)

$$|B|_\mu^{\alpha+\frac{1}{r}-\frac{1}{s}}\left(|B|_\mu^{-1}\int_B \mathcal{U}(y)d\mu(y)\right)^{\frac{1}{r}}\left(|B|_\mu^{-1}\int_B \mathcal{V}^{\varepsilon(1-s')}(y)d\mu(y)\right)^{\frac{1}{\varepsilon s'}} \leq A$$

for all balls $B = B(n,t)$. Therefore, similarly as the above C. Pérez's result, we are led to claim that condition (4.16) implies the boundedness $\mathcal{M}_{\alpha,\mu} : L^s(X, \mathcal{V}(m)d\mu(m)) \to L^r(X, \mathcal{U}(m)d\mu(m))$. None of papers (on weighted inequalities) quoted in this work contains such a statement, although it remains true.

Indeed the proof of this statement can be done by adapting the C. Pérez arguments [13], [14] given for the euclidean space $\mathbb{R}^d$. To extend his technic in the spaces of homogeneous type, a modification of the Calderón-Zygmund cubes is required. This was first done by H. Aimar and R. Macias [1] and next extended by A. Bernardis and O. Salinas [3]. The crucial point for the proof is a Decomposition Lemma (see [3], Lemma 2.2, pp. 203-204) of the sets $\{y \in X|\ \ b^{k+1} \geq (\mathcal{M}_{\alpha,\mu}\mathcal{F})(y) > b^k\}$, $k \in \mathbb{Z}$, by means of suitable balls which replace the role of Calderón-Zygmund cubes in the C. Pérez's proof [13] (see Proof of Theorem 2.11, pp. 680-682). So with Lemma

2.2 given in [3] (pp. 203-204), the fact that (2.12) implies $\mathcal{M}_{\alpha,\mu} : L^s(X, \mathcal{V}(m)d\mu(m)) \to L^r(X, \mathcal{U}(m)d\mu(m))$ can be done verbatim as in the proof of Theorem 2.11 in [13] (see pp. 680-682). □

Proof of Lemma 2.4. Consider, in (2.9), an interval $I =]x_0 - t, x_0 + t[$ with $x_0 \in \mathbb{R}$ and $0 < 2t \leq 3$. For $\alpha + \frac{1}{q} - \frac{1}{p} < 0$ and letting $t \to 0$, the Lebesgue differentiation theorem implies necessarily $u(.) = 0$ or $v^{1-p'}(.) = 0$ a.e. On the other hand taking $x_0 = 0$ then (2.14) follows.

Conversely to get condition (2.9), consider an arbitrary interval $I =]x_0 - t, x_0 + t[$ with $x_0 \in \mathbb{R}$ and $0 < 2t \leq 3$.

For $|x_0| \leq 4t$ then

$$|I|^{\alpha+\frac{1}{q}-\frac{1}{p}} \left(\frac{1}{|I|}\int_I u(y)dy\right)^{\frac{1}{q}} \left(\frac{1}{|I|}\int_I v^{1-p'}(y)dy\right)^{\frac{1}{p'}}$$
$$\leq cR^{\alpha+\frac{1}{q}-\frac{1}{p}} \left(\frac{1}{R}\int_{|y|<R} u(y)dy\right)^{\frac{1}{q}} \left(\frac{1}{R}\int_{|y|<R} v^{1-p'}(y)dy\right)^{\frac{1}{p'}} = cA(R)$$

with $R = 6t \leq 9$ and $c > 0$ is a constant which only depends on p, q and α. By condition (2.14), the term $A(R)$ is bounded by a fixed constant $A > 0$ whenever $R \leq \frac{3}{2}$, i.e., $0 < t \leq \frac{1}{4}$. For the remaining case $\frac{1}{4} < t \leq \frac{3}{2}$, $A(R)$ is trivially bounded by a fixed constant since $u(.)$ and $v^{1-p'}(.)$ are locally integrable functions.

Finally if $4t < |x_0|$ then $\frac{1}{2}|x_0| < |y| < 2|x_0|$ for each $y \in I =]x_0 - t, x_0 + t[$. Consequently using (2.15) (with $R_1 = t$ and $R = |x_0|$) then

$$|I|^{\alpha+\frac{1}{q}-\frac{1}{p}} \left(\frac{1}{|I|}\int_I u(y)dy\right)^{\frac{1}{q}} \left(\frac{1}{|I|}\int_I v^{1-p'}(y)dy\right)^{\frac{1}{p'}}$$
$$\leq |I|^{\alpha+\frac{1}{q}-\frac{1}{p}} \left(\sup_{\frac{1}{2}|x_0|<|y|<2|x_0|} u(y)\right)^{\frac{1}{q}} \left(\sup_{\frac{1}{2}|x_0|<|z|<2|x_0|} v^{1-p'}(z)\right)^{\frac{1}{p'}} \leq cA.$$

□

Proof of Corollary 2.5.

