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ON EXTREMAL ELLIPTIC $K3$ SURFACES

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ABSTRACT. In this paper, we first classify the possible configurations of fibrations which are not semi-stable on extremal elliptic $K3$ surfaces. Then we give a complete list of extremal elliptic $K3$ surfaces whose singular fibers are all not of type I_n.

0. Introduction

Let C be a smooth projective curve over an algebraically closed field of characteristic 0, and X an *elliptic surface* over C. By this we mean the following: X is a smooth projective surface with a relatively minimal elliptic fibration

$$f : X \longrightarrow C.$$

In this paper, we also assume:
(i) f has a global section $\mathcal{O}$, and
(ii) f is not smooth, i.e., there is at least one singular fibre.

To every (Jacobian) elliptic fibration X there is a group of sections $\Phi(X)$ with the distinguished section $\mathcal{O}$ as zero. Up to a finite group, $\Phi(X)$ is identified with the relative automorphism group of the fibration.

Due to a formula of Shioda-Tate we have the basic inequalities

$$0 \leq \text{rank}\Phi \leq \rho(X) - 2$$

where ρ is the Picard number and the discrepancy in the upper bound is related to the degree of reducibility of the fibres.

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DEFINITION 0.1. An elliptic fibration X is called *extremal* if and only if $\rho(X) = h^{1,1}(X)$ (*maximal Picard number*) and rank $\Phi(X) = 0$.

Let $f : X \longrightarrow \mathbf{P}^1$ be an *extremal* elliptic $K3$ surface. A fibration f is called *semi-stable* if each singular fiber of f is of type I_n [7]. Here we call a fibration *unsemi-stable* if it is not semi-stable.

In [6], R. Miranda and U. Persson have classified possible semi-stable fibrations. The determination of all semi-stable fibrations has been done in [7] and [1]. In this paper, we first classify all possible configurations of unsemi-stable fibrations (cf. Theorem 2.4). Then we calculate the possible Mordell-Weil Groups for Case(A) (cf. Theorem 3.1), i.e., the case where each singular fibre of f is *not* of type I_n. Finally, by using the method in [1], we will precisely determine which cases in *Table* 1 are actually realizable (cf. Theorem 0.4).

Let i_n denote the number of singular fibres of f of type I_n. Similarly we define i_n^*, ii, iii, iv, iv^*, iii^*, ii^* (cf. [5]). Then we have the following Theorem 0.2.

THEOREM 0.2. *Let $f : X \longrightarrow \mathbf{P}^1$ be an extremal elliptic $K3$ surface with* $\deg J \neq 0$. *Then*

$$\begin{aligned}\deg J &= \sum_{n\geq 1} n(i_n + i_n^*)\\ &= 6\sum_{n\geq 1}(i_n + i_n^*) + 4(ii + iv^*) + 3(iii + iii^*) + 2(iv + ii^*) - 12.\end{aligned}$$

REMARK 0.3. From [5, Lemma 3.1 and Proposition 3.4], we know that the second equality in Theorem 0.2 is replaced by "$\leq$" in general cases, for example, the fiber type $(II^*, II^*, I_2, I_1, I_1)$ (cf. [11, Lemma 3.1]). Thus it (Theorem 0.2) justifies their naming "extremal".

THEOREM 0.4. *Let $f : X \longrightarrow \mathbf{P}^1$ be an extremal elliptic $K3$ surface and each singular fibre of f is* not *of type I_n. Then there exists exactly* 11 *fiber types as given below* (table 1). *In particular, Mordell-Weil Group is uniquely determined by the fiber type of f.*

Table 1

$\sharp$	the fibre type	MW(f)	$\sharp$	the fibre type	MW(f)
1	(II^*, I_1^*, I_1^*)	(0)	7	(IV^*, IV^*, IV^*)	$\mathbf{Z}/3\mathbf{Z}$
2	(II^*, II^*, IV)	(0)	8	(IV^*, IV^*, I_2^*)	(0)
3	(II^*, IV^*, I_0^*)	(0)	9	(IV^*, I_3^*, I_1^*)	(0)
4	(III^*, III^*, I_0^*)	$\mathbf{Z}/2\mathbf{Z}$	10	(I_4^*, I_1^*, I_1^*)	$\mathbf{Z}/2\mathbf{Z}$
5	(III^*, IV^*, I_1^*)	(0)	11	(I_2^*, I_2^*, I_2^*)	$\mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$
6	(III^*, I_2^*, I_1^*)	$\mathbf{Z}/2\mathbf{Z}$			

All the above 11 fiber types are realizable.

REMARK 0.5. At the same time, and independently, I. Shimada and D. Q. Zhang present a complete list of extremal elliptic $K3$ surfaces [9]. But our method is different and without the help of computer.

This paper is organized as follows. In Section 1, we introduce some basic notation and theorems which will be used in the paper. In Section 2, we first prove Theorem 0.2. Then we give the combinatorical classification of the possible unsemi-stable fibration (cf. Theorem 2.4). In Section 3, we calculate all possible Mordell-Weil Groups for Case(A) (cf. Theorem 3.1). In Section 4, we prove Theorem 0.4.

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1. Preliminaries

(a) Lattices

Let L be a lattice, i.e.,

(i) L is a free finite $\mathbf{Z}$ module and

(ii) L is equipped with a non-degenerate bilinear symmetric pairing $\langle , \rangle$.

The determinant of L, $\det L$, is defined as the determinant of the matrix $I = (\langle x_i, x_j\rangle)$ where $\{x_1, \dots, x_r\}$ is a $\mathbf{Z}$-basis of L ($r=$ the rank of L):

$$\det L = \det(\langle x_i, x_j\rangle).$$

We define the positive- (or negative-) definiteness or the signature of a lattice by that of the matrix I, noting that these properties are independent of the choice of a basis. An lattice L is called *even* if $\langle x, x\rangle \in 2\mathbf{Z}$ for all $x \in L$. We call L *unimodular* if $\det L$=1. Let J be a sublattice of L. We denote its orthogonal complement with respect to $\langle , \rangle$ by $J^{\perp}$.

For a lattice L, we denote its *dual* lattice by $L^{\vee}$. By using pairing, L is embedded in $L^{\vee}$ as a sublattice with the same rank. Hence the quotient group $L^{\vee}/L$ is a finite abelian group, which we denote by G_J.

