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LITTLEWOOD-PALEY TYPE ESTIMATES FOR BESOV SPACES ON A CUBE BY WAVELET COEFFICIENTS

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ABSTRACT. This paper deals with Littlewood-Paley type estimates of the Besov spaces $B^{\alpha}_{p,q}$ on the d-dimensional unit cube for $0 < p, q < \infty$ by two certain classes. These classes are including biorthogonal wavelet systems or dual multiscale systems but not necessarily obtained as the dilates or translates of certain fixed functions. The main assumptions are local supports of both classes, sufficient smoothness for one class, and sufficiently many vanishing moments for the other class. With these estimates, we characterize the Besov spaces by coefficient norms of decompositions with respect to biorthogonal wavelet systems on the cube.

1. Introduction

For practical applications, many researchers [1, 4, 6, 20, 22] have constructed wavelet bases on a closed interval or cube Ω with the standard wavelet theory on the real line $\mathbb{R}$. Their constructions provide a wavelet decomposition of the space $L_2(\Omega)$. Some of them [6, 22] concerned with wavelet bases characterizing Hölder-Zygmund spaces $C^{\alpha}(\Omega)$. We observe that their characterizations can be described in terms of Littlewood-Paley expressions with wavelet coefficients.

Littlewood-Paley type characterization [16, 17] for a function space by wavelets is useful in not only theory but also many applications. Once such a characterization is established, the smoothness and local property of a given function can be described by the size or the decay of wavelet coefficients. In addition, the characterization ensures that wavelets form

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an unconditional basis for many function spaces. In applications to numerical solutions to differential equations, specifically, elliptic differential equations, the characterization of Sobolev spaces with wavelet coefficients offers a good preconditioner (that produces uniformly bounded condition numbers) and an adaptive algorithm (see, for example, [11, 7]). In particular, the characterization based on a wavelet decomposition implicitly provides an optimal algorithm of compression [8, 9] for a certain large class of spaces such as Besov spaces.

Besov spaces include several function spaces such as Sobolev spaces H^α and Hölder-Zygmund spaces [23, 17] as well as have many applications to approximation theory [10] and to the study of partial differential equations [2, 12]. The definition of these spaces will be briefly reviewed in Section 2.

R. DeVore et al. [9, 13] have provided the coefficient characterization of Besov spaces on d-dimensional unit cube Ω with B-splines and wavelets. Their wavelet coefficients are however implicitly defined via a certain extension method such as quasi-interpolants (see, for example, [13, 3]), so that it is very difficult to obtain the dual wavelet basis corresponding to the coefficients, especially near the boundaries of Ω. We have recently found the report by A. Cohen et al. [5] where biorthogonal wavelet systems for rather general domain are concerned and some approximation properties of the multiresolution analysis on the domain are considered for the characterization of Besov spaces. The characterizations of this present paper are independently studied (see [19]) and also are vaild for general classes of functions with local supports but without being necessarily obtained as the dilates or translates of certain fixed functions. Moreover, our characterizations can be trivially extended for general domains such as a bounded, simply connected domain satisfying the uniform cone condition [15] (examine the estimates of this paper).

In Section 3, we establish Littlewood-Paley type estimates for Besov spaces $B^\alpha_{p,q}$, $0 < \alpha, p, q < \infty$, on the cube Ω by coefficients norms of decompositions with respect to two certain classes. These classes are including biorthogonal wavelet systems or dual multiscale systems (possibly, multiwavelet systems) but not necessarily obtained as the dilates or translates of fixed functions. The main assumptions are local supports (with correct scaling in certain norms) of both classes, sufficient smoothness for one class, and sufficiently many vanishing moments for the other class. When $p < 1$, we establish the estimates for the family

$B^{\alpha}_{q,q}$ of Besov spaces with $q > (\alpha/d+1)^{-1}$ which has specific applications to nonlinear approximation theory (see [14, 9, 12]). Using our estimates, in the final section, we characterize Besov spaces with wavelet coefficients of the decomposition by biorthogonal wavelet systems on Ω.

2. Preliminaries

This section briefly reviews the main idea of wavelet bases on $[0,1]$ from [1, 4, 6] as well as the definitions of Besov spaces and sequence spaces from [23, 17]. It is assumed that the standard wavelet theory and multiresolution analysis of $L_2(\mathbb{R})$ are well understood.

One of the main different features of multiresolution analysis on the interval from those on $\mathbb{R}$ is caused by that the space $L_2([0,1])$ is not translation and dilation invariant. One [4, 6] starts from an initial closed subspace $\mathcal{V}_{k_0}$ of $L_2([0,1])$ at a certain level $k_0 \geq 0$ and investigates closed subspaces $\mathcal{V}_k$ only for $k \geq k_0$ such that

$$\mathcal{V}_k \subset \mathcal{V}_{k+1} \quad \text{and} \quad \overline{\bigcup_{k \geq k_0} \mathcal{V}_k} = L_2([0,1]) \tag{2.1}$$

where the closure is in $L_2([0,1])$. The closed space $\mathcal{V}_k$ can be obtained by a finite set of dyadic translations $\phi_{k,j} := 2^{k/2}\phi(2^k \cdot -j)$ of a standard scaling function ϕ in the interior and certain edge functions $\phi^{\mathrm{e}}_{k,\nu}$ adapted from a set $\{\phi_{k,j}\}$ near the edges. Here we assume that ϕ has compact support and its integer translates are orthogonal.