The Sufficient Part

By Lemma 2.4, the task remains to check conditions (2.14) and (2.15). Let $A(R)$ be the left member of (2.14). Then $A(R) \approx R^{\alpha+\frac{1}{q}-\frac{1}{p}} \times$

$R^{(\beta-1)\frac{1}{q}} \times R^{-(\delta-1)\frac{1}{p}} = R^{\alpha+\frac{\beta}{q}-\frac{\delta}{p}}$ whenever $\beta > 0$ and $\delta < p$. So (2.14) is satisfied if and only if $0 \leq \alpha + \frac{\beta}{q} - \frac{\delta}{p}$. Next let $B(R, R_1)$ be the left member of (2.15). Easy computations lead to $B(R, R_1) \approx R_1^{\alpha+\frac{1}{q}-\frac{1}{p}} \times R^{(\beta-1)\frac{1}{q}-(\delta-1)\frac{1}{p}}$ for $4R_1 < R$. Inequality (2.15) is true for $R \leq 1$, since $B(R, R_1) \approx (\frac{R_1}{R})^{\alpha+\frac{1}{q}-\frac{1}{p}} \times R^{\alpha+\frac{\beta}{q}-\frac{\delta}{p}} \leq 1$ when $0 \leq \alpha + \frac{1}{q} - \frac{1}{p}$ and $0 \leq \alpha + \frac{\beta}{q} - \frac{\delta}{p}$. For $R > 1$ inequality (2.15) is also true, because $B(R, R_1) \leq c(\frac{3}{2})^{\alpha+\frac{1}{q}-\frac{1}{p}}$ whenever $R_1 \leq \frac{3}{2}$ and $(\beta-1)\frac{1}{q} - (\delta-1)\frac{1}{p} \leq 0$.

The Necessary Part

The assumption $\delta < p$ follows by the local integrability required for $v^{1-p'}(.)$. In view of the above consideration, $0 \leq \alpha + \frac{\beta}{q} - \frac{\delta}{p}$ is a condition required for (2.9). Taking $I =]x_0 - t, x_0 + t[$ in (2.9) with $t = \frac{3}{2}$ and $|x_0| = R \geq 6$ then $A \geq cR^{(\beta-1)\frac{1}{q}-(\delta-1)\frac{1}{p}}$ since $4t \leq |x_0|$. So letting $R \to \infty$, it is necessary that $(\beta-1)\frac{1}{q} - (\delta-1)\frac{1}{p} \leq 0$ which implies $\alpha + \frac{\beta}{q} - \frac{\delta}{p} \leq \alpha + \frac{1}{q} - \frac{1}{p}$. □

Proof of Lemma 2.6. The fact that (2.16) is a necessary condition for (2.10) can be seen as (2.13) by using the Lebesgue differentiation theorem. Condition (2.17) follows from (2.10) by taking $n_0 = 0$ and restricting the summations about m to $N_0 \leq |m| \leq N$.

Conversely to get (2.10) consider arbitrary n_0 and N, and denote by $\Theta(n_0, N)$ the left member of this condition (we have to bound by a fixed constant).

For $|n_0| \leq 4(2N+1)$ then

(4.17)

$$\begin{aligned}\Theta(n_0, N) &\leq c_1(2N+1)^{\alpha+\frac{1}{r}-\frac{1}{s}} \left[\frac{1}{5(2N+1)} \sum_{|m|\leq 5(2N+1)} \left(\int_m^{m+1} u(y)dy\right)^{\frac{r}{q}}\right]^{\frac{1}{r}} \\ &\quad \times \left[\frac{1}{5(2N+1)} \sum_{|m|\leq 5(2N+1)} \left(\int_{m-1}^{m} v^{1-p'}(y)dy\right)^{\frac{s'}{p'}}\right]^{\frac{1}{s'}}.\end{aligned}$$

So for $N \leq N_0$, this inequality yields

$$\Theta(n_0, N) \leq c_2 \left(\int_{|y|<6(2N_0+1)} u(y)dy\right)^{\frac{1}{q}} \left(\int_{|y|<6(2N_0+1)} v^{1-p'}(y)dy\right)^{\frac{1}{p'}}.$$

Here the integer $N_0 \geq 4$ is defined as in condition (2.17), and c_1, $c_2 > 0$ depend only on α, r, s, q and p. Consequently it can be assumed now that $N > N_0$. Splitting the sum $\sum_{|m|\leq 5(2N+1)}$ as $\sum_{|m|\leq 5(2N+1)} = \sum_{|m|<N_0} + \sum_{N_0\leq|m|\leq 5(2N+1)}$, the estimate (4.17) can be continued to get

