J. Korean Math. Soc. **36** (1999), No. 6, pp. 1169–1180

A GEOMETRIC APPROACH TO TWO-POINT COMPARISONS FOR HYPERBOLIC AND EUCLIDEAN GEOMETRY

SEONG-A KIM AND DAVID MINDA

ABSTRACT. Two-point comparison theorems between hyperbolic and euclidean geometry for convex regions in the complex plane $\mathbb{C}$ are known ([5], [6]). We give new geometric proofs of sharp two-point comparison theorems for convex regions.

1. Introduction

Sharp two-point comparison theorems between hyperbolic and euclidean geometry are known for various types of regions in the complex plane $\mathbb{C}$ ([5], [6], [7], [8]). These comparison theorems were motivated by work of Blatter [1] dealing with a characterization of univalent functions. The proofs of these results rely upon coefficient estimates for certain classes of functions defined on the unit disk $\mathbb{D}$. The purpose of this note is to present new geometric proofs of two-point comparison theorems for convex regions. Our proofs show that these comparisons can be derived from the fact that the reciprocal of the density of the hyperbolic metric is a concave function on convex regions [10]. This concavity property actually characterizes convex regions [4]. Other characterizations of convex regions in terms of hyperbolic geometry are given in [3].

We recall the known comparisons for convex regions. We begin with a brief discussion of hyperbolic geometry. Suppose Ω is a convex region in $\mathbb{C}$ with $\Omega \neq \mathbb{C}$. Let $\lambda_\Omega(w)|dw|$ denote the hyperbolic metric on Ω. For the unit disk $\lambda_{\mathbb{D}}(z)|dz| = |dz|/(1-|z|^2)$. The density λ_Ω is determined from $\lambda_\Omega(f(z)) = 1/[(1-|z|^2)|f'(z)|]$, where f is any conformal mapping

Received June 1, 1999.
1991 Mathematics Subject Classification: 30C45.
Key words and phrases: the hyperbolic metric, convex region.
This research was supported in part by a grant from the Korean Council for University Education, 1998.

of the unit disk onto Ω. The hyperbolic distance between $A, B \in \Omega$ is given by

$$d_\Omega(A,B) = \inf \int_\gamma \lambda_\Omega(w)|dw|,$$

where the infimum is taken over all rectifiable paths γ in Ω joining A and B. A path δ connecting A and B is called a hyperbolic geodesic if

$$d_\Omega(A,B) = \int_\delta \lambda_\Omega(w)|dw|.$$

For the unit disk

$$d_{\mathbb{D}}(a,b) = \operatorname{artanh}\left|\frac{a-b}{1-\bar{a}b}\right|$$

and hyperbolic geodesics are arcs of circles that are orthogonal to the unit circle $\partial\mathbb{D}$. Hyperbolic geodesics exist in Ω and are the images of hyperbolic geodesics in $\mathbb{D}$ under a conformal map $f : \mathbb{D} \to \Omega$.

In addition to supporting hyperbolic geometry, Ω carries the euclidean geometry that it inherits as a subset of $\mathbb{C}$. Two-point comparison theorems bound the euclidean distance $|A-B|$ above and below in terms of the hyperbolic distance $d_\Omega(A,B)$ and the values $\lambda_\Omega(A)$ and $\lambda_\Omega(B)$. For a convex region Ω and any $p \geq 1$ the lower bound

$$\frac{\sinh(d_\Omega(A,B))}{[2\cosh(pd_\Omega(A,B))]^{1/p}}\left[\frac{1}{\lambda_\Omega(A)^p} + \frac{1}{\lambda_\Omega(B)^p}\right]^{1/p} \leq |A-B|$$

and the upper bound

$$|A-B| \leq \frac{[2\cosh(pd_\Omega(A,B))]^{1/p}\sinh(d_\Omega(A,B))}{[\lambda_\Omega(A)^p + \lambda_\Omega(B)^p]^{1/p}}$$

are known ([5] and [6]). Both bounds are sharp: equality holds in either for distinct $A, B \in \Omega$ if and only if Ω is a half-plane and the euclidean line through A and B is perpendicular to the edge of the half-plane. The lower (upper) bound is a nonincreasing (nondecreasing) function of $p \geq 1$. Therefore, the bounds for $p = 1$, namely,