For an even lattice L, we define a quadratic form q_L with values in $\mathbf{Q}/2\mathbf{Z}$ as follows:

$$q_L(x \bmod L) = \langle x, x\rangle \bmod 2\mathbf{Z}.$$

LEMMA 1.1. *For $j = 1,2$, let $\Delta_j = \Delta(1)_j \oplus \cdots \oplus \Delta(r_j)_j$ be a lattice where each $\Delta(i)_j$is of Dynkin type A_a, D_d or E_e.*

(1) *Suppose that Φ: $\Delta_1 \longrightarrow \Delta_2$ is a lattice-isometry. Then $r_1 = r_2$ and $\Phi(\Delta(i)_1) = \Delta(i)_2$ after relabelling.*

(2) *Let $\mathbf{B}(6) = E_7 \oplus D_6 \oplus D_5$, $\mathbf{B}(10) = D_8 \oplus D_5 \oplus D_5$, $\mathbf{B}(11) = D_6 \oplus D_6 \oplus D_6$.*

Then we have

(*i*) *$\mathbf{B}(6) \subset E_7 \oplus D_{11}$ is an index-2 lattice extension.*

(*ii*) *$\mathbf{B}(10) \subset D_5 \oplus D_{13}$ is an index-2 extension, $\mathbf{B}(10) \subset D_8 \oplus D_{10}$ is an index-2 extension and $\mathbf{B}(10) \subset D_{18}$ is an index-4 extension.*

(*iii*) *$\mathbf{B}(11) \subset D_6 \oplus D_{12}$ is an index-2 extension, $\mathbf{B}(11) \subset D_{18}$ is an index-4 extension.*

Proof. We observe that

$$|\det(A_n)| = n+1,\ |\det(D_n)| = 4,\ |\det(E_6)| = 3,$$
$$|\det(E_7)| = 2, \qquad |\det(E_8)| = 1,$$

and for an index-n lattice extension $L \subset M$ one has

$$|\det(L)| = n^2 |\det(M)|.$$

Then (1) comes from [1, Lemma 1.3], and (2) can be obtained by an easy calculation. □

DEFINITION 1.2. (The lattice D_n) [3] For $n \geq 3$,

$$D_n = \{(x_1, x_2, ..., x_n) \in \mathbf{Z}^n : x_1 + \cdots + x_n \text{ is even } \}.$$

REMARK 1.3. From Lemma 1.1, we know $D_8 \oplus D_5 \oplus D_5 \subset D_{18}$ is an index-4 extension. Thus $D_{18}/(D_8 \oplus D_5 \oplus D_5)$ maybe $\mathbf{Z}/4\mathbf{Z}$ or $\mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$. In the following Lemma 1.4, We shall prove that, $D_{18}/(D_8 \oplus D_5 \oplus D_5) = \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$.

LEMMA 1.4. *For any lattice-isometric embedding i: $D_5 \oplus D_5 \oplus D_8 \longrightarrow D_{18}$, we have*

$$D_{18}/(D_5 \oplus D_5 \oplus D_8) = \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}.$$

Proof. We denote $A \oplus B \oplus C = D_5 \oplus D_5 \oplus D_8$ and let $i : A \oplus B \oplus C \longrightarrow D_{18}$ be a lattice-isometric embedding. For one generator e of the lattices A, B or C, we assume that $i(e) = (x_1, x_2, ..., x_{18}) \in D_{18}$. Since $2 = \langle e, e \rangle = \langle i(e), i(e) \rangle = \sum_{i=1}^{18} x_i^2$ and x_i is integer, we have

CLAIM 1.5. There are exactly two coordinates of $i(e)$ which are non-zero and each of which is 1 or -1.

Thus by relabelling the coordinates, we may assume that one generators e_1 of A satisfies

$$i(e_1) = (1, 1, 0, ..., 0) \quad \text{or} \quad (1, -1, 0, ..., 0).$$

Then we can use the connections among the generators in Dynkin diagram of D_5 to get the possible coordinates of the generators e_1, e_2, e_3, e_4, e_5 of A. After a simple calculation, we find that, by rebelling the coordinates, we may assume that

$$i(A) \subset (x_1, x_2, x_3, x_4, x_5, 0, ..., 0) \cap D_{18} := L_1.$$

On the other hand, we know L_1 is a D_5 type lattice and i is lattice isometry, thus we get

$$i(A) = L_1.$$

By using the same method, we may assume that

$$i(B) = (0, .., 0, x_6, x_7, .., x_{10}, 0, .., 0) \cap D_{18} := L_2$$

and

$$i(C) = (0, .., 0, x_{11}, .., x_{18}) \cap D_{18} := L_3.$$

Here we will use the orthonormal conditions among $i(A)$, $i(B)$ and $i(C)$.

A direct computation shows that

$$D_{18}/(L_1 \oplus L_2 \oplus L_3) = \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}.$$

Thus we prove the Lemma 1.4. □

By using the same idea as above proof, we get

THEOREM 1.6. *For* $m = \sum_{i=1}^{k} n_i$, *we have*

$$D_m/(\oplus_{i=1}^{k} D_{n_i}) = \oplus_{i=1}^{k} \mathbf{Z}/2\mathbf{Z}$$

for any lattice-isometric embedding $i : \oplus_{i=1}^{k} D_{n_i} \longrightarrow D_m$.

(b) Mordell-Weil lattices of elliptic surface

Given an elliptic surface f: $X \longrightarrow C$, let $F_\nu = f^{-1}(\nu)$ denote the fibre over $\nu \in C$, and let

$Sing(f) = \{\nu \in C | F_\nu \text{ is singular}\}$.

$\mathbf{R} = \mathrm{Red}(f) = \{\nu \in C | F_\nu \text{ is reducible}\}$.

For each $\nu \in \mathbf{R}$, let

$$F_\nu = f^{-1}(\nu) = \Theta_{\nu,0} + \sum_{i=1}^{m_\nu - 1} \mu_{\nu,i}\Theta_{\nu,i} \quad (\mu_{\nu,0} = 1)$$

where $\Theta_{\nu,i}$ $(0 \leq i \leq m_\nu - 1)$ are the irreducible components of F_ν, m_ν being their number, such that $\Theta_{\nu,0}$ is the unique component of F_ν meeting the zero section.

Here we denote

$$E(K) = \text{the group of sections of } f,$$

and

$$NS(X) = \text{the group of divisors on } X \text{ modulo algebraic equivalence.}$$

THEOREM 1.7. (cf. [10, Theorem 1.1, 1.2, 1.3]) *Under the assumptions for the elliptic surfaces in Introduction, we have*

(1) $E(K)$ is a finite generated abelian group.

(2) $N(X)$ is finitely generated and torsion-free.