(4.18)

$$\Theta(n_0, N) \leq c_0 A\Big(\frac{2N+1}{2N_0+1}\Big)^{\alpha-1} + C_1\Theta_1(n_0,N) + C_2\Theta_2(n_0,N) + c_3\Theta_3(n_0,N)$$

where

$$\Theta_1(n_0,N) = (2N+1)^{\alpha-1}\Big[\sum_{N_0\leq|m|\leq 5(2N+1)}\Big(\int_m^{m+1} u(y)dy\Big)^{\frac{r}{q}}\Big]^{\frac{1}{r}}$$

$$\Theta_2(n_0,N) = (2N+1)^{\alpha-1}\Big[\sum_{N_0\leq|m|\leq 5(2N+1)}\Big(\int_{m-1}^{m} v^{1-p'}(y)dy\Big)^{\frac{s'}{p'}}\Big]^{\frac{1}{s'}}$$

and

$$\Theta_3(n_0,N) = (2N+1)^{\alpha-1}\Big[\sum_{N_0\leq|m|\leq 5(2N+1)}\Big(\int_m^{m+1} u(y)dy\Big)^{\frac{r}{q}}\Big]^{\frac{1}{r}} \times \Big[\sum_{N_0\leq|m|\leq 5(2N+1)}\Big(\int_{m-1}^{m} v^{1-p'}(y)dy\Big)^{\frac{s'}{p'}}\Big]^{\frac{1}{s'}}.$$

The constants c_0 and c_3 depend on r and s. The constant C_1 (resp. C_2) depend on α, r, s, q, p and $v(.)$ (resp. $u(.)$). By condition (2.17): $\Theta_3(n_0,N) \leq A$. Using the same condition then $\Theta_1(n_0,N) \leq C_3A$ and $\Theta_2(n_0,N) \leq C_4A$. Here C_3, $C_4 > 0$ depend on N_0, r, s, p, q, $v(.)$ and $u(.)$. All of these last considerations and (4.18) lead to the expected estimate $\Theta(n_0,N) \leq CA$.

The case $4(2N+1) < |n_0|$ (which implies $4 < |n_0|$) is now examined. Here $\frac{1}{2}|n_0| < |m| < 2|n_0|$ for each m with $m \in \{n_0 - N, \dots, n_0, \dots, n_0+N\}$. Also $\frac{1}{4}|n_0| < |y| \leq 4|n_0|$ for each y with $m < y < m+1$ or $m-1 < y < m$ and $m \in \{n_0-N, \dots, n_0, \dots, n_0+N\}$. The conclusion follows from these last observations and condition (2.18) (with $R = |n_0|$) since

$$\begin{aligned}
&\Theta(n_0, N)\\
&\leq (2N+1)^{\alpha+\frac{1}{r}-\frac{1}{s}}\Big(\sup_{\frac{1}{4}|n_0|<|y|<4|n_0|} u(y)\Big)^{\frac{1}{q}}\Big(\sup_{\frac{1}{4}|n_0|<|z|<4|n_0|} v^{1-p'}(z)\Big)^{\frac{1}{p'}}\\
&\leq A.
\end{aligned}$$

□

Proof of Corollary 2.7.

The Sufficient Part

To get $M_\alpha : \ell^s(L^p_v) \to \ell^r(L^q_u)$, by Part B in Theorem 2.3, it is sufficient to check (2.9) and (2.10) and to justify that $v^{1-p'}(.) \in A_\infty$ and $\mathcal{V}^{1-s'}(.) \in A_\infty(d\mu)$.

Condition (2.9) is satisfied because of Corollary 2.5 and hypothesis (2.19).

To get condition (2.10), by Lemma 2.6, it remains to prove (2.17) and (2.18). Condition (2.18) is satisfied because $0 \leq \alpha + \frac{1}{r} - \frac{1}{s}$ and $\left[\alpha+\frac{1}{r}-\frac{1}{s}\right] + \left[(\beta-1)\frac{1}{q} - (\delta-1)\frac{1}{p}\right] \leq 0$. Indeed for $R > 0$ and $N \in \mathbb{N}$ with $4(2N+1) < R$ then

$$\begin{aligned}
&(2N+1)^{\alpha+\frac{1}{r}-\frac{1}{s}}\Big(\sup_{\frac{1}{4}R<|y|<4R} u(y)\Big)^{\frac{1}{q}}\Big(\sup_{\frac{1}{4}R<|z|<4R} v^{(1-p')}(z)\Big)^{\frac{1}{p'}}\\
&\leq c_1 N^{\alpha+\frac{1}{r}-\frac{1}{s}} \times R^{(\beta-1)\frac{1}{q}} \times R^{-(\delta-1)\frac{1}{p}}\\
&\leq c_2 R^{\alpha+\frac{1}{r}-\frac{1}{s}+(\beta-1)\frac{1}{q}-(\delta-1)\frac{1}{p}} \leq c_3.
\end{aligned}$$

To derive (2.17), consider $N_0 \geq 4$ and a nonnegative integer k sufficiently large. Since $1 + (\beta-1)\frac{r}{q} > 0$ then

$$\begin{aligned}
&\sum_{N_0\leq|m|\leq N}\Big(\int_m^{m+1} u(y)dy\Big)^{\frac{r}{q}} = \sum_{N_0\leq|m|\leq N}\Big(\int_m^{m+1} |y|^{(\beta-1)}dy\Big)^{\frac{r}{q}}\\
&\approx \sum_{m=N_0}^{N} |m|^{(\beta-1)\frac{r}{q}} \approx N^{1+(\beta-1)\frac{r}{q}} \quad \text{for } N > kN_0
\end{aligned}\tag{4.19}$$