$$(1) \qquad \frac{\sinh(d_\Omega(A,B))}{2\cosh(d_\Omega(A,B))}\left[\frac{1}{\lambda_\Omega(A)} + \frac{1}{\lambda_\Omega(B)}\right] \leq |A-B| \leq \frac{2\cosh(d_\Omega(A,B))\sinh(d_\Omega(A,B))}{\lambda_\Omega(A)+\lambda_\Omega(B)}$$

are the strongest. The limiting cases $p = \infty$ are the weakest:

$$\frac{\sinh(d_\Omega(A,B))}{\exp(d_\Omega(A,B))\min\{\lambda_\Omega(A),\lambda_\Omega(B)\}} \leq |A-B| \leq \frac{\exp(d_\Omega(A,B))\sinh(d_\Omega(A,B))}{\max\{\lambda_\Omega(A),\lambda_\Omega(B)\}}.$$

These weakest comparisons are invariant versions of the classical growth theorem

$$\frac{|z|}{1+|z|} \leq |g(z)| \leq \frac{|z|}{1-|z|}$$

for normalized ($g(0) = 0, g'(0) = 1$) convex univalent functions g defined on $\mathbb{D}$. We shall provide a new, geometric proof of the strongest inequalities (1).

2. Preliminaries

Two quantities associated with the hyperbolic metric play an important role in our work. These quantities are the connection

$$\Gamma_\Omega(w) = 2\frac{\partial \log \lambda_\Omega}{\partial w}(w)$$

and the Schwarzian

$$\begin{aligned} S_\Omega(w) &= 2\left[\frac{\partial^2 \log \lambda_\Omega}{\partial w^2}(w) - \left(\frac{\partial \log \lambda_\Omega}{\partial w}(w)\right)^2\right] \\ &= \frac{\partial \Gamma_\Omega}{\partial w}(w) - \frac{1}{2}\Gamma_\Omega(w)^2. \end{aligned}$$

Note that

$$\frac{\partial \Gamma_\Omega}{\partial w}(w) = S_\Omega(w) + \frac{1}{2}\Gamma_\Omega(w)^2.$$

Also,

$$\frac{\partial \Gamma_\Omega}{\partial \overline{w}}(w) = 2\lambda_\Omega(w)^2$$

follows from the fact that $\lambda_\Omega(w)|dw|$ has curvature -4; that is,

$$-4 = -\frac{\Delta \log \lambda_\Omega(w)}{\lambda_\Omega(w)^2} = -\frac{4\frac{\partial^2 \log \lambda_\Omega}{\partial \overline{w}\partial w}(w)}{\lambda_\Omega(w)^2}.$$

Various characterizations of convexity are conveniently expressed in terms of the connection and the Schwarzian.

PROPOSITION 1. *Suppose Ω is a hyperbolic region in $\mathbb{C}$. Then the following are equivalent.*

(i) *Ω is convex.*
(ii) *$\frac{1}{\lambda_\Omega}$ is concave on Ω.*
(iii) *$|S_\Omega(w)| + \frac{1}{2}|\Gamma_\Omega(w)|^2 \leq 2\lambda_\Omega(w)^2$ for $w \in \Omega$.*
(iv) *$|\Gamma_\Omega(w)| \leq 2\lambda_\Omega(w)$ for $w \in \Omega$.*

The equivalence of (i), (ii) and (iii) is established in [4], but in different notation. The simple identities

$$\frac{1}{\lambda_\Omega(w)}\left|\frac{\partial^2\left(\frac{1}{\lambda_\Omega}\right)}{\partial w^2}(w)\right| = \frac{|S_\Omega(w)|}{2\lambda_\Omega(w)^2},$$

$$2\left|\frac{\partial}{\partial w}\left(\frac{1}{\lambda_\Omega}\right)(w)\right| = \frac{|\Gamma_\Omega(w)|}{\lambda_\Omega(w)},$$

show that Theorem 1 of [4] contains the equivalence of (i), (ii) and (iii). Note that (iii) implies (iv). The equivalence of (i) and (iv) is given in [2]. See [9] for a geometric proof that (i) implies (iv), as well as for a proof that equality holds in (iv) if and only if Ω is a half-plane. In other words, $|\Gamma_\Omega| \equiv 2\lambda_\Omega$ when Ω is a half-plane.