(3) Let T denote the subgroup of $NS(X)$ generated by the zero section (O) and all the irreducible components of fibres. Then, there is a natural isomorphism

$$E(K) \cong NS(X)/T,$$

which maps $P \in E(K)$ to (P) mod T.

THEOREM 1.8. (cf. [10, Lemma 8.1]) *For any P, $Q \in E(K)$, let*

$$\langle P, Q\rangle = -(\varphi(P) \cdot \varphi(Q)) \qquad (*)$$

where $\varphi(P)$ (resp. $\varphi(Q)$), satisfying the condition:

(1) $\varphi(P) \equiv (P)$ mod $T_{\mathbf{Q}}$, *and*

(2) $\varphi(P) \perp T$.

Then it defines a symmetric bilinear pairing on $E(K)$, which induces the structure of a positive-definite lattice on $E(K)/E(K)_{tor}$.

DEFINITION 1.9. The pairing $(*)$ on the Mordell-Weil group $E(K)$ is called the *height pairing*, and the lattice

$$(E(K)/E(K)_{tor}, \langle , \rangle)$$

is called the *Mordell-Weil Lattice* of the elliptic curve E/K or of the elliptic surface $f\colon S \longrightarrow C$.

THEOREM 1.10. (Explicit formula for the height pairing) [10, Theorem 8.6] *For any P,$Q \in E(K)$, we have*

$$\langle P, Q\rangle = \chi + (PO) + (QO) - (PQ) - \sum_{\nu\in R} contr_\nu(P, Q),$$

$$\langle P, P\rangle = 2\chi + 2(PO) - \sum_{\nu\in R} contr_\nu(P).$$

REMARK 1.11. Here χ is the arithmetic genus of S, and (PO) is the intersection number of the sections (P) and (O), and similarly for (QO),(PQ). The term $contr_\nu(P,Q)$ stands for the local contribution ar $\nu \in R$, which is defined as follows: suppose that (P) interests $\Theta_{\nu,i}$ and (Q) intersects $\Theta_{\nu,j}$. Then we let

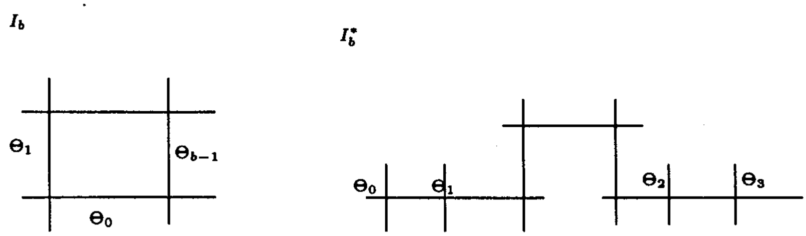
$$contr_\nu(P,Q) = \begin{cases} (-A_\nu^{-1})_{i,j}, & \text{if } i \geq 1,\ j \geq 1, \\ 0, & \text{otherwise} \end{cases}$$

where the first one means the (i,j)-entry of the matrix $(-A_\nu^{-1})$. Further we set

$$contr_\nu(P) = contr_\nu(P,P).$$

Arrange $\Theta_i = \Theta_{\nu,i}$ $(i = 0, 1, \cdots, m_\nu - 1)$ so that the simple components are numbered as in the figure below.

Figure 1



For the other types of reducible fibres, the numbering is irrelevant. Assume that (P) intersects $\Theta_{\nu,i}$ and (Q) intersect $\Theta_{\nu,j}$ with $i > 1$, $j > 1$. Then we have the following table: the forth row is for the case $i < j$ (interchange P, Q if necessary).

Table 2

T_ν^-	A_1	E_7	A_2	E_6	A_{b-1}	D_{b+4}
type of F_ν	III	III^*	IV	IV^*	$I_b(b \geq 2)$	$I_b^*(b \geq 0)$
$contr_\nu(P)$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{2}{3}$	$\frac{4}{3}$	$\frac{i(b-i)}{b}$	$\begin{cases} 1, & i=1 \\ 1+\frac{b}{4}, & i>1 \end{cases}$
$contr_\nu(P,Q)(i<j)$	-	-	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{i(b-i)}{b}$	$\begin{cases} \frac{1}{2}, & i=1 \\ \frac{(2+b)}{4}, & i>1 \end{cases}$

THEOREM 1.12. (cf. [5, Lemma 3.1 and Proposition 3.4]) *For an elliptic fibration* π: $X \longrightarrow \mathbf{P}^1$, *we have the following formulas:*

$$\deg J = \sum_{n \geq 1} n(i_n + i_n^*),$$

and if furthermore $\deg J \neq 0$, *then we also have*

$$\deg J \leq 6 \sum_{n \geq 1} (i_n + i_n^*) + 4(ii + iv^*) + 3(iii + iii^*) + 2(iv + ii^*) - 12.$$

2. The possible configurations of the unsemi-stable fibrations

We shall prove Theorem 0.2 in the present section.

Let f: $X \longrightarrow \mathbf{P}^1$ be a (relatively) minimal elliptic surface over $\mathbf{P}^1$ with a distinguished section $\mathcal{O}$. The complete list of possible fibers has been given by Kodaira [4]. It encompasses two infinite families $(I_n, I_n^*, n \geq 0)$ and six exceptional cases $(II, III, IV, II^*, III^*, IV^*)$. And they can be considered as sublattices of the Neron-Severi group of X and as such they have rank $(= r(F))$. If $e(F)$ denotes the Euler number of the fiber as a reduced divisor, we can set up the following table.

Table 3

	I_0	$I_n (n \geq 1)$	$I_n^* (n \geq 0)$	II	III	IV	IV^*	III^*	II^*
e	0	n	$n+6$	2	3	4	8	9	10
r	0	$n-1$	$n+4$	0	1	2	6	7	8

LEMMA 2.1. (cf. [5, Corollary 1.3]) *In all cases* $0 \leq e - r \leq 2$. *Moreover,*

(1) $e - r = 0 \Longleftrightarrow$ *the fibre* F *is smooth, i.e., of type* I_0;
(2) $e - r = 1 \Longleftrightarrow$ *the fibre* F *is semi-stable, i.e., of type* I_n ,$n \geq 1$;
(3) $e - r = 2 \Longleftrightarrow$ *the fibre* F *is unstable.*

In the following discussion, we denote

$$[Q1] := \sum_{n\geq 1} n i_n + \sum_{n\geq 1}(n+6)i_n^* \\ + 6i_0^* + 10ii^* + 9iii^* + 8iv^* + 4iv + 3iii + 2ii.$$

$$[Q2] := \sum_{n\geq 1}(n-1)i_n + \sum_{n\geq 1}(n+4)i_n^* \\ + 4i_0^* + 8ii^* + 7iii^* + 6iv^* + 2iv + iii.$$

$$[Q3] := \sum_{n\geq 1} i_n + 2(\sum_{n\geq 1} i_n^* + i_0^* + ii^* + iii^* + iv^* + iv + iii + ii).$$

$$[Q4] := 6\sum_{n\geq 1}(i_n + i_n^*) + 4(ii + iv^*) + 3(iii + iii^*) + 2(iv + ii^*) - 12.$$

$$[Q5] := i_0^* + iv + iii + ii.$$

It is easy to see $[Q1] = 24$, $[Q2] = \rho(X) - 2$ and $[Q1] - [Q2] = [Q3]$.