The basic idea of wavelet bases on $[0,1]$ is to construct good edge functions $\{\phi^{\mathrm{e}}_{k,\nu}\}_\nu$ near the edges so that

$$\text{(2.2)} \quad \begin{array}{ll} \text{(i)} & \phi^{\mathrm{e}}_{k,\nu} \in W^s_\infty([0,1]), \text{ whenever } \phi \in W^s_\infty(\mathbb{R}); \\ \text{(ii)} & \mathcal{P}_r([0,1]) \subset \mathcal{V}_k, \text{ whenever } \phi \text{ locally reproduces } \mathcal{P}_r; \\ \text{(iii)} & |\operatorname{supp} \phi^{\mathrm{e}}_{k,\nu}| \text{ is comparable with } 2^{-k}; \\ \text{(iv)} & \text{the number of edge functions } \phi^{\mathrm{e}}_{k,\nu} \text{ is minimal} \end{array}$$

where $\mathcal{P}_r$ is the collection of all polynomials of total degree less than r and W^s_∞ is the Sobolev space in L_∞. For a comprehensive study of these bases, we refer to the papers [1, 4, 5, 6, 20]. Inspired from the conditions (2.2)(i)–(iii), we shall in Section 3 introduce two general classes. Once such a basis for $\mathcal{V}_k$ is determined, $L_2([0,1])$ is decomposed as

$$L_2([0,1]) = \mathcal{V}_{k_0} \oplus \left(\bigoplus_{k \geq k_0} \mathcal{W}_k \right), \quad \mathcal{V}_{k+1} = \mathcal{V}_k \oplus \mathcal{W}_k, \qquad k \geq k_0.$$

The complement space $\mathcal{W}_k$ is also generated by interior wavelets $\psi_{k,j}$ and certain edge functions $\psi^{\mathrm{e}}_{k,\nu}$ adapted from $\psi_{k,j}$ near the edges. With a certain index set Λ_k (depending on the construction of edge functions) at the level k, a wavelet decomposition of $L_2([0,1])$ has the form

$$f = \sum_{j \in \Lambda_{k_0}} \langle f, \widetilde{\Phi}_{k_0,j} \rangle \, \Phi_{k_0,j} + \sum_{k \geq k_0} \sum_{j \in \Lambda_k} \langle f, \widetilde{\Psi}_{k,j} \rangle \, \Psi_{k,j} \tag{2.3}$$

where $\Phi_{k,j}$ is $\phi_{k,j}$ or $\phi^{\mathrm{e}}_{k,\nu}$, and $\Psi_{k,j}$ is $\psi_{k,j}$ or $\psi^{\mathrm{e}}_{k,\nu}$ for $j, \nu \in \Lambda_k$. In the case of biorthogonal wavelet systems, $\widetilde{\Phi}_{k_0,j}$ and $\widetilde{\Psi}_{k,j}$ will be the duals of $\Phi_{k_0,j}$ and $\Psi_{k,j}$, respectively. For our general setting, we shall focus on the biorthogonal wavelet decomposition of the form (2.3).

To review the Besov spaces, we recall the modulus of smoothness of order r of $f \in L_p(\Omega)$ defined by

$$\omega_r(f,t)_p := \omega_r(f,t,\Omega)_p := \sup_{|h|<t} \|\Delta^r_h(f,\cdot)\|_{L_p(\Omega(rh))}, \qquad t > 0 \tag{2.4}$$

on $\Omega(rh) := \{x \,|\, [x, x+rh] \subset \Omega\}$. Here

$$\Delta^k_h(f,x) = \Delta^{k-1}_h(f,x+h) - \Delta^{k-1}_h(f,x), \quad \Delta^0_h(f,x) := f(x)$$

is the kth difference of f in the direction $h \in \mathbb{R}^d$.

Definition 2.1. Let $0 < \alpha < \infty$, $0 < p, q \leq \infty$, and r be a positive integer with $\alpha < r$. The Besov space $B^\alpha_{p,q}(\Omega)$ is defined as the set of all functions $f \in L_p(\Omega)$ for which

$$|f|_{B^\alpha_{p,q}(\Omega)} := \left(\sum_{\nu \geq 0} \left[2^{\nu\alpha} \omega_r(f, 2^{-\nu})_p \right]^q \right)^{1/q} \tag{2.5}$$

is finite. When $q = \infty$, the sum above is replaced by sup. A (quasi-)norm for $B^\alpha_{p,q}(\Omega)$ is defined by $\|f\|_{B^\alpha_{p,q}(\Omega)} := \|f\|_{L_p(\Omega)} + |f|_{B^\alpha_{p,q}(\Omega)}$.

This (quasi-)norm has equivalent integral versions (see [23, 17]). It is well known that different values of $r > \alpha$ result in equivalent (quasi-) norms.

Remark. There is another version of Besov spaces defined as a collection of distributions (see [23, 17]). We notice from [23] that the spaces with that version are identical with $B^\alpha_{p,q}$ as long as $\alpha > d(1/\min(1,p) - 1)$.

We next review from [13, 17] sequence spaces. For given $0 < \alpha < \infty$, $0 < q \le \infty$ and a sequence $(c_k)_{k\in\mathbb{Z}}$ of real numbers, we define a weighted (quasi-)norm

$$\big\|(c_k)\big\|_{l^\alpha_q(\Lambda)} := \Big(\sum_{k\in\Lambda}\big[2^{k\alpha}|c_k|\big]^q\Big)^{1/q} \tag{2.6}$$

with the usual change to sup when $q = \infty$. Here Λ is a subset of $\mathbb{Z}$. Let us introduce a useful inequality for the weighted (quasi-)norm.