In addition to these characterizations of convex regions we require an elementary result for a differential inequality.

PROPOSITION 2. *Suppose $u, v \in C^2[-L, L]$, $v'' \leq 4v$ and $u'' = 4u$. If $u(L) = v(L)$ and $u(-L) = v(-L)$, then either $v = u$ on $[-L, L]$ or $v > u$ on $(-L, L)$.*

For a proof of this result see [6].

3. Main result

THEOREM 1. *Suppose Ω is a convex region in $\mathbb{C}$ with $\Omega \neq \mathbb{C}$.*

(i) *For $A, B \in \Omega$*

$$\frac{\sinh(d_\Omega(A,B))}{2\cosh(d_\Omega(A,B))}\left[\frac{1}{\lambda_\Omega(A)} + \frac{1}{\lambda_\Omega(B)}\right] \leq |A - B|. \tag{2}$$

(ii) *For* $A, B \in \Omega$

$$|A-B| \leq \frac{2\cosh(d_\Omega(A,B))\sinh(d_\Omega(A,B))}{\lambda_\Omega(A)+\lambda_\Omega(B)}. \tag{3}$$

Equality holds in (2) *or* (3) *for distinct* A *and* B *if and only if* Ω *is a half-plane and the euclidean line through* A *and* B *is perpendicular to the edge of* Ω.

Proof. (i) Fix $A, B \in \Omega$ with $A \neq B$. Because Ω is convex, the straight line segment $\gamma := [A, B]$ is contained in Ω. Let $\gamma : w = w(s)$, $-L \leq s \leq L$, be a hyperbolic arclength parametrization of γ. This means that $w'(s) = e^{i\theta}/\lambda_\Omega(w(s))$, where θ is the argument of $B - A$, and $2L$ is the hyperbolic length of γ. Set

$$v(s) := \frac{1}{\lambda_\Omega(w(s))}, \quad -L \leq s \leq L.$$

Then

$$\begin{aligned} v'(s) &= -\frac{2}{\lambda_\Omega(w(s))^2}\mathrm{Re}\left\{\frac{\partial\lambda_\Omega(w(s))}{\partial w}w'(s)\right\} \\ &= -\frac{1}{\lambda_\Omega(w(s))}\mathrm{Re}\left\{\Gamma_\Omega(w(s))e^{i\theta}\right\} \\ &= -v^2(s)\mathrm{Re}\left\{\Gamma_\Omega(w(s))e^{i\theta}\right\}. \end{aligned}$$

Next, we compute the second derivative of v.

$$\begin{aligned} v''(s) &= -2v(s)v'(s)\mathrm{Re}\left\{\Gamma_\Omega(w(s))e^{i\theta}\right\} - v^2(s)\mathrm{Re}\left\{e^{i\theta}\frac{d}{ds}\Gamma(w(s))\right\} \\ &= 2v^3(s)\mathrm{Re}^2\left\{\Gamma_\Omega(w(s))e^{i\theta}\right\} - v^2(s)\mathrm{Re}\left\{e^{i\theta}\frac{d}{ds}\Gamma_\Omega(w(s))\right\}. \end{aligned}$$

Now,

$$\begin{aligned} \frac{d}{ds}\Gamma_\Omega(w(s)) &= \frac{\partial\Gamma_\Omega}{\partial w}(w(s))w'(s) + \frac{\partial\Gamma_\Omega}{\partial\overline{w}}(w(s))\overline{w'(s)} \\ &= \left[S_\Omega(w(s)) + \frac{1}{2}\Gamma_\Omega(w(s))^2\right]\frac{e^{i\theta}}{\lambda_\Omega(w(s))} + 2\lambda_\Omega(w(s))e^{-i\theta}, \end{aligned}$$