Similarly, for the same $N > kN_0$, and after using $1-(\delta-1)\frac{s'}{p} > 0$ then
(4.20)

$$\sum_{N_0\le|m|\le N}\Big(\int_{m-1}^{m} v^{1-p'}(y)dy\Big)^{\frac{s'}{p'}} \approx \sum_{m=N_0}^{N}|m|^{-(\delta-1)\frac{s'}{p}} \approx N^{1-(\delta-1)\frac{s'}{p}}.$$

The nonnegative integer k depends on β, δ, r, s, p and q and is chosen sufficiently big in order to insure both the equivalences in (4.19) and (4.20). Now inequality (2.17) for $N > kN_0$ follows after using (4.19), (4.20) and the fact that $\Big[\alpha+\frac{1}{r}-\frac{1}{s}\Big]+\Big[(\beta-1)\frac{1}{q}-(\delta-1)\frac{1}{p}\Big] \le 0$. Indeed if the left member of (2.17) is denoted by $\Lambda(N, N_0)$ then

$$\begin{aligned}\Lambda(N, N_0) \le & c_4 N^{\alpha+\frac{1}{r}-\frac{1}{s}} \times N^{(\beta-1)\frac{1}{q}} \times N^{-(\delta-1)\frac{1}{p}}\\ = & N^{\alpha+\frac{1}{r}-\frac{1}{s}+(\beta-1)\frac{1}{q}-(\delta-1)\frac{1}{p}} \le c_5.\end{aligned}$$

Of course inequality (2.17) is trivially satisfied whenever $N_0 < N \le kN_0$.

Clearly $v^{1-p'}(.) \in A_\infty$ since $v^{1-p'}(.) \in A_{p'}$ (see the definition in (2.7)).

Finally to prove $\mathcal{V}^{1-s'}(.) \in A_\infty(d\mu)$ it is sufficient to apply a similar argument as in the converse part in Lemma 2.6, with this time $\alpha = 0$ and $r = s$. Indeed for some $t > 1$ (with $t' < 1 + \frac{p}{s'}(1-\delta)$ if $\delta < 1$) and using inequalities like in (2.17) and (2.18) then $\mathcal{V}^{1-s'}(.) \in A_t(d\mu)$.

The Necessary Part

Following Part A in Theorem 2.3, the boundedness $M_\alpha : \ell^s(L^p_v) \to \ell^r(L^q_u)$ implies conditions (2.9) and (2.10). Condition (2.19) is implied by (2.9) because of Corollary 2.5. The inequality $0 \le \alpha + \frac{1}{r} - \frac{1}{s}$ follows from (2.10) and Lemma 2.6. And the inequality $\alpha + \frac{1}{r} - \frac{1}{s} \le (\delta-1)\frac{1}{p} - (\beta-1)\frac{1}{q}$ will be derived from (2.17) which is a consequence of (2.10) (see Lemma 2.6). Indeed if $\Lambda(N, N_0)$ denotes the left member of (2.17), then equivalences (4.19) and (4.20) lead to $\Lambda(N, N_0) \approx N^{[\alpha+\frac{1}{r}-\frac{1}{s}]+[(\beta-1)\frac{1}{q}-(\delta-1)\frac{1}{p}]} \le A$ for $N > kN_0$ with k sufficiently big. The last inequality means that necessarily $[\alpha + \frac{1}{r} - \frac{1}{s}] + [(\beta-1)\frac{1}{q} - (\delta-1)\frac{1}{p}] \le 0$. □

Proof of Lemma 2.8. To see that $\mathcal{W}(.) \in A_\infty(d\mu)$, we first state and prove elementaries points. So it is crucial to observe that for $w(.) \in RH_{\tau'}$ $(\tau > 1)$

(4.21)

$$\Big(|I|^{-1}\int_I w(y)dy\Big)^{-\tau} \le c\Big(|I|^{-1}\int_I w^{-\tau}(y)dy\Big) \quad \text{for all } I = [a,b].$$

Indeed by the Hölder inequality and the condition $w(.) \in RH_{\tau'}$

$$\begin{aligned} 1 \le &\Big(|I|^{-1}\int_I w^{\tau'}(y)dy\Big)^{\frac{1}{\tau'}}\Big(|I|^{-1}\int_I w^{-\tau}(y)dy\Big)^{\frac{1}{\tau}} \\ &\le c\Big(|I|^{-1}\int_I w(y)dy\Big)\Big(|I|^{-1}\int_I w^{-\tau}(y)dy\Big)^{\frac{1}{\tau}}. \end{aligned}$$