LEMMA 2.2. *Let $f\colon X \longrightarrow \mathbf{P}^1$ be an elliptic surface over $\mathbf{P}^1$ with $\rho(X) = a(\leq 20)$. If $\deg J \neq 0$, then we have*

(1) $[Q4] - \deg J = 6(20 - a - [Q5])$.
(2) $0 \leq [Q5] \leq 20 - a$.

Proof. By Theorem 1.12, we have

$$\begin{aligned}
&[Q4] - \deg J \\
&= [Q4] - \sum_{n\geq 1} n(i_n + i_n^*) \\
&= [Q4] - 24 + \sum_{n\geq 1} 6i_n^* + 6i_0^* + 10ii^* + 9iii^* + 8iv^* + 4iv + 3iii + 2ii \\
&= 6\sum_{n\geq 1} i_n + 12(\sum_{n\geq 1} i_n^* + ii^* + iii^* + iv^*) + 6[Q5] - 36 \\
&= 6([Q3] - [Q5] - 6) \\
&= 6(20 - a - [Q5]) \geq 0.
\end{aligned}$$

□

Proof of Theorem 0.2. In this case, $a = 20$, by Lemma 2.2, we get the result. □

LEMMA 2.3. *Assume X is an extremal elliptic $K3$ surface with an unsemi-stable fibration $f : X \longrightarrow \mathbf{P}^1$. Let m be the number of singular fibers of f. Then*
(a) $\sum(e(F) - r(F)) = 6$.
(b) $3 \leq m \leq 5$.

Proof. Since X is $K3$ surface, $\sum e(F) = 24$. Also since X is extremal, $\sum r(F) = \rho - 2 = 18$. This proves (a). (b) follows from Lemma 2.1 and our definition of *unsemi-stable fibration.* □

In the following discussion, we denote F_i (i=1,2,3) as the singular fiber of f which is not of type I_n.

THEOREM 2.4. *Let $f : X \longrightarrow \mathbf{P}^1$ be an extremal elliptic $K3$ unsemi-stable fibration. Then the number m of the singular fibers of f is* 3, 4 *or* 5.
(A) *If $m = 3$, then the possible fiber types of (F_1, F_2, F_3) are listed in the following.*
(II^*, I_1^*, I_1^*), (II^*, II^*, IV), (II^*, IV^*, I_0^*), (III^*, III^*, I_0^*),
(III^*, IV^*, I_1^*), (III^*, I_2^*, I_1^*), (IV^*, IV^*, IV^*), (IV^*, IV^*, I_2^*),
(IV^*, I_2^*, I_2^*), (IV^*, I_3^*, I_1^*),(I_2^*, I_2^*, I_2^*), (I_2^*, I_3^*, I_1^*), (I_4^*, I_1^*, I_1^*).
(B) *If $m = 4$, then the possible fiber types of $(F_1, F_2, I_{n_3}, I_{n_4})$ are listed in the following.*
(B.1) $(I_{n_1}^*, I_{n_2}^*, I_{n_3}, I_{n_4})$ *where* $n_1 \geq n_2 \geq 1$ *and* $\sum_{i=1}^4 n_i = 12$.
(B.2)
(i) $(I_{n_1}^*, II^*, I_{n_3}, I_{n_4})$ *where* $n_1 + n_3 + n_4 = 8$ *and* $n_1 \geq 1$.
(ii) $(I_{n_1}^*, III^*, I_{n_3}, I_{n_4})$ *where* $n_1 + n_3 + n_4 = 9$ *and* $n_1 \geq 1$.
(iii) $(I_{n_1}^*, IV^*, I_{n_3}, I_{n_4})$ *where* $n_1 + n_3 + n_4 = 10$ *and* $n_1 \geq 1$.
(B.3.1)
(i) $(II^*, II^*, I_{n_3}, I_{n_4})$ *where* $n_3 + n_4 = 4$.
(ii) $(II^*, III^*, I_{n_3}, I_{n_4})$ *where* $n_3 + n_4 = 5$.
(iii) $(II^*, IV^*, I_{n_3}, I_{n_4})$ *where* $n_3 + n_4 = 6$.
(B.3.2)
(i) $(III^*, III^*, I_{n_3}, I_{n_4})$ *where* $n_3 + n_4 = 6$.
(ii) $(III^*, IV^*, I_{n_3}, I_{n_4})$ *where* $n_3 + n_4 = 7$.

(B.3.3)
$(IV^*, IV^*, I_{n_3}, I_{n_4})$ where $n_3 + n_4 = 8$.

(C) *If* $m = 5$, *then the possible fiber types of* $(F_1, I_{n_2}, I_{n_3}, I_{n_4}, I_{n_5})$ *are listed in the following.*

(*i*) $(I^*_{n_1}, I_{n_2}, I_{n_3}, I_{n_4}, I_{n_5})$ where $\sum_{i=1}^5 n_i = 18$ and $n_1 \geq 1$.
(*ii*) $(II^*, I_{n_2}, I_{n_3}, I_{n_4}, I_{n_5})$ where $\sum_{i=2}^5 n_i = 14$.
(*iii*) $(III^*, I_{n_2}, I_{n_3}, I_{n_4}, I_{n_5})$ where $\sum_{i=2}^5 n_i = 15$.
(*iv*) $(IV^*, I_{n_2}, I_{n_3}, I_{n_4}, I_{n_5})$ where $\sum_{i=2}^5 n_i = 16$.

Proof. We discuss $\deg J = 0$ and $\deg J \neq 0$ separately.