so that

$$\begin{aligned} v''(s) &= 2v^3(s)\mathrm{Re}^2\left\{\Gamma_\Omega(w(s))e^{i\theta}\right\} \\ &\quad -\frac{v^2(s)}{\lambda_\Omega(w(s))}\mathrm{Re}\left\{\left[S_\Omega(w(s))+\frac{1}{2}\Gamma_\Omega(w(s))^2\right]e^{2i\theta}\right\} \\ &\quad -2v^2(s)\lambda_\Omega(w(s)) \\ &= v^3(s)\left[2\mathrm{Re}^2\left\{\Gamma_\Omega(w(s))e^{i\theta}\right\}\right. \\ &\qquad \left.-\mathrm{Re}\left\{\left[S_\Omega(w(s))+\frac{1}{2}\Gamma_\Omega(w(s))^2\right]e^{2i\theta}\right\}\right]-2v(s) \\ &= v^3(s)\left[|\Gamma_\Omega(w(s))|^2\right. \\ &\qquad \left.-\mathrm{Re}\left\{\left[S_\Omega(w(s))-\frac{1}{2}\Gamma_\Omega(w(s))^2\right]e^{2i\theta}\right\}\right]-2v(s) \end{aligned}$$

since

$$2\mathrm{Re}^2\left\{\Gamma_\Omega(w(s))e^{i\theta}\right\}=|\Gamma_\Omega(w(s))|^2+\mathrm{Re}\left\{\left[\Gamma_\Omega(w(s))e^{i\theta}\right]^2\right\}.$$

Because Ω is convex we obtain

$$\begin{aligned} v''(s) &\leq v^3(s)\left[|\Gamma_\Omega(w(s))|^2+\left|S_\Omega(w(s))-\frac{1}{2}\Gamma_\Omega(w(s))^2\right|\right]-2v(s) \\ &\leq v^3(s)\left[|\Gamma_\Omega(w(s))|^2+|S_\Omega(w(s))|+\frac{1}{2}|\Gamma_\Omega(w(s))|^2\right]-2v(s) \\ &\leq v^3(s)\left[(2\lambda_\Omega(w(s)))^2+2\lambda_\Omega(w(s))^2\right]-2v(s) \\ &= 4v(s). \end{aligned}$$

Note that if $v''(s)=4v(s)$, then $|\Gamma_\Omega(w(s))|=2\lambda_\Omega(w(s))$, and so Ω must be a half-plane.

Let $u(s)=c\cosh(2s)+d\sinh(2s)$ be the solution of $u''(s)=4u(s)$ that satisfies the boundary conditions $u(-L)=v(-L)$ and $u(L)=v(L)$. Then

$$\begin{aligned} c &= \frac{v(L)+v(-L)}{2\cosh(2L)}=\frac{\frac{1}{\lambda_\Omega(A)}+\frac{1}{\lambda_\Omega(B)}}{2\cosh(2L)}, \\ d &= \frac{v(L)-v(-L)}{2\sinh(2L)}. \end{aligned}$$

Proposition 2 implies that either $v = u$ on $[-L, L]$ or $v > u$ on $(-L, L)$. Now,

$$\begin{aligned}
|A - B| &= \int_\gamma |dw| = \int_{-L}^{L} |w'(s)|ds \\
&= \int_{-L}^{L} \frac{ds}{\lambda_\Omega(w(s))} = \int_{-L}^{L} v(s)ds \\
&\geq \int_{-L}^{L} u(s)ds = c\sinh(2L) \\
&= \frac{\sinh(2L)}{2\cosh(2L)}\left[\frac{1}{\lambda_\Omega(A)} + \frac{1}{\lambda_\Omega(B)}\right] \\
&= \frac{1}{2}\tanh(2L)\left[\frac{1}{\lambda_\Omega(A)} + \frac{1}{\lambda_\Omega(B)}\right].
\end{aligned}$$

Note that equality implies $v = u$ and so $v''(s) = 4v(s)$ on $[-L, L]$; this implies Ω is a half-plane. Since $2L \geq d_\Omega(A, B)$ with equality if and only if γ is a hyperbolic geodesic, we conclude that

$$\begin{aligned}
|A - B| &\geq \frac{1}{2}\tanh(d_\Omega(A, B))\left[\frac{1}{\lambda_\Omega(A)} + \frac{1}{\lambda_\Omega(B)}\right] \\
&= \frac{\sinh(d_\Omega(A, B))}{2\cosh(d_\Omega(A, B))}\left[\frac{1}{\lambda_\Omega(A)} + \frac{1}{\lambda_\Omega(B)}\right].
\end{aligned}$$

Equality implies that Ω is a half-plane and the straight line segment $\gamma = [A, B]$ is a hyperbolic geodesic. In a half-plane the hyperbolic geodesics are arcs of circles orthogonal to the edge of the half-plane and segments of lines perpendicular to the edge of the half-plane. Hence, if equality holds, then Ω is a half-plane and $[A, B]$ is part of a line perpendicular to the edge of the half-plane.