LEMMA 2.2 (The discrete Hardy inequality). *Let (c_k) and (d_k), $k \in \mathbb{Z}$ be two given sequences. Also, let $0 < \alpha, \mu, \lambda < \infty$ and $0 < q \le \infty$. For any $k \in \mathbb{Z}$, if either*

$$\begin{aligned} &\text{(i)} \quad |d_k| \le C2^{-k\lambda}\big\|(c_j)\big\|_{l^\lambda_\mu(\{j\le k\})} \quad \textit{with } 0<\alpha<\lambda \quad \textit{or} \\ &\text{(ii)} \quad |d_k| \le C\big\|(c_j)\big\|_{l_\mu(\{j>k\})} \quad \textit{with } \alpha > 0, \end{aligned} \tag{2.7}$$

then

$$\big\|(d_k)\big\|_{l^\alpha_q} \le C\big\|(c_k)\big\|_{l^\alpha_q} \tag{2.8}$$

where the constant C in (2.8) depends only on μ and $1/(\lambda-\alpha)$ in case (i) and $1/\alpha$ in case (ii). Note that $\mu \le q$ as long as $\alpha < \lambda$ in (i) and $\alpha > 0$ in (ii).

An associated sequence space with $B^\alpha_{p,q}(\mathbb{R}^d)$ is the space $\dot{b}^\alpha_{p,q}(\mathbb{R}^d)$ of sequences $(c_{k,j})$ with two indices $k \in \mathbb{Z}$, $j \in \mathbb{Z}^d$.

DEFINITION 2.3. Let $\alpha \in \mathbb{R}$ and $0 < p, q \le \infty$. Then the space $\dot{b}^\alpha_{p,q}(\mathbb{R}^d)$ consists of all sequences $c = (c_{k,j})$ such that

$$\|c\|_{\dot{b}^\alpha_{p,q}(\mathbb{R}^d)} = \left\|\left(2^{kd(1/2-1/p)}\Big(\sum_{j\in\mathbb{Z}^d}|c_{k,j}|^p\Big)^{1/p}\right)\right\|_{l^\alpha_q(\mathbb{Z})} \tag{2.9}$$

is finite.

Under the condition $\alpha > d(1/\min(p,1) - 1)$, if $f = \sum_{k,j} c_{k,j} m_{k,j}$ with a family $\{m_{k,j}\}$ of certain smooth molecules (see [17]), then one can have

$$|f|_{B^\alpha_{p,q}(\mathbb{R}^d)} \le C\big\|(c_{k,j})\big\|_{\dot{b}^\alpha_{p,q}(\mathbb{R}^d)} \tag{2.10}$$

with C independent of f.

For fixed α, q, and p, let $J = d/\min(1,p,q)$ and $N = \max(\lfloor J-d-\alpha\rfloor, -1)$ where $\lfloor x\rfloor$ is the largest integer less than or equal to x. We consider a family $\{\varphi_{k,j}\}$of functions satisfying the following (see [16]): for some ρ and M with $J-\alpha-\lfloor J-\alpha\rfloor < \rho \le 1$ and $M > J$,

(2.11)

$$\text{(i)} \quad \int_{\mathbb{R}^d} x^\gamma \varphi_{k,j}(x)dx = 0 \qquad \text{if } |\gamma| \le \lfloor\alpha\rfloor,$$

$$\text{(ii)} \quad |\varphi_{k,j}(x)| \le 2^{kd/2}(1+2^k|x-x_{k,j}|)^{-\max(M,M+d+\alpha-J)},$$

$$\text{(iii)} \quad |D^\gamma\varphi_{k,j}(x)| \le 2^{k(d/2+|\gamma|)}(1+2^k|x-x_{k,j}|)^{-M} \quad \text{if } |\gamma| \le N,$$

$$\text{(iv)} \quad |D^\gamma\varphi_{k,j}(x) - D^\gamma\varphi_{k,j}(y)| \le 2^{k(d/2+|\gamma|+\rho)}|x-y|^\rho \sup_{|z|\le|x-y|}(1+2^k|x-z-x_{k,j}|)^{-M} \quad \text{if } |\gamma| = N.$$

Here $x_{k,j}$ can be any point in the support of $\varphi_{k,j}$. By convention, (2.11)(i) is void if $\alpha < 0$; similarly, (2.11)(iii) and (iv) are void if $N < 0$. The following is a special case of the inverse estimate of (2.10).

LEMMA 2.4. *Suppose $\alpha > 0$, $0 < p < \infty$ and $\alpha > d(1/\min(1,p)-1)$. If $f \in B^\alpha_{p,p}(\mathbb{R}^d)$ and $\{\varphi_{k,j}\}$ is a family of functions satisfying* (2.11)(i)–(iv), *then*

$$\|(\langle f, \varphi_{k,j}\rangle)\|_{\dot{b}^\alpha_{p,p}(\mathbb{R}^d)} \le C|f|_{B^\alpha_{p,p}(\mathbb{R}^d)}, \tag{2.12}$$

where C is independent of f.