If $\deg J = 0$, then $m = 3$. By $[Q2] = 18$, we have

$$18 = iii + 2iv + 6iv^* + 7iii^* + 8ii^* + 4i_0^*.$$

Thus we have the following possible fiber types:

$$(II^*, II^*, IV),\ (II^*, IV^*, I_0^*),\ (III^*, III^*, I_0^*),\ (IV^*, IV^*, IV^*).$$

If $\deg J \neq 0$, then by Lemma 2.2, we have $i_0^* + iv + iii + ii = 0$ and

$$24 = \sum_{n\geq 1} ni_n + \sum_{n\geq 1}(n+6)i_n^* + 10ii^* + 9iii^* + 8iv^* := [a].$$

If $m = 3$, then we have

$$24 = \sum_{n\geq 1}(n+6)i_n^* + 10ii^* + 9iii^* + 8iv^*.$$

Thus we have the following possible fiber types:

$$(II^*, I_1^*, I_1^*),\ (III^*, IV^*, I_1^*),\ (III^*, I_2^*, I_1^*),$$

$$(IV^*, IV^*, I_2^*),\ (IV^*, I_2^*, I_2^*),\ (IV^*, I_3^*, I_1^*),$$

$$(I_2^*, I_2^*, I_2^*),\ (I_2^*, I_3^*, I_1^*),\ (I_4^*, I_1^*, I_1^*).$$

Combining the above results, we prove Case(A).

If $m = 4$, then with $[Q3] = 6$, we have

$$2 = \sum_{n \geq 1} i_n^* + ii^* + iii^* + iv^* := [b]$$

and

$$[a] - 6 \times [b] = \sum_{n \geq 1} ni_n + \sum_{n \geq 1} ni_n^* + 4ii^* + 3iii^* + 2iv^* = 12.$$

This proves Case(B).

If $m = 5$, then with $[Q3] = 6$, we have

$$1 = \sum_{n \geq 1} i_n^* + ii^* + iii^* + iv^* := [c]$$

and

$$[a] - 6 \times [c] = \sum_{n \geq 1} ni_n + \sum_{n \geq 1} ni_n^* + 4ii^* + 3iii^* + 2iv^* = 18.$$

This proves Case(C). □

3. The possible Mordell-Weil Groups for Case(A)

We shall prove the following Theorem 3.1 in the present section. For simplicity, we label the fiber types which appeared in Case(A) of Theorem 2.4.

THEOREM 3.1. *The possible Mordell-Weil Groups for Case*(A) *are listed in the following table:*

Table 4

♯	the fibre type	$MW(f)$	♯	the fibre type	$MW(f)$
1	(II^*, I_1^*, I_1^*)	(0)	8	(IV^*, IV^*, I_2^*)	(0)
2	(II^*, II^*, IV)	(0)	9	(IV^*, I_3^*, I_1^*)	(0)
3	(II^*, IV^*, I_0^*)	(0)	10	(I_4^*, I_1^*, I_1^*)	(0), $\mathbf{Z}/2\mathbf{Z}$
4	(III^*, III^*, I_0^*)	(0), $\mathbf{Z}/2\mathbf{Z}$	11	(I_2^*, I_2^*, I_2^*)	(0), $\mathbf{Z}/2\mathbf{Z}$, $\mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$
5	(III^*, IV^*, I_1^*)	(0)	12	(I_2^*, I_3^*, I_1^*)	(0)
6	(III^*, I_2^*, I_1^*)	(0), $\mathbf{Z}/2\mathbf{Z}$	13	(IV^*, I_2^*, I_2^*)	(0)
7	(IV^*, IV^*, IV^*)	(0), $\mathbf{Z}/3\mathbf{Z}$			

We now explain the outline of the proof of Theorem 3.1. Firstly, we deal with types 1,2,3,5,8,9,13 (cf. Lemma 3.2). Then we calculate the possible nontrivial Mordell-Weil Groups of types 4,6,7,10,11 (cf. Lemma 3.4). Finally we deal with type 12 (cf. Lemma 3.5).

LEMMA 3.2. *For type m, where $m = 1,2,3,5,8,9$ or 13, the possible Mordell-Weil Group is* (0).

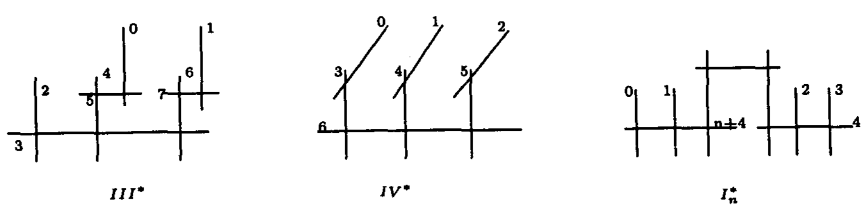
Proof. With Definition 1.9, Theorem 1.10 and Remark 1.11, it is easy to prove Lemma 3.2. For example, $m = 1$, if there is a non zero section, say, $P_1 \in E(K)_{tor}$, where the i-th component of P_1 is indicated in Remark 1.11, then by Theorem 1.10, we have

$$0 = \langle P_1, P_1\rangle = 2\chi(\mathcal{O}_X)+2(P_1\mathcal{O})-\begin{cases} 0, & i=0, \\ 1, & i=1, \\ 1+\frac{1}{4}, & i>1. \end{cases} -\begin{cases} 0, & i=0, \\ 1, & i=1, \\ 1+\frac{1}{4}, & i>1. \end{cases}$$

With $2\chi(\mathcal{O}_X) = 4$ and $(P_1\mathcal{O}) \geq 0$, we get a contradiction. The others can be proved by the same method. □

REMARK 3.3. In the following calculation, we let G_i, H_i and J_i be the i-th component in the corresponding fiber type F_1, F_2, F_3 respectively (cf. Theorem 2.4). The numbering of the singular fiber is defined as following diagrams. Meanwhile, "P_1 pass through the (i, j, k) component" means P_1 only intersect G_i, H_j and G_k in the corresponding fiber type F_1, F_2 and F_3 respectively.

Figure 2



LEMMA 3.4. *The possible nontrivial Mordell-Weil Group of type* 4 (resp. 6,7,10,13) *is* $\mathbf{Z}/2\mathbf{Z}$ (resp. $\mathbf{Z}/2\mathbf{Z}$, $\mathbf{Z}/3\mathbf{Z}$, $\mathbf{Z}/2\mathbf{Z}$, $\mathbf{Z}/2\mathbf{Z}$ or $\mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$).

Proof. We only show how to deal with type 6, the others can be done by the same way.