For $q_1 < r_1$ there is a real t such that the following are satisfied

(i) $$1 < t < \frac{r_1}{q_1};$$

(ii) $$1 < \frac{1}{(t-1)}\frac{r_1}{q_1};$$

(iii) $$\Big((t'-1)\frac{r_1}{q_1}\Big)' = \Big(\frac{1}{(t-1)}\frac{r_1}{q_1}\Big)' = \frac{r_1}{r_1-(t-1)q_1} < \frac{r_1}{q_1};$$

(iv) $$w^{\frac{r_1}{q_1}}(.) \in A_t;$$

and

(v) $$w(.) \in RH_{((t'-1)\frac{r_1}{q_1})'}.$$

Indeed since $w(.) \in A_{1+\frac{q_1}{r_1}(\frac{r_1}{q_1}-1)} \cap RH_{\frac{r_1}{q_1}}$ implies $w^{\frac{r_1}{q_1}}(.) \in A_{\frac{r_1}{q_1}}$ (see [10], p. 641) then necessarily $w^{\frac{r_1}{q_1}}(.) \in A_t$ for some t with $1 < t < \frac{r_1}{q_1}$. Thus (i) and (iv) are satisfied. Inequality (ii) is also true, since from (i) then $t-1 > 0$ and $t < 1+\frac{r_1}{q_1}$. The inequality in (iii) can be seen by using (ii) and (i). Finally (v) holds since $w(.) \in RH_{\frac{r_1}{q_1}} \subset RH_{((t'-1)\frac{r_1}{q_1})'}$ because $((t'-1)\frac{r_1}{q_1})' < \frac{r_1}{q_1}$ by (iii).

For $r_1 \leq q_1$, the hypothesis on $w(.)$ is also equivalent to

$$(vi) \qquad \begin{array}{l} w(.) \in A_t \cap RH_{(t'-1)'} \\ \text{for some } 1 < t < 2 \text{ and where } (t'-1)' = \dfrac{1}{2-t}. \end{array}$$

Now, for $q_1 < r_1$, it is time to prove that $\mathcal{W}(.) \in A_\infty(d\mu)$, i.e., condition (2.8) for some $t > 1$. Really take $t > 1$ as in (i) to (v). The conclusion appears since

$$\begin{aligned}
\Theta(n_0, N) &= \left[\frac{1}{(2N+1)} \sum_{m=n_0-N}^{n_0+N} \Big(\int_{m-1}^{m} w(y)dy\Big)^{\frac{r_1}{q_1}}\right]^{\frac{1}{t}} \\
&\quad \left[\frac{1}{(2N+1)} \sum_{m=n_0-N}^{n_0+N} \Big(\int_{m-1}^{m} w(y)dy\Big)^{(1-t')\frac{r_1}{q_1}}\right]^{\frac{1}{t'}} \\
&\leq c_1 \left[\frac{1}{(2N+1)} \sum_{m=n_0-N}^{n_0+N} \int_{m-1}^{m} w^{\frac{r_1}{q_1}}(y)dy\right]^{\frac{1}{t}} \\
&\quad \times \left[\frac{1}{(2N+1)} \sum_{m=n_0-N}^{n_0+N} \int_{m-1}^{m} w^{(1-t')\frac{r_1}{q_1}}(y)dy\right]^{\frac{1}{t'}}
\end{aligned}$$

by the Hölder inequality and by (4.21)
since $w(.) \in RH_{((t'-1)\frac{r_1}{q_1})'}$ (see v)

$$\begin{aligned}
&= c_1 \left[\frac{1}{(2N+1)} \int_{n_0-N-1}^{n_0+N} w^{\frac{r_1}{q_1}}(y)dy\right]^{\frac{1}{t}} \\
&\quad \times \left[\frac{1}{(2N+1)} \int_{n_0-N-1}^{n_0+N} w^{(1-t')\frac{r_1}{q_1}}(y)dy\right]^{\frac{1}{t'}} \\
&\leq c_2 \qquad \text{since } w^{\frac{r_1}{q_1}}(.) \in A_t \text{ (see } (iv)).
\end{aligned}$$

The case $r_1 \leq q_1$ can be treated similarly. Indeed

$$\Theta(n_0, N) \leq \left[\frac{1}{(2N+1)} \sum_{m=n_0-N}^{n_0+N} \left(\int_{m-1}^{m} w(y)dy\right)\right]^{\frac{r_1}{q_1 t}}$$

$$\left[\frac{1}{(2N+1)} \sum_{m=n_0-N}^{n_0+N} \left(\int_{m-1}^{m} w(y)dy\right)^{(1-t')}\right]^{\frac{r_1}{q_1 t'}}$$

by using the Hölder inequality for $\dfrac{q_1}{r_1} > 1$

$$\leq c_3 \left[\frac{1}{(2N+1)} \sum_{m=n_0-N}^{n_0+N} \int_{m-1}^{m} w(y)dy\right]^{\frac{r_1}{q_1 t}}$$

$$\times \left[\frac{1}{(2N+1)} \sum_{m=n_0-N}^{n_0+N} \int_{m-1}^{m} w^{(1-t')}(y)dy\right]^{\frac{r_1}{q_1 t'}}$$

by (4.21) and since $w(.) \in RH_{(t'-1)'}$ (see vi)

$$= c_3 \left[\frac{1}{(2N+1)} \int_{n_0-N-1}^{n_0+N} w(y)dy\right]^{\frac{r_1}{q_1 t}}$$

$$\times \left[\frac{1}{(2N+1)} \int_{n_0-N-1}^{n_0+N} w^{(1-t')}(y)dy\right]^{\frac{r_1}{q_1 t'}}$$