Proof. For a proof, we refer to Section 3 of [16]. □

3. Littlewood-Paley Type Estimates

Inspired from the usual properties of wavelets, we introduce two classes $\mathbf{\Psi}_s$, $\widetilde{\mathbf{\Psi}}_r$ that are characterized by smoothness conditions and vanishing moment conditions, respectively. For given $k \in \mathbb{N}$, let Λ_k be a multi-index subset of $\mathbb{Z}^d \times \{1,\cdots,2^d-1\}$ such that the number of elements of Λ_k is comparable to 2^{kd}; that is, $\#(\Lambda_k) \approx 2^{kd}$. We typically think of Λ_k including all multi-integers $j = (j_d, j_e)$ with $j_d \in \mathbb{Z}^d$ such that $2^{-k}j_d + 2^{-k}\Omega \subseteq \Omega$, plus some other indices near the boundary of Ω. The second component $j_e \in \{1,\cdots 2^d-1\}$ is the secondary index for d-dimensional wavelets. Let us now consider compactly supported functions, $\varphi_{k,j} \in L_\infty(\Omega)$, indexed by the set $\Lambda := \{\Lambda_k\}_{k\in\mathbb{N}}$ such that

$$(3.1)\qquad \begin{aligned} &\text{(i)}\quad \text{the support of each } \varphi_{k,j} \text{ is contained in a union of at most} \\ &\qquad C_1 \text{ dyadic cubes in } \Omega \text{ with side length } 2^{-k}, \\ &\text{(ii)}\quad \text{for each } x \in \Omega,\ \ \varphi_{k,j}(x) \neq 0 \ \text{ for at most } C_2 \text{ of } j \in \Lambda_k \end{aligned}$$

where the constants C_1, C_2 are independent of k, j, and x. (3.1) will be a basic condition to establish our Littlewood-Paley type estimates.

For a fixed positive integer s, let $\boldsymbol{\Psi}_s$ be a class of functions $\psi_{k,j}$, $k \in \mathbb{N}$, $j \in \Lambda_k$, satisfying (3.1) and

$$(3.2)\qquad |\psi_{k,j}|_{W^\alpha_\infty(\Omega)} \leq C 2^{k\alpha} 2^{kd/2}, \qquad 0 \leq \alpha \leq s,$$

where α is not necessarily an integer. Also, for a fixed positive integer r, let $\widetilde{\boldsymbol{\Psi}}_r$ be a class of functions $\widetilde{\psi}_{k,j}$, $k \in \mathbb{N}$, $j \in \Lambda_k$ satisfying (3.1) and

$$(3.3)\qquad \begin{aligned} &\text{(i)}\quad \|\widetilde{\psi}_{k,j}\|_{L_\infty(\Omega)} \leq C 2^{kd/2}, \\ &\text{(ii)}\quad \int_\Omega x^\gamma\, \widetilde{\psi}_{k,j}\, dx = 0, \qquad |\gamma| < r. \end{aligned}$$

Here the constant C is independent of k and j. The conditions (3.2) and (3.3) are associated with the smoothness and the vanishing moment conditions of wavelets, respectively. Note that the (biorthogonal) wavelet systems on Ω of [1, 4, 6, 20, 22] are special cases of the classes $\boldsymbol{\Psi}_s$, $\widetilde{\boldsymbol{\Psi}}_r$ with suitable s and r.

We next introduce a sequence space on Ω. For a given sequence $c = (c_{k,j})$ indexed by the family $\Lambda = \{\Lambda_k\}_{k\in\mathbb{N}}$, we define

$$(3.4)\qquad \|c\|_{b^\alpha_{p,q}(\Omega)} = \left\| \left(2^{kd(1/2-1/p)} \Big(\sum_{j\in\Lambda_k} |c_{k,j}|^p \Big)^{1/p} \right) \right\|_{l^\alpha_q(k\geq 0)}.$$

This is an analog of the quasi-norm $\|\cdot\|_{\dot{b}^\alpha_{p,q}(\mathbb{R}^d)}$. Indeed, once an index family Λ is fixed, the set $b^\alpha_{p,q}(\Omega)$ of all sequences $c = (c_{k,j})$, $k \in \mathbb{N}$, $j \in \Lambda_k$ such that (3.4) is finite can be a sequence space with the quasi-norm $\|\cdot\|_{b^\alpha_{p,q}(\Omega)}$.

The following theorems establish our Littlewood-Paley type estimates for the Besov spaces $B^\alpha_{p,q}(\Omega)$ by the sequence quasi-norm $\|\cdot\|_{b^\alpha_{p,q}(\Omega)}$ with $\boldsymbol{\Psi}_s$ and $\widetilde{\boldsymbol{\Psi}}_r$. Let us fix the positive integers s and r.

THEOREM 3.1. *Let $0 < \alpha < s$, $1 \le p \le \infty$, and $0 < q \le \infty$. If a function f on Ω has a representation with respect to $\mathbf{\Psi}_s$:*

$$f = \sum_{k \ge 0} \sum_{j \in \Lambda_k} c_{k,j} \psi_{k,j}, \tag{3.5}$$

then

$$\|f\|_{B^\alpha_{p,q}(\Omega)} \le C \|(c_{k,j})\|_{b^\alpha_{p,q}(\Omega)} \tag{3.6}$$

with C independent of f, k, and j. Moreover when $0 < p < 1$, for $p = q$

$$\|f\|_{B^\alpha_{q,q}(\Omega)} \le C \|(c_{k,j})\|_{b^\alpha_{q,q}(\Omega)}. \tag{3.7}$$

REMARK. The space $B^\alpha_{q,q}$ with $q > (\alpha/d + 1)^{-1}$ has a useful application to nonlinear approximation theory that has many applications such as to data compressions and adaptive algorithms (see [14, 9, 12]).