For m=6, the fiber type is (III^*, I_2^*, I_1^*). If the Mordell-Weil Group is nontrivial, then for a non zero section P_1, by Theorem 1.10, we have

$$0 = \langle P_1, P_1 \rangle = 2\chi(\mathcal{O}_X) + 2(P_1O) - \begin{cases} 0, & i=0, \\ 1, & i=1, \\ 1+\frac{1}{2}, & i>1, (*) \end{cases}$$

$$- \begin{cases} 0, & i=0, \\ 1, & i=1, \\ 1+\frac{1}{2}, & i>1, (*) \end{cases} - \begin{cases} 0, & i=0, \\ 1, & i=1, (*) \\ 1+\frac{1}{2}, & i>1. \end{cases}$$

Thus we may assume that the section P_1 pass through the $(1,2,1)$ component. An easy calculation shows

$$P_1 = \mathcal{O} + 2F + \sum_{i=1}^{7} \alpha_i G_i + \sum_{j=1}^{6} \beta_j H_j + \sum_{k=1}^{5} \gamma_k J_k$$

where

$$\begin{aligned}
(\alpha_i) &= (-\frac{3}{2}, -\frac{3}{2}, -3, -2, -1, -\frac{5}{2}, -2), \\
(\beta_j) &= (-\frac{1}{2}, -\frac{3}{2}, -1, -2, -\frac{3}{2}, -1), \\
(\gamma_k) &= (-1, -\frac{1}{2}, -\frac{1}{2}, -1, -1).
\end{aligned}$$

and there doesn't exist another non-zero section. Thus the possible nontrivial Mordell-Weil Group of type 6 is $\mathbf{Z}/2\mathbf{Z}$. □

LEMMA 3.5. *The possible Mordell-Weil Group of type* 12 *is trivial, i.e.,* (0).

Proof. Assume Lemma 3.5 is false. By the same discussion as above, we may assume that there is a nonzero section P passing through the $(1,2,2)$ component. An easy calculation shows

$$P = \mathcal{O} + 2F + \sum_{i=1}^{6} G_i\theta_i + \sum_{j=1}^{7} \beta_j H_j + \sum_{k=1}^{5} \gamma_k J_k$$

where

$$\begin{aligned}
(\alpha_i) &= (-1, -\frac{1}{2}, -\frac{1}{2}, -1, -1, -1),\\
(\beta_j) &= (-\frac{1}{2}, -\frac{7}{4}, -\frac{5}{4}, -\frac{3}{2}, -2, -\frac{5}{2}, -1),\\
(\gamma_k) &= (-\frac{1}{2}, -\frac{5}{4}, -\frac{3}{4}, -\frac{3}{2}, -1).
\end{aligned}$$

Thus the possible nontrivial Mordell-Weil Group of type 12 is $\mathbf{Z}/4\mathbf{Z}$. That is to say, this group has at least two nonzero distinct sections, say, P_1, P_2. On the other hand, with Theorem 1.10, we have

$$0 = \langle P_1, P_2 \rangle = 2 - (P_1P_2) - 1 - \begin{cases} 1 + \frac{3}{4}, & i = j > 1, \\ \frac{5}{4}, & j > i > 1. \end{cases}$$

$$- \begin{cases} 1 + \frac{1}{4}, & i = j > 1, \\ \frac{3}{4}, & j > i > 1. \end{cases} < 0.$$

Thus we get a contradiction and prove Lemma 3.5. □

Combining Lemma 3.2, 3.4 and 3.5, we prove Theorem 3.1.

4. The complete determination of the Mordell-Weil Groups for Case (A)

We shall prove Theorem 0.4 in the present section.

LEMMA 4.1. (cf. [1, Lemma 3.1]) *Let S be an even symmetric lattice of rank 20 and signature $(1,19)$ and T a positive definite even* *symmetric lattice of rank 2. Assume that φ: $T^\vee/T \longrightarrow S^\vee/S$ is an isomorphism which induces the following equality involving $\mathbf{Q}/2\mathbf{Z}$-valued discriminant (quadratic) forms:*

$$q_S = -q_T.$$

Let X be the unique $K3$ surface (up to isomorphisms) with the transcendental lattice $T_X = T$. Then the Picard lattice $PicX$ is isometric to S.

LEMMA 4.2. *Let f: $X \longrightarrow \mathbf{P}^1$ be of type m where $m = 4$, 6,7,10,11, 12 and 13. Then*
(1) $MW(f_m) \neq (0)$, and further
(2) $MW(f_{11}) \neq \mathbf{Z}/2\mathbf{Z}$.

Proof. Suppose the contrary that f: $X \longrightarrow \mathbf{P}^1$ is of the corresponding type with $MW(f) = (0)$. Let (b_{ij}) be the intersection metrix of the transcendental lattice $T = T_X$, then $\det(b_{ij}) = |\det(PicX)|$ (cf.[2]). Modulo congruent action of $SL(2,\mathbf{Z})$, we may assume that $-b_{11} < 2|b_{12}| \leq b_{11} \leq b_{22}$, and that $b_{12} \geq 0$ when $b_{11} = b_{22}$.

Embed T, as a sublattice, into $T^\vee = Hom_{\mathbf{Z}}(T,\mathbf{Z})$. Then $T^\vee/T \cong (PicX)^\vee/(PicX)$. On the other hand, $T^\vee$ has a $\mathbf{Z}$-basis $(e_1, e_2)(b_{ij})^{-1} = (g_1, g_2)$, where e_1,e_2 form a canonical basis of T. Then comparing the order of (g_i) ($i = 1$,2) with $T^\vee/T$, we will get a contraction. For simplicity, we only show the case for $m = 4$, the others can be done by the same way.

Table 5

m	$T^\vee/T$	the possible $T = (b_{ij})$	order of g_1, g_2
4	$\mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z} \oplus$	diag $[2,8]$	2,8
	$(\mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z})$	diag $[4,4]$	4,4

□

REMARK 4.3. From Theorem 3.1 and Lemma 4.2, we know that there does not exist type 12 or 13. The existence of type $m = 1$,3,4,7 can be found in [11] (page 121, 131 and 132). Thus the Mordell-Weil Group of type 1 (resp. 3,4,7) is (0) (resp. (0), $\mathbf{Z}/2\mathbf{Z}$, $\mathbf{Z}/3\mathbf{Z}$).

LEMMA 4.4. *Consider the pairs below:*

$$(m, G_m) = (2, (0)), (5, (0)), (6, \mathbf{Z}/2\mathbf{Z}), (8, (0)), (9, (0)), \\ (10, \mathbf{Z}/2\mathbf{Z}), (11, \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}).$$

For each of these seven pairs (m, G_m), there is a Jacobian elliptic $K3$ surface $f_m\colon X_m \longrightarrow \mathbf{P}^1$ of type m such that $(m, MW(f_m)) = (m, G_m)$.