All that remains is to show that equality holds when Ω is a half-plane and $[A, B]$ is perpendicular to $\partial\Omega$. It suffices to consider the particular half-plane $\mathbb{H} = \{w : \operatorname{Im}\{w\} > 0\}$. Then $\lambda_\mathbb{H}(w) = 1/[2\operatorname{Im}\{w\}]$. Fix $u \in \mathbb{R}$ and $0 < a < b$. Then for $A = u+ia$ and $B = u+ib$, $|A-B| = b-a$, $d_\mathbb{H}(A, B) = \frac{1}{2}\log\frac{b}{a}$, and $\lambda_\mathbb{H}(A) = 1/(2a)$, $\lambda_\mathbb{H}(B) = 1/(2b)$. It is now straightforward to check that equality holds in (2).

(ii) Consider $A, B \in \Omega$ with $A \neq B$. Let δ be the hyperbolic geodesic joining A to B. Suppose $\delta : w = w(s)$, $-L \leq s \leq L$, is a hyperbolic arclength parametrization of δ. Then $2L = d_\Omega(A, B)$, the hyperbolic length of δ, and $w'(s) = e^{i\theta(s)}/\lambda_\Omega(w(s))$, where $e^{i\theta(s)}$ is a unit tangent to

δ at $w(s)$. Set

$$V(s) := \lambda_\Omega(w(s)).$$

Then

$$\begin{aligned} V'(s) &= 2\mathrm{Re}\left\{\frac{\partial\lambda_\Omega}{\partial w}(w(s))w'(s)\right\} \\ &= 2\mathrm{Re}\left\{\frac{\partial\lambda_\Omega}{\partial w}(w(s))\frac{e^{i\theta(s)}}{\lambda_\Omega(w(s))}\right\} \\ &= \mathrm{Re}\left\{\Gamma_\Omega(w(s))e^{i\theta(s)}\right\}. \end{aligned}$$

Because Ω is convex we obtain

$$|V'(s)| \le |\Gamma_\Omega(w(s))| \le 2\lambda_\Omega(w(s)) = 2V(s). \tag{4}$$

Next, we calculate the second derivative of V.

$$\begin{aligned} V''(s) &= \mathrm{Re}\left\{e^{i\theta(s)}\frac{d}{ds}\Gamma_\Omega(w(s))\right\} + \mathrm{Re}\left\{\Gamma_\Omega(w(s))\frac{d}{ds}e^{i\theta(s)}\right\} \\ &= \frac{1}{\lambda_\Omega(w(s))}\mathrm{Re}\left\{\left[S_\Omega(w(s)) + \frac{1}{2}\Gamma_\Omega(w(s))^2\right]e^{2i\theta(s)}\right\} \\ &\quad +2\lambda_\Omega(w(s)) + \frac{d\theta(s)}{ds}\mathrm{Re}\left\{ie^{i\theta(s)}\Gamma_\Omega(w(s))\right\}. \end{aligned}$$

The hyperbolic curvature of δ is

$$\begin{aligned} \kappa_h(w(s),\delta) &= \frac{\kappa_e(w(s),\delta) + 2\,\mathrm{Im}\left\{\frac{\partial \log\lambda_\Omega}{\partial w}(w(s))e^{i\theta(s)}\right\}}{\lambda_\Omega(w(s))} \\ &= \frac{\kappa_e(w(s),\delta) + \mathrm{Im}\left\{\Gamma_\Omega(w(s))e^{i\theta(s)}\right\}}{\lambda_\Omega(w(s))}, \end{aligned}$$

where

$$\kappa_e(w(s),\delta) = \frac{1}{|w'(s)|}\,\mathrm{Im}\left\{\frac{w''(s)}{w'(s)}\right\}$$

is the euclidean curvature of γ at $w(s)$. Since δ is a hyperbolic geodesic, $\kappa_h(w(s),\delta) = 0$, so

$$\mathrm{Im}\left\{\Gamma_\Omega(w(s))e^{i\theta(s)}\right\} = -\kappa_e(w(s),\delta).$$