$\leq c_4$ since $w(.) \in A_t$ (see (vi)). □

Proof of Corollary 2.9. This result can be obtained by applying Part B of Theorem 2.2 and Lemma 2.8. Indeed first note that in each case $v^{1-p'}(.) \in A_\infty$. On the other hand $\mathcal{V}^{1-s'}(m) = \left(\int_{m-1}^{m} v^{1-p'}(y)dy\right)^{\frac{s'}{p'}}$, therefore the fact that $\mathcal{V}^{1-s'}(.) \in A_\infty(d\mu)$ will be given by Lemma 2.8 with $w(.) = v^{1-p'}(.)$, $r_1 = s'$ and $q_1 = p'$. □

Proof of Corollary 2.10. In view of Corollary 2.9 and since the global condition (2.21) implies (2.9), the problem remains to check condition (2.10) under the extra-hypotheses on $v(.)$ or $u(.)$.

The left member in condition (2.10) (to bound by a fixed constant) can be written as

$(2N+1)^{\alpha+\frac{1}{r}-\frac{1}{s}}\Theta(n_0,N)$ where $\Theta(n_0,N)$ is

$$\left[\frac{1}{(2N+1)}\sum_{m=n_0-N}^{n_0+N}\left(\int_m^{m+1}u(y)dy\right)^{\frac{r}{q}}\right]^{\frac{1}{r}}$$
$$\left[\frac{1}{(2N+1)}\sum_{m=n_0-N}^{n_0+N}\left(\int_{m-1}^{m}v^{1-p'}(y)dy\right)^{\frac{s'}{p'}}\right]^{\frac{1}{s'}}.$$

The arguments are close to those used in Lemma 2.8. Indeed for $\frac{r}{q}>1$, using the Hölder inequality and $u(.)\in RH_{\frac{r}{q}}$ then

$$\begin{aligned}(4.22)\qquad &\left[\frac{1}{(2N+1)}\sum_{m=n_0-N}^{n_0+N}\left(\int_m^{m+1}u(y)dy\right)^{\frac{r}{q}}\right]^{\frac{1}{r}}\\ &\leq c\left[\frac{1}{(2N+1)}\int_{n_0-N}^{n_0+N+1}u(y)dy\right]^{\frac{1}{q}}.\end{aligned}$$

This estimate is trivial for $\frac{r}{q}=1$, and remains true for $\frac{r}{q}<1$ (with no assumption on $u(.)$) after using the Hölder inequality (since $\frac{q}{r}>1$). Similarly

$$\begin{aligned}(4.23)\qquad &\left[\frac{1}{(2N+1)}\sum_{m=n_0-N}^{n_0+N}\left(\int_{m-1}^{m}v^{1-p'}(y)dy\right)^{\frac{s'}{p'}}\right]^{\frac{1}{s'}}\\ &\leq c\left[\frac{1}{(2N+1)}\int_{n_0-N-1}^{n_0+N}v^{1-p'}(y)dy\right]^{\frac{1}{p'}}\end{aligned}$$

after using the condition $v^{1-p'}(.)\in RH_{\frac{p'}{s'}}$ for $s<p$ (i.e., $\frac{s'}{p'}>1$), and the Hölder inequality when $p<s$ (i.e., $\frac{p'}{s'}>1$). Of course the case $s=p$ is trivial.

Now using estimates (4.22), (4.23) and the global condition (2.21) then

$$(2N+1)^{\alpha+\frac{1}{r}-\frac{1}{s}}\Theta(n_0,N)\leq C$$

for a fixed constant C (which does not depend on n_0 and N). $\square$

5. Proofs of Results for I_α

Proof of Theorem 3.1.

The Necessary Part

The fact that $s \le r$ and $I_\alpha : \ell^s(L^p_v) \to \ell^r(L^q_u)$ imply $(loc)I_\alpha : L^p_v \to L^q_u$ can be seen as in the case of M_α. The details are not given. The main points to get inequalities (3.3) and (3.4) are the equivalences

$$(I_\alpha f 1\!\!1_{]-\infty,n-1[})(x) \approx \sum_{m=-\infty}^{n-1} (n-m)^{\alpha-1}\mathcal{G}(m) \tag{5.1}$$

and

$$(I_\alpha f 1\!\!1_{]n+2,\infty[})(x) \approx \sum_{m=n+3}^{\infty} (m-n)^{\alpha-1}\mathcal{G}(m) \tag{5.2}$$

where $x \in [n, n+1]$ and $\mathcal{G}(m) = \int_{m-1}^{m} f(y)dy$. Indeed take an arbitrary function $\mathcal{F}(.) \ge 0$ and define $f(.)$ from $\mathcal{F}(.)$ as in (4.1). Inequality (3.3) appears by using (4.4), (5.1) and (4.2) as follows