Proof. Let T_m, $m = 2, 5, 6, 8, 9, 10$ and 11 be the positive define symmetric lattice of rank 2 with the following intersection form, respectively:

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 12 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}, \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}, \begin{pmatrix} 4 & 0 \\ 0 & 12 \end{pmatrix}, \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

For $m = 2,5,8,9$, let S_m be the lattice of rank 20 and signature $(1, 19)$ with the following intersection form, respectively

$$U \oplus E_8 \oplus E_8 \oplus A_2, U \oplus E_7 \oplus E_6 \oplus D_5, U \oplus E_6 \oplus E_6 \oplus D_6, U \oplus E_6 \oplus D_7 \oplus D_5.$$

We now show how to define S_6. S_{10} and S_{11} can be defined by the similar way. □

Let Γ_6 be the lattice $U \oplus E_7 \oplus D_6 \oplus D_5$, with $G_i (1 \le i \le 7)$, $H_j (1 \le j \le 6)$, $J_k (1 \le k \le 5)$as the canonical basis of $E_7 \oplus D_6 \oplus D_5$ which are indicated in *Section* 3, and $\mathcal{O}$, F as a basis of U such that $\mathcal{O}^2 = -2$, $F^2 = 0$, $\mathcal{O}F = 1$.

We extend Γ_6 to an index-2 integral over lattice $S_6 = \Gamma_6 + \mathbf{Z}s_6$, where

$$\begin{aligned} s_6 = \mathcal{O} + 2F &+ [-\frac{3}{2}G_1 - \frac{3}{2}G_2 - 3G_3 - 2G_4 - G_5 - \frac{5}{2}G_6 - 2G_7] \\ &+ [-\frac{1}{2}H_1 - \frac{3}{2}H_2 - H_3 - 2H_4 - \frac{3}{2}H_5 - H_6] \\ &+ [-J_1 - \frac{1}{2}J_2 - \frac{1}{2}J_3 - J_4 - J_5]. \end{aligned}$$

It is easy to see that intersection form on Γ_6 can be extend to an integral even symmetric lattice of signature $(1, 19)$. Indeed, setting $s = s_6$, we have

$$s^2 = -2, s \cdot F = s \cdot G_1 = s \cdot H_2 = s \cdot J_1 = 1,$$

$$s \cdot G_i = s \cdot H_j = s \cdot J_k = 0 \ (\forall i \neq 1, j \neq 2, k \neq 1).$$

Moreover, $|\det(S_6)| = |\det(\Gamma_6)|/2^2 = 8$.

Note that $\Gamma_6^\vee = \mathrm{Hom}_{\mathbf{Z}}(\Gamma_6, \mathbf{Z})$ contains Γ_6 as a sublattice with $E_7 \oplus D_6 \oplus D_5$ as the factor group, and is generated by the following, modulo Γ_6:

$$\begin{aligned} h_1 &= (1/2)(G_1 + G_2 + G_6), \\ h_2 &= (1/2)(H_1 + H_2 + H_5), \\ h_3 &= (1/2)(H_1 + H_3 + H_5), \\ h_4 &= (1/4)(2J_1 + J_2 - J_3 + 2J_4). \end{aligned}$$

Since $(S_6)^\vee$ is an (index-2) sublattice of $(\Gamma_6^\vee)$, an element x is in $(S_6)^\vee$ if and only if $x = \sum_{i=1}^4 a_i h_i (\bmod \Gamma_6)$ such that x is integral on S_6, i.e.,

$$x \cdot s = (a_1 + a_2 + a_4)/2$$

is an integer. Hence $(S_6)^\vee$ is generated by the following module Γ_6:

$$h_1 + h_2, h_1 + h_4, h_2 + h_4, h_3.$$

Noting that $2h_1$,$2h_2 \in S_6$ and $h_1 + h_2 + 2h_4$ is equal to s $(\bmod \Gamma_6)$ and hence contained in S_6, we find that $(S_6)^\vee$ is generated by the following, modulo Γ_6:

$$\epsilon_1 = h_3, \ \epsilon_2 = h_1 + h_4.$$

Now the fact that $|(S_6)^\vee/S_6| = 8$ and that $2\epsilon_1$,$4\epsilon_2 \in S_6$ imply that $(S_6)^\vee/S_6$ is a direct sum of its cyclic subgroups which are of order 2,4, and generated by ϵ_1, ϵ_2, modulo S_6.

We note that the negative of the discriminant form

$$-q_{(S_6)} = (-(\epsilon_1)^2) \oplus (-(\epsilon_2)^2) = (3/2) \oplus (3/4).$$

Similarly, we can get

$$\begin{aligned} -q_{(S_{10})} &= (-(\epsilon_1)^2) \oplus (-(\epsilon_2)^2) = (5/4) \oplus (5/4), \\ -q_{(S_{11})} &= (-(\epsilon_1)^2) \oplus (-(\epsilon_2)^2) = (1/2) \oplus (1/2), \end{aligned}$$

for suitable generators ϵ_1, ϵ_2 in the corresponding cases.

Claim 4.5. The pair (S_m, T_m) satisfies the conditions of Lemma 4.1 and hence if we let X_m be the unique $K3$ surface with $T_{X_m} = T_m$ then Pic$X_m = S_m$ (both two equalities here are modulo isometries).

Proof of the claim. We need to show that $q_{T_m} = -q_{S_m}$.

(1) $m \neq 2$. $(S_m)^\vee/S_m$ is generated by two elements ϵ_i (i=1,2) (ϵ_i is a simple sum of the natural generators of $(S_m)^\vee/S_m$) such that for every a,$b \in \mathbf{Z}$ one has $-q_{S_m}(a\epsilon_1 + b\epsilon_2) = -a^2(\epsilon_1)^2 - b^2(\epsilon_2)^2$. For all six m where $m \neq 2$, ϵ_i can be chosen such that $(-\epsilon_1^2, -\epsilon_2^2)$ is respectively given as follows:

$$(3/2, 7/12),\ (3/2, 3/4),\ (5/6, 5/6),\ (7/12, 7/4),\ (5/4, 5/4),\ (1/2, 1/2).$$

On the other hand, $(T_m)^\vee$ $(m \neq 2)$ is generated by $(g_1, g_2) = (e_1, e_2)T_m^{-1}$, where e_1, e_2 form a canonical basis of T_m which gives rise to the intersection matrix of T_m shown before this claim. Now the claim follows from the existence of the following isomorphism, which induces $q_{T_m} = -q_{S_m}$:

$$\phi\ :\ (T_m)^\vee/T_m \longrightarrow (S_m)^\vee/S_m, \quad (g_1, g_2) = (\epsilon_1, \epsilon_2)B_m.$$

Here B_m is respectively given as:

$$\begin{pmatrix} 1 & 1 \\ 6 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 5 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

for $m =$ 5,6,8,9,10 and 11 respectively.