From $w'(s) = e^{i\theta(s)}/\lambda_\Omega(w(s))$, we obtain

$$\mathrm{Im}\left\{\frac{w''(s)}{w'(s)}\right\} = \frac{d\theta(s)}{ds}$$

and so

$$\kappa_e(w(s),\delta) = \frac{d\theta(s)}{ds}\lambda_\Omega(w(s)).$$

Thus,

$$\frac{d\theta(s)}{ds} = -\frac{1}{\lambda_\Omega(w(s))}\,\mathrm{Im}\,\left\{\Gamma_\Omega(w(s))e^{i\theta(s)}\right\}$$

and so

$$\begin{aligned} V''(s) &= \frac{1}{\lambda_\Omega(w(s))}\mathrm{Re}\left\{\left[S_\Omega(w(s)) + \frac{1}{2}\Gamma_\Omega(w(s))^2\right]e^{2i\theta(s)}\right\} \\ &\quad +2\lambda_\Omega(w(s)) + \frac{1}{\lambda_\Omega(w(s))}\,\mathrm{Im}^2\left\{\Gamma_\Omega(w(s))e^{i\theta(s)}\right\} \end{aligned}$$

since

$$\mathrm{Re}\left\{i\Gamma_\Omega(w(s))e^{i\theta(s)}\right\} = -\,\mathrm{Im}\,\left\{\Gamma_\Omega(w(s))e^{i\theta(s)}\right\}.$$

Finally,

$$V''(s) = \frac{1}{\lambda_\Omega(w(s))}\left(\frac{1}{2}\left|\Gamma_\Omega(w(s))\right|^2 + \mathrm{Re}\,\left\{S_\Omega(w(s))e^{2i\theta(s)}\right\}\right) + 2\lambda_\Omega(w(s)),$$

since

$$\frac{1}{2}\mathrm{Re}\,\left\{\left(\Gamma_\Omega(w(s))e^{i\theta(s)}\right)^2\right\} + \mathrm{Im}^2\left\{\Gamma_\Omega(w(s))e^{i\theta(s)}\right\} = \frac{1}{2}\left|\Gamma_\Omega(w(s))\right|^2.$$

Because Ω is convex, we get

$$\begin{aligned} V''(s) &\le \frac{1}{\lambda_\Omega(w(s))}\left(\frac{1}{2}\left|\Gamma_\Omega(w(s))\right|^2 + \left|S_\Omega(w(s))\right|\right) + 2\lambda_\Omega(w(s)) \\ &\le 4\lambda_\Omega(w(s)) = 4V(s). \end{aligned}$$

Now, let $U(s) = C\cosh(2s) + D\sinh(2s)$ be the solution of $U''(s) = 4U(s)$ that satisfies the boundary conditions $U(-L) = V(-L)$ and $U(L) = V(L)$. Then

$$\begin{aligned} C &= \frac{V(L)+V(-L)}{2\cosh(2L)} = \frac{\lambda_\Omega(A)+\lambda_\Omega(B)}{2\cosh(d_\Omega(A,B))} > 0, \\ D &= \frac{V(L)-V(-L)}{2\sinh(2L)}. \end{aligned}$$

Proposition 2 implies $V = U$ on $[-L, L]$ or $V > U$ on $(-L, L)$. Now, if $U > 0$ on $[-L, L]$

$$\begin{aligned} |A - B| &\leq \int_\delta |dw| = \int_{-L}^{L} |w'(s)| ds \\ &= \int_{-L}^{L} \frac{ds}{\lambda_\Omega(w(s))} = \int_{-L}^{L} \frac{ds}{V(s)} \\ &\leq \int_{-L}^{L} \frac{ds}{C \cosh(2s) + D \sinh(2s)}, \end{aligned}$$