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}}\Big[\sum_{m=-\infty}^{n-1}(n-m)^{\alpha-1}\mathcal{F}(m)\Big]^r \mathcal{U}(n)\\
=&\sum_{n\in\mathbb{Z}}\Big[\sum_{m=-\infty}^{n-1}(n-m)^{\alpha-1}\mathcal{G}(m)\Big]^r\Big(\int_n^{n+1}u(x)dx\Big)^{\frac{r}{q}} \quad \text{by (4.4)}\\
=&\sum_{n\in\mathbb{Z}}\Big(\int_n^{n+1}\Big[\sum_{-\infty}^{n-1}(n-m)^{\alpha-1}\mathcal{G}(m)\Big]^q u(x)dx\Big)^{\frac{r}{q}}\\
\le& c_1\sum_{n\in\mathbb{Z}}\Big(\int_n^{n+1}(I_\alpha f 1\!\!1_{]-\infty,n-1[})^q(x)u(x)dx\Big)^{\frac{r}{q}} \text{ by (5.1)}\\
\le& c_2\Big[\sum_{n\in\mathbb{Z}}\Big(\int_{n-1}^{n}f^p(y)v(y)dy\Big)^{\frac{s}{p}}\Big]^{\frac{r}{s}} \text{since } I_\alpha : \ell^s(L^p_v) \to \ell^r(L^q_u)\\
=& c_2\Big[\sum_{n\in\mathbb{Z}}\mathcal{F}^s(n)\Big(\int_{n_1}^{n}v^{1-p'}(y)dy\Big)^{-\frac{s}{p'}}\Big]^{\frac{r}{s}} \text{ by (4.2)}
\end{aligned}$$

$$=c_2\left[\sum_{n\in\mathbb{Z}}\mathcal{F}^s(n)\mathcal{V}(n)\right]^{\frac{r}{s}}.$$

Inequality (3.4) can be proved as in the same manner.

The elementary point to get (5.2) is that $2^{-1}k \leq (y-x) \leq 2k$ whenever $x \in [n, n+1]$, $y \in [n+k, n+k+1]$ for some integer $k \geq 2$. Indeed

$$\begin{aligned}
&(I_\alpha f \mathbb{I}_{]n+2,\infty[})(x)\\
&= \int_{n+2}^{\infty}(y-x)^{\alpha-1}f(y)dy = \sum_{k=2}^{\infty}\int_{n+k}^{n+k+1}(y-x)^{\alpha-1}f(y)dy\\
&\approx \sum_{k=2}^{\infty}k^{\alpha-1}\mathcal{G}(n+k+1) \approx \sum_{k=2}^{\infty}(k+1)^{\alpha-1}\mathcal{G}(n+k+1)\\
&\approx \sum_{l=3}^{\infty}l^{\alpha-1}\mathcal{G}(n+l) = \sum_{m=n+3}^{\infty}(m-n)^{\alpha-1}\mathcal{G}(m).
\end{aligned}$$

The Sufficient Part

The boundedness $I_\alpha : \ell^s(L^p_v) \to \ell^r(L^q_u)$ will be obtained by first writing an analogue of (4.5) where this time $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$ are defined by means of I_α instead of M_α. $\mathcal{S}_2$ is estimated as in the case of M_α by using the local boundedness $(loc)I_\alpha : L^p_v \to L^q_u$ and the fact that $s \leq r$. The term $\mathcal{S}_1$ (resp. $\mathcal{S}_3$) can be handled by using condition (3.3) (resp. (3.4)) and mainly the equivalence (5.1) (resp. (5.2)). The details are omitted since the proof follows the same pattern as in the case of M_α.□

Proof of Theorem 3.2. Since $(M_\alpha g)(.) \leq c(I_\alpha g)(.)$ and by Part A in Theorem 2.3 then (2.4) and (2.5) are necessary conditions for the boundedness $I_\alpha : \ell^s(L^p_v) \to \ell^r(L^q_u)$ to hold.

Conversely to get $I_\alpha : \ell^s(L^p_v) \to \ell^r(L^q_u)$, by Theorem 3.1 and Remarks after this result, it is sufficient to prove $(loc)I_\alpha : L^p_v \to L^q_u$ and $\mathcal{I}_{\alpha,\mu} : L^s(X, \mathcal{V}(m)d\mu(m)) \to L^r(X, \mathcal{U}(m)d\mu(m))$.