The following theorem is a dual of Theorem 3.1 with the class $\widetilde{\mathbf{\Psi}}_r$.

THEOREM 3.2. *Let $0 < \alpha < r$, $1 \le p \le \infty$ and $0 < q \le \infty$. If $f \in B^\alpha_{p,q}(\Omega)$ and $c_{k,j} = \langle f, \widetilde{\psi}_{k,j} \rangle$ with $\widetilde{\psi}_{k,j} \in \widetilde{\mathbf{\Psi}}_r$, then*

$$\|(c_{k,j})\|_{b^\alpha_{p,q}(\Omega)} \le C \|f\|_{B^\alpha_{p,q}(\Omega)} \tag{3.8}$$

with C independent of f, k, and j. Moreover when $0 < p < 1$, for $p = q$ and $q > (\alpha/d + 1)^{-1}$

$$\|(c_{k,j})\|_{b^\alpha_{q,q}(\Omega)} \le C \|f\|_{B^\alpha_{q,q}(\Omega)}. \tag{3.9}$$

In order to establish Theorem 3.1, we need a preliminary estimate for the difference operator Δ^s_h. This estimate gives a Bernstein inequality for the modulus of smoothness (see [9]).

LEMMA 3.3. *Let $\psi_{k,j} \in \mathbf{\Psi}_s$, $k \in \mathbb{N}$, $j \in \Lambda_k$. Then for $k \in \mathbb{N}$, $j \in \Lambda_k$,*

$$|\Delta^r_h(\psi_{k,j}, x)| \le C \min(1, 2^{kr}|h|^r) 2^{kd/2} \chi_{Q^r_h(\psi_{k,j})}(x), \quad 0 \le r \le s$$

with C independent of k and $x \in \Omega$. Here $\chi_{Q^r_h(\psi_{k,j})}$ is the characteristic function of $Q^r_h(\psi_{k,j}) := \mathrm{supp} \Delta^r_h(\psi_{k,j}, \cdot)$.

Proof. First, note the following formula: for any $f \in W^r_\infty$

$$\Delta^r_h(f, x) = \int_{-\infty}^{\infty} N_r(\xi) \sum_{|\nu| = r} \frac{r!}{\nu!} D^\nu f(x + \xi h) h^\nu d\xi \tag{3.10}$$

where N_r is the B-spline of order of r. This formula can be easily proved by the induction on r. To employ (3.10), we need to extend $\psi_{k,j}$ from Ω to $\mathbb{R}^d$ by the Whitney type extension (see [15]). The Whitney type extension operator $\mathcal{E}$ is a bounded operator from $W^r_\infty(\Omega)$ to $W^r_\infty(\mathbb{R}^d)$, so that $|\mathcal{E}\psi_{k,j}|_{W^r_\infty(\mathbb{R}^d)} \le C|\psi_{k,j}|_{W^r_\infty(\Omega)}$ with C independent of $\psi_{k,j}$. Now since $\int_{-\infty}^{\infty} N_r(\xi)d\xi = 1$, $r \in \mathbb{N}$, for fixed $x \in Q^r_h(\psi_{k,j})$

$$\begin{aligned}
|\Delta^r_h(\psi_{k,j}, x)| &= \left| \int_{-\infty}^{\infty} N_r(\xi) \sum_{|\nu|=r} \frac{r!}{\nu!} D^\nu \psi_{k,j}(x+\xi h)h^\nu d\xi \right| \\
&\le C|\psi_{k,j}|_{W^r_\infty(\Omega)}|h|^r \int_{-\infty}^{\infty} N_r(\xi)d\xi \\
&\le C2^{kr}|h|^r 2^{kd/2}, \qquad 0 \le r \le s
\end{aligned}$$

where the constant C depends only on r and d. In addition, if $|h| \ge 2^{-k}$, by the definition of the difference operator Δ^r_h,

$$|\Delta^r_h(\psi_{k,j}, x)| \le C2^{kd/2}, \qquad 0 \le r \le s$$

where the constant C depends only on r. This completes the proof. □

Notice that for each $x \in \Omega$, $\Delta^s_h(\psi_{k,j}, x) \neq 0$ for at most C of $j \in \Lambda_k$ with C independent of k and j (see (3.1)(ii)). Then for $g_k = \sum_{j\in\Lambda_k} c_{k,j}\psi_{k,j}$,

$$\text{(3.11)} \qquad |\Delta^s_h(g_k, x)|^p \le C \sum_{j\in\Lambda_k} |c_{k,j}|^p |\Delta^s_h(\psi_{k,j}, x)|^p \quad 0 < p \le \infty.$$

The following (Bernstein type inequality) estimate follows from Lemma 3.3, (3.11), and (3.1)(i).

Lemma 3.4. *Let $\psi_{k,j} \in \mathbf{\Psi}_s$, $k \in \mathbb{N}$, $j \in \Lambda_k$. Also let $0 < p \le \infty$. Then for $g_k = \sum_{j\in\Lambda_k} c_{k,j}\psi_{k,j}$,*

$$\text{(3.12)} \qquad \omega_s(g_k, 2^{-\nu})_p \le C2^{s\min(k-\nu,0)}2^{kd(1/2-1/p)} \|(c_{k,j})\|_{l_p(j\in\Lambda_k)}$$

where the constant C depends only on s, d, and p.