with equality if and only if the hyperbolic geodesic δ is the euclidean line segment $[A, B]$. We show that $U > 0$ on $[-L, L]$. Note that

$$\begin{aligned} \frac{1}{U(s)} &= \frac{C \cosh(2s) - D \sinh(2s)}{C^2 \cosh^2(2s) - D^2 \sinh^2(2s)} \\ &= \frac{1}{C} \frac{\cosh(2s) - \tau \sinh(2s)}{1 + (1 - \tau^2) \sinh^2(2s)}, \end{aligned}$$

where

$$\tau := \frac{D}{C} = \frac{V(L) - V(-L)}{V(L) + V(-L)} \, \frac{1}{\tanh(2L)}.$$

We will show that $|\tau| \leq 1$; in particular, this shows that $U > 0$ on $[-L, L]$. From (4) we have

$$-2 \leq \frac{V'(s)}{V(s)} \leq 2$$

for $s \in [-L, L]$. If we integrate these inequalities over the interval $[-L, L]$, then we obtain

$$-4L \leq \log \frac{V(L)}{V(-L)} \leq 4L,$$

or

$$e^{-4L} \leq \frac{V(L)}{V(-L)} \leq e^{4L}.$$

The function $h(t) = (t - 1)/(t + 1)$ is increasing for $t > -1$ since $h'(t) = 2/(t + 1)^2 > 0$. Thus,

$$h(e^{-4L}) \leq h\left(\frac{V(L)}{V(-L)}\right) \leq h(e^{4L}),$$

or

$$-\tanh(2L) \leq \frac{V(L)-V(-L)}{V(L)+V(L)} \leq \tanh(2L).$$

This demonstrates that $|\tau| \leq 1$. Also, we conclude that $\tau = \pm 1$ implies $|V'(s)| = 2|V(s)|$, or $|\Gamma_\Omega(w(s))| = 2\lambda_\Omega(w(s))$, which means Ω is a half-plane. Now

$$\begin{aligned}\int_{-L}^{L} \frac{ds}{C\cosh(2s)+D\sinh(2s)} &= \frac{1}{C}\int_{-L}^{L} \frac{\cosh(2s)-\tau\sinh(2s)}{1+(1-\tau^2)\sinh^2(2s)}\,ds \\ &= \frac{1}{C}\int_{-L}^{L} \frac{\cosh(2s)}{1+(1-\tau^2)\sinh^2(2s)}\,ds\end{aligned}$$

since $\sinh(2s)/[1+(1-\tau^2)\sinh^2(2s)]$ is an odd function. Also,

$$\begin{aligned}\int_{-L}^{L} \frac{\cosh(2s)}{1+(1-\tau^2)\sinh^2(2s}\,ds &\leq \int_{-L}^{L} \cosh(2s)\,ds \\ &= \sinh(2L) \\ &= \sinh(d_\Omega(A,B)),\end{aligned}$$

and equality implies $\tau = \pm 1$. By combining our inequalities, we obtain.

$$|A-B| \leq \frac{\sinh(d_\Omega(A,B))}{C} = \frac{2\cosh(d_\Omega(A,B))\sinh(d_\Omega(A,B))}{\lambda_\Omega(A)+\lambda_\Omega(B)}$$

and equality implies Ω is a half-plane and $\delta = [A,B]$ is a hyperbolic geodesic, so $[A,B]$ must be perpendicular to $\partial\Omega$.

Conversely, it is routine to show that if Ω is a half-plane and $[A,B]$ lies on a line orthogonal to $\partial\Omega$, then equality holds in (3).

4. Acknowledgement

Both authors want to thank Professor Kang Tae Kim for the kind invitation to visit POSTECH May 2-3, 1997. This work was begun during the visit. The second author thanks Professor Jae Keol Park of Pusan National University for arranging his visit to Korea in May, 1997. Also, the first author thanks Department of Mathematics of POSTECH for their allowing her visit to the Department during the 1998 academic year.

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Department of Mathematics
Woosuk University, Wanju-gun
Cheonbuk 565-701, Korea
E-mail: sakim@core.woosuk.ac.kr

Department of Mathematical Sciences
University of Cincinnati
Cincinnati, OH 45221-0025, USA
E-mail: David.Minda@math.uc.edu