As in the proof of Part B in Theorem 2.3, the local boundedness $(loc)I_\alpha : L^p_v \to L^q_u$ can be seen by proving inequality (3.5). As in (4.14) the assumption $u(.) \in A_\infty$ implies that $\widetilde{u}(.) \in A_\infty$ with a A_∞-constant which only depends on $u(.)$ but not on n. Since $\widetilde{\sigma}(.)$, $\widetilde{u}(.) \in A_\infty$ and

(4.13) is satisfied, then by a result of C. Pérez [13] (Corollary 2.9, p. 669), inequality (3.5) appears.

With $\mathcal{U}(.) \in A_\infty(d\mu)$ and the well-known Muckenhoupt-Wheeden inequality (see [3], Remark 1.10, p.203) then $\mathcal{I}_{\alpha,\mu} : L^s(X, \mathcal{V}(m)d\mu(m)) \to L^r(X,\mathcal{U}(m)d\mu(m))$ is implied by the maximal boundedness $\mathcal{M}_{\alpha,\mu} : L^s(X, \mathcal{V}(m)d\mu(m)) \to L^r(X,\mathcal{U}(m)d\mu(m))$. And as in the proof of Part B in Theorem 2.3, this last is assured by condition (2.10) since $\mathcal{V}^{1-s'}(.) \in A_\infty(d\mu)$. □

Proof of Corollary 3.3. By Lemma 2.8, conditions (2.23) and (3.7) imply that $\mathcal{U}(.) \in A_\infty(d\mu)$ and $\mathcal{V}^{1-s'}(.) \in A_\infty(d\mu)$. On the other hand $v^{1-p'}(.) \in A_\infty$ and $u(.) \in A_\infty$. So, by Theorem 3.2, the boundedness $I_\alpha : \ell^s(L^p_v) \to \ell^r(L^q_u)$ holds provided that (2.9) and (2.10) are assumed. But, as in Corollary 2.10, these last conditions are implied by (2.21), (2.22) and (3.7). □

References

[1] H. Aimar, R. Macias, *Weighted norm inequalities for the Hardy-Littlewood maximal operator on spaces of homogeneous type*, Proc. Amer. Math. Soc. **91** (1984), 213-216.

[2] K. Andersen, H. Heinig, *Weighted norm inequalities for certain integral operators*, SIAM J. Math. **14** (1983), 834-844.

[3] A. Bernardis, O. Salinas, *Two-weight norm inequalities for the fractional maximal operator on spaces of homogeneous type*, Studia Math. **108** (1994), 201-207.

[4] C. Carton-Lebrun, H. Heinig, S. Hofmann, *Integral operators on weighted amalgams*, Studia Math **109** (1994), 133-157.

[5] R. Coifman, G. Weiss, *Analyse harmonique non-commutative sur certains espaces homogènes*, Lecture Notes in Math. Springer Verlag, Berlin and New York **242** (1971).

[6] J. Fournier, J. Stewart, *Amalgams of L^p and ℓ^q*, Bull. Amer. Math Soc. **13** (1985), 1-21.

[7] I. Genebashvili, A. Gogatishvili, V. Kokilashvili, *Solution of two-weight problems for integral transforms with positive kernels*, Georgian Math. J. **3** (1996), 319-342.

[8] A. Gogatishvili, V. Kokilashvili, *Criteria of strong type two weighted inequalities for fractional maximal functions*, Georgian Math. J. **3** (1996), 423-446.

[9] F. Holland, *Harmonic analysis on amalgams of L^p and ℓ^q*, J. London Math Soc. **10** (1975), 295-305.

[10] R. Johnson, C. Neugebauer, *Change of variable results for A_p and reverse Hölder RH_r-classes*, Trans. Amer. Math Soc. **328** (1991), 639-666.

[11] J. Lakey, *Weighted restriction for curves*, Canad. Math. Bull. **36** (1993), 87-95.

[12] ______, *Weighted Fourier transform inequalities via mixed norm Hausdorff-Young inequalities*, Canad. J. Math. **46** (1994), 585-601.

[13] C. Pérez, *Two weighted inequalities for potential and fractional type maximal operators*, Indiana univ. Math. J. **43** (1994), 663-683.

[14] ______, *On sufficient conditions for the boundedness of the Hardy-Littlewood maximal operator between weighted L^p-spaces with different weights*, Proc. London Math. Soc. **71** (1995), 135-157.

[15] Y. Rakotondratsimba, *Weighted inequalities for maximal and fractional integral operators*, Zeit. Anal. Anw. **15** (1996), 309-328.

[16] E. Sawyer, *A characterization of a two weight norm inequality for maximal operators*, Studia Math. **75** (1982), 1-11.

[17] E. Sawyer, R. L. Wheeden, *Weighted inequalities for fractional integrals on euclidean and homogeneous spaces*, Amer. J. Math. **114** (1992), 813-874.

[18] I. E. Verbitski, *Weighted norm inequalities for maximal operators and Pisier's theorem on factorization through $L^{p\infty}$*, Integr. Equat. Oper. Th. **15** (1992), 124-153.

[19] R. Wheeden, *A characterization of some weighted norm inequalities for fractional maximal functions*, Studia Math. **107** (1993), 251-272.

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