Applying the sub-additivity of norm, the discrete Hardy inequality, and Lemma 3.4, we shall prove Theorem 3.1. In approximation theory, the estimate of Theorem 3.1 is referred as an inverse theorem.

Proof of Theorem 3.1. Let us first consider the case $1 \le p \le \infty$. Using the representation (3.5) of f, the sub-additivity of the norm, and

(3.12), we can write

$$\omega_s(f, 2^{-\nu})_p \leq C \sum_{k\geq 0} 2^{s\min(k-\nu,0)} 2^{kd(1/2-1/p)} \left\| (c_{k,j}) \right\|_{l_p(j\in\Lambda_k)}. \tag{3.13}$$

By splitting the right hand side of (3.13) into two parts as $d_\nu + e_\nu$ with

$$\begin{aligned} d_\nu &= \sum_{0\leq k\leq \nu} 2^{(k-\nu)s} 2^{kd(1/2-1/p)} \left\| (c_{k,j}) \right\|_{l_p(j\in\Lambda_k)}, \\ e_\nu &= \sum_{k>\nu} 2^{kd(1/2-1/p)} \left\| (c_{k,j}) \right\|_{l_p(j\in\Lambda_k)}, \end{aligned}$$

we obtain

$$\begin{aligned} |f|^q_{B^\alpha_{p,q}(\Omega)} &= \|\omega_s(f, 2^{-\nu})_p\|^q_{l^\alpha_q(\nu\geq 0)} \\ &\leq C\left\{ \left\| (d_\nu) \right\|^q_{l^\alpha_q(\nu\geq 0)} + \left\| (e_\nu) \right\|^q_{l^\alpha_q(\nu\geq 0)} \right\}. \end{aligned} \tag{3.14}$$

Notice that for $\mu = \min(1, q)$

$$\begin{aligned} |d_\nu| &= 2^{-\nu s} \sum_{0\leq k\leq \nu} \left[2^{ks} \left(2^{kd(1/2-1/p)} \left\| (c_{k,j}) \right\|_{l_p(j\in\Lambda_k)} \right) \right] \\ &\leq 2^{-\nu s} \left(\sum_{0\leq k\leq \nu} \left[2^{ks} \left(2^{kd(1/2-1/p)} \left\| (c_{k,j}) \right\|_{l_p(j\in\Lambda_k)} \right) \right]^\mu \right)^{1/\mu}. \end{aligned}$$

Since $0 < \alpha < s$, by Lemma 2.2 with (2.7)(i)

$$\begin{aligned} \left\| (d_\nu) \right\|_{l^\alpha_q(\nu\geq 0)} &\leq C \left\| \left(2^{kd(1/2-1/p)} \left\| (c_{k,j}) \right\|_{l_p(j\in\Lambda_k)} \right) \right\|_{l^\alpha_q(k\geq 0)} \\ &= C \left\| (c_{k,j}) \right\|_{b^\alpha_{p,q}(\Omega)}. \end{aligned}$$

By similar argument using Lemma 2.2 with (2.7)(ii),

$$\left\| (e_\nu) \right\|_{l^\alpha_q(\nu\geq 0)} \leq C \left\| (c_{k,j}) \right\|_{b^\alpha_{p,q}(\Omega)}.$$

Here the constant C at each step is independent of f, k, and j.

It remains to estimate $\|f\|_{L_p(\Omega)}$, $1 \le p \le \infty$. For $1 \le q < \infty$, using the sub-additivity of $\|\cdot\|_{L_p(\Omega)}$ and the Hölder inequality on q,

$$\begin{aligned}
\|f\|_{L_p} &\le \sum_{k\ge 0} 2^{-\alpha k} 2^{\alpha k} \Big\| \sum_{j\in\Lambda_k} c_{k,j}\psi_{k,j} \Big\|_{L_p(\Omega)} \\
&\le C\Big(\sum_{k\ge 0} \Big[2^{\alpha k} \Big\| \sum_{j\in\Lambda_k} c_{k,j}\psi_{k,j} \Big\|_{L_p(\Omega)} \Big]^q \Big)^{1/q} \\
&\le C \Big\| \Big(2^{kd(1/2-1/p)} \|(c_{k,j})\|_{l_p(j\in\Lambda_k)} \Big) \Big\|_{l_q^\alpha(k\ge 0)}
\end{aligned}$$

where for the third inequality, (3.1) and (3.2) are used. When $0 < q < 1$, we raise $\|f\|_{L_p}$ to the qth power to obtain the same result; that is,

$$\begin{aligned}
\|f\|_{L_p}^q &\le \sum_{k\ge 0} 2^{\alpha k q} \Big\| \sum_{j\in\Lambda_k} c_{k,j}\psi_{k,j} \Big\|_{L_p(\Omega)}^q \\
&\le C \Big\| \Big(2^{kd(1/2-1/p)} \|(c_{k,j})\|_{l_p(j\in\Lambda_k)} \Big) \Big\|_{l_q^\alpha(k\ge 0)}^q .
\end{aligned}$$

For the case $q = \infty$, the same argument with the usual change to a supremum norm establishes the estimate. Finally, the case $0 < p < 1$ for the estimate (3.7) can be established by a modification of the proof above with $\|\cdot\|_{L_q}^q$-norm in the place of $\|\cdot\|_{L_p}$-norm. □

Theorem 3.2 will be established by the vanishing moment conditions (3.3)(ii) and a Jackson type estimate for the modulus of smoothness (see [10, 9]). Let us recall from [3, 13] a fundamental result between smoothness and polynomial approximations of $f \in L_p(I)$, $0 < p \le \infty$, where I is a cube in $\mathbb{R}^d$.