(2) $m = 2$. In this case, we know $(T_2)^\vee$ is generated by $(g_1, g_2) = (e_1, e_2)T_2^{-1}$, where e_1, e_2 form a canonical basis of T_2 which gives rise to the intersection matrix of T_2 shown before this claim. In fact we have

$$g_1 \equiv g_2 \ (\bmod T_2).$$

Thus $(T_2)^\vee/T_2$ is generated by one element g_1, and the natural isomorphism

$$\phi : g_1 \longrightarrow \epsilon_1$$

will give $q_{T_2} = -q_{S_2}$, where ϵ_1 is a canonical $\mathbf{Z}$-basis of A_2 and such that for every $a \in \mathbf{Z}$, we have $-q_{S_2}(a\epsilon_1) = -a^2(\epsilon_1)^2$. $\square$

Write S_m (resp. Γ_m) as $U \oplus \mathbf{B}(m)$ with $\mathbf{B}(m)$ as in the definitions of them. Let $\mathcal{O}$, F be a $\mathbf{Z}$-basis of U for all m. By [8, p. 573, Theorem 1], after an (isometric) action of reflections on $S_m = PicX_m$, we may assume at the beginning that F is a fibre of elliptic fibration f_m: $X_m \longrightarrow \mathbf{P}^1$. Since $\mathcal{O}^2 = -2$, Riemann-Roch Theorem implies that $\mathcal{O}$ is an effective divisor for $\mathcal{O} \cdot F > 0$. Moreover, $\mathcal{O} \cdot F = 1$ implies that $\mathcal{O} = \mathcal{O}_1 + F'$ where $\mathcal{O}_1$ is a cross-section of f_m and F' is an effective divisor contained in fibres. So f_m is a Jacobian elliptic fibration and we can choose $\mathcal{O}_1$ as the zero element of $MW(f_m)$.

Let Λ_m be the lattice generated by all fiber components of f_m. Clearly, $\Lambda_m = \mathbf{Z}F \oplus \Delta$, $\Delta = \Delta(1) \oplus \cdots \oplus \Delta(r)$ (depending on m), where each $\Delta(i)$ is a negative definite even lattice of Dynkin type A_p, D_q, or E_r, contained in a single reducible fibre F_i of f_m and spanned by smooth components of F_i disjoint from $\mathcal{O}_1$.

CLAIM 4.6. We have

(1) $Span_{\mathbf{Z}}\{x \in S_m | x \cdot F = 0, x^2 = -2\} = \Lambda_m = \mathbf{Z}F \oplus \mathbf{B}(m)$; in particular, there are lattice-isometries: $\Delta \cong \mathbf{B}(m)$.

(2) $MW(f_m) = (0)$ for $m = 2,5,8$ and 9.

(3) $MW(f_m) = \mathbf{Z}/2\mathbf{Z}$ for $m = 6,10$.

(4) $MW(f_m) = \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$ for $m = 11$.

Proof of the claim. The first equality in (1) is from Kodaira's classification of elliptic fibres and the Riemann-Roch Theorem as used prior to this claim to deduce $\mathcal{O} \geq 0$. The second equality is clear for that cases of $m = 2, 5, 8$ and 9, because then $\mathrm{Pic}X_m = S_m = (\mathbf{Z}\mathcal{O} + \mathbf{Z}F) \oplus \mathbf{B}(m)$.

We now show the second equality for $m = 6,10$ and 11 using Lemma 1.1. Clearly, $\mathbf{Z}F \oplus \mathbf{B}(m)$ is contained in the first term of (1) and hence in Λ_m. We also have

$$19 = \mathrm{rank}S_m - 1 \geq \mathrm{rank}\Lambda_m = 1 + \mathrm{rank}\Delta = 1 + \mathrm{rank}\mathbf{B}(m) = 19.$$

Hence $\Delta = \Delta(1) \oplus \cdots \oplus \Delta(r) \cong \Lambda_m/\mathbf{Z}F$ contains a finite-index sublattice $(\mathbf{Z}F \oplus \mathbf{B}(m))/\mathbf{Z}F \cong \mathbf{B}(m)$.

Suppose the contrary that the second equality in (1) is not true. Then $\mathbf{B}(m)$ is an index-n ($n > 1$) sublattice of Δ.

For $m = 6$, by Lemma 1.1, we know $\Delta = E_7 \oplus D_{11}$. On the other hand, if we denote $s'_6 = s_6 - \mathcal{O} - 2F$, then we have

$$\Lambda_6 \subset Span_{\mathbf{Z}}\{x \in S_m | x \cdot F = 0\} = Span_{\mathbf{Z}}\{F, G_i, H_j, J_k, s'_6\}$$

where $1 \leq i \leq 7$, $1 \leq j \leq 6$ and $1 \leq k \leq 5$.

Thus we get (mod $\mathbf{Z}F$)

$$E_7 \oplus D_{11} \subset Span_{\mathbf{Z}}\{G_i, H_j, J_k, s'_6\}.$$

By a simple calculation, we find that for any element $e \in D_{11} - (D_6 \oplus D_5)$, $e \notin Span_{\mathbf{Z}}\{G_i, H_j, J_k, s'_6\}$. Thus we get a contradiction.

Similarly, we can prove the second equality in (1) for $m = 10$, 11. The assertion (2), (3) and (4) follow from the fact in [10, Theorem 1.3], that $MW(f_m)$ is isomorphic to the factor group of PicX_m modulo $(\mathbf{Z}\mathcal{O}_1 + \mathbf{Z}F) \oplus \Delta$, where the latter is equal to

$$(\mathbf{Z}\mathcal{O} + \mathbf{Z}F) + \Delta = (\mathbf{Z}\mathcal{O} + \mathbf{Z}F) \oplus \mathbf{B}(m) = U \oplus \mathbf{B}(m).$$

This proves the claim. And this completes the lattice-theoretical proof of Lemma 4.4. □

Combining Remark 4.3 and Lemma 4.4, we prove Theorem 0.4.

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