Let $E_r(f,I)_p := \inf_{Q\in\mathcal{P}_r} \|f - Q\|_{L_p(I)}$ be the local error of approximation of f by the elements of the collection $\mathcal{P}_r$. The Whitney theorem [3] gives an estimate of $E_r(f,I)_p$ in terms of the modulus of smoothness; that is,

$$E_r(f,I)_p \le C\omega_r(f,\ell_I,I)_p \tag{3.15}$$

where ℓ_I is the side length of I and C is a constant depending only on r and d (and p if p is close to 0). Using this estimate with the averaged modulus of smoothness (see [13]), we obtain the following lemma from which the first part of Theorem 3.2 is straightforward.

LEMMA 3.5. *Let $1 \le p \le \infty$ and $r \in \mathbb{N}$. If $f \in L_p(\Omega)$ and $c_{k,j} = \langle f, \widetilde{\psi}_{k,j} \rangle$ with $\widetilde{\psi}_{k,j} \in \widetilde{\mathbf{\Psi}}_r$, then*

$$2^{kd(1/2-1/p)} \| (c_{k,j}) \|_{l_p(j\in\Lambda_k)} \le C\omega_r(f, 2^{-k})_p \tag{3.16}$$

with C independent of k, j, and f.

Proof. Let us first estimate $|c_{k,j}|$. Using (3.3)(ii) and the Hölder inequality with $1/p + 1/p' = 1$, we have for all $Q \in \mathcal{P}_r$

$$\begin{aligned} |c_{k,j}| &= |\langle f - Q, \widetilde{\psi}_{k,j} \rangle| \\ &\le \| f - Q \|_{L_p(\mathrm{supp}\widetilde{\psi}_{k,j})} \| \widetilde{\psi}_{k,j} \|_{L_{p'}(\Omega)}. \end{aligned}$$

Let $J_{k,j}$ be the smallest cube containing all cubes I in Ω with side length $\ell_I = 2^{-k}$ such that $\mathrm{supp}\widetilde{\psi}_{k,j} \cap I \neq \emptyset$. Notice from (3.1) that

$$\mathrm{supp}\widetilde{\psi}_{k,j} \subseteq J_{k,j} \subseteq \Omega, \quad |J_{k,j}| \le C2^{-kd}, \qquad \text{for} \quad j \in \Lambda_k,$$

with C depending only on d and r. Then, by (3.1), (3.3), and (3.15), for all k and $j \in \Lambda_k$

$$\begin{aligned} |c_{k,j}| &\le C2^{kd(1/2-1/p')} \| f - Q \|_{L_p(J_{k,j})} \\ &\le C2^{kd(1/2-1/p')} \omega_r(f, \ell_{J_{k,j}}, J_{k,j})_p \\ &\le C2^{kd(1/2-1/p')} \omega_r(f, 2^{-k}, J_{k,j})_p \end{aligned}$$

with C independent of f, k, and j. For the last inequality, we have used the fact that $\omega_r(f, \lambda t)_p \le C\omega_r(f,t)_p$ for $\lambda > 0$ with C depending only on r, λ, d, and p.

Since each point $x \in \Omega$ appears in at most C cubes $J_{k,j}$ (see (3.1)(ii)), adding up $|c_{k,j}|^p$ over $j \in \Lambda_k$ provides

$$\sum_{j\in\Lambda_k} |c_{k,j}|^p \le C2^{kdp(1/2-1/p')} \omega_r(f, 2^{-k}, \Omega)_p^p. \tag{3.17}$$

Therefore, (3.16) immediately follows. Indeed, the estimate (3.17) can be proved via the averaged modulus of smoothness (see [13]). □

Notice that the proof of Lemma 3.5 can not be applied for $0 < p < 1$. However, we can extend the first part of Theorem 3.2 to $0 < p < 1$ taken $p = q > (\alpha/d + 1)^{-1}$ by using the extension operator of [15] and Lemma 2.4.

Proof of Theorem 3.2. Since the first part of Theorem 3.2 immediately follows from Lemma 3.5, we shall focus on the case $0 < p < 1$. Let us first extend $\psi_{k,j} \in \widetilde{\Psi}_r$, $k \in \mathbb{N}$, $j \in \Lambda_k$ to a function $\widetilde{\Psi}_{k,j}$, $k \in \mathbb{Z}$, $j \in \mathbb{Z}^d \times \{1, \cdots, 2^d - 1\}$, defined on $\mathbb{R}^d$ as follows:

$$\widetilde{\Psi}_{k,j}(x) = \begin{cases} \widetilde{\psi}_{k,j}(x) & \text{if } x \in \Omega \text{ and } (k,j) \in \mathbb{N} \times \Lambda_k, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\mathrm{supp}\widetilde{\psi}_{k,j} \subseteq \Omega$,

$$\|(\langle f, \widetilde{\psi}_{k,j}\rangle)\|_{b^{\alpha}_{q,q}(\Omega)} = \|(\langle f, \widetilde{\Psi}_{k,j}\rangle)\|_{\dot{b}^{\alpha}_{q,q}(\mathbb{R}^d)} = \|(\langle \mathcal{E} f, \widetilde{\Psi}_{k,j}\rangle)\|_{\dot{b}^{\alpha}_{q,q}(\mathbb{R}^d)}. \tag{3.18}$$

Here $\mathcal{E}$ is the extension operator of [15] such that for $\alpha > d(1/\min(p,1) - 1)$, $0 < p, q \leq \infty$,

$$\|f\|_{B^{\alpha}_{p,q}(\Omega)} \approx \|\mathcal{E} f\|_{B^{\alpha}_{p,q}(\mathbb{R}^d)}. \tag{3.19}$$

To employ Lemma 2.4, note that $J = d/\min(1,q) = d/q$ and $1/q < \alpha/d + 1$, so that $N = \max(\lfloor J - d - \alpha \rfloor, -1) = -1$. We then need the conditions (2.11)(i) and (ii). Indeed, (2.11)(i) and (ii) are straightforward from (3.3) and (3.1). Thus, we have

$$\|(\langle \mathcal{E} f, \widetilde{\Psi}_{k,j}\rangle)\|_{\dot{b}^{\alpha}_{q,q}(\mathbb{R}^d)} \leq C \|\mathcal{E} f\|_{B^{\alpha}_{q,q}(\mathbb{R}^d)}$$

and complete the proof with (3.19) and (3.18). □

4. Wavelet Coefficients and Besov Spaces

In the standard wavelet theory, it has been known that a wavelet decomposition for $L_2(\mathbb{R}^d)$ is valid for $L_p(\mathbb{R}^d)$, $1 < p < \infty$, with the convergence in L_p if certain conditions (e.g., stability conditions and decay conditions [18]) on the wavelet ψ and its dual $\widetilde{\psi}$ hold (see [18, 21]). For instance, Meyer [21] has shown that every orthogonal wavelet for $L_2(\mathbb{R}^d)$ is an L_p-stable basis for $L_p(\mathbb{R}^d)$. On the interval $[0,1]$, the decomposition (2.3)

$$f = \sum_{j \in \Lambda_{k_0}} \langle f, \widetilde{\Phi}_{k_0,j}\rangle \, \Phi_{k_0,j} + \sum_{k \geq k_0} \sum_{j \in \Lambda_k} \langle f, \widetilde{\Psi}_{k,j}\rangle \, \Psi_{k,j}$$

of $L_2([0,1])$ still converges in L_p for any $f \in L_p([0,1])$, $1 < p < \infty$, as long as the wavelet basis and its dual basis satisfy the condition (3.1) of local supports (see [19]). This argument can be also extended for the unit cube Ω (or a cube) in $\mathbb{R}^d$ by the tensor product.

In this section, we establish that a wavelet basis and its dual basis for $L_2(\Omega)$ characterize the smoothness space $B^\alpha_{p,q}(\Omega)$ in $L_p(\Omega)$, $1 < p < \infty$, if they satisfy the smoothness conditions and vanishing moment conditions, respectively, plus the local supports. This implies that the wavelet basis forms an unconditional basis for the Besov space.

Let us assume that a wavelet decomposition

$$f = \sum_{j\in\Lambda_{k_0}} \langle f, \widetilde{\phi}_{k_0,j}\rangle\, \phi_{0,j} + \sum_{k\geq k_0}\sum_{j\in\Lambda_k} \langle f, \widetilde{\psi}_{k,j}\rangle\, \psi_{k,j} \tag{4.1}$$

converges in $L_p(\Omega)$, $1 < p < \infty$. For our notational convenience, we are employing the notations for the case of one-dimension instead of those for Ω.

THEOREM 4.1. *Assume that $0 < \alpha < \infty$, $1 < p < \infty$, and $0 < q \leq \infty$. Let s and r be the positive integers associated with the classes $\mathbf{\Psi}_s$ and $\widetilde{\mathbf{\Psi}}_r$. Also let $\phi_{k,j}$, $\psi_{k,j} \in \mathbf{\Psi}_s$ and $\widetilde{\phi}_{k,j}$, $\widetilde{\psi}_{k,j} \in \widetilde{\mathbf{\Psi}}_r$, $j \in \Lambda_k$, $k \geq k_0$. If $f \in L_p(\Omega)$ enjoys the decomposition (4.1), then for $\alpha < \min(r, s)$*

$$\|f\|_{B^\alpha_{p,q}(\Omega)} \approx 2^{k_0 d(1/2-1/p)} \big\| (\langle f, \widetilde{\phi}_{k_0,j}\rangle) \big\|_{l_p(j\in\Lambda_{k_0})} + \big\| (\langle f, \widetilde{\psi}_{k,j}\rangle) \big\|_{b^\alpha_{p,q}(\Omega)}. \tag{4.2}$$

Proof. Let us assume $0 < q < \infty$. The case $q = \infty$ can be handled in same way as below. The direction

$$\|f\|_{B^\alpha_{p,q}(\Omega)} \leq C\Big(2^{k_0 d(1/2-1/p)} \big\| (\langle f, \widetilde{\phi}_{k_0,j}\rangle) \big\|_{l_p(j\in\Lambda_{k_0})} + \big\| (\langle f, \widetilde{\psi}_{k,j}\rangle) \big\|_{b^\alpha_{p,q}(\Omega)} \Big)$$

is straightforward from Theorem 3.1.

By virtue of Theorem 3.2, the other direction of (4.2) is reduced to estimate

$$2^{k_0 d(1/2-1/p)} \big\| (\langle f, \widetilde{\phi}_{k_0,j}\rangle) \big\|_{l_p(j\in\Lambda_{k_0})} \leq C \|f\|_{L_p(\Omega)}. \tag{4.3}$$

Indeed, (4.3) is straightforward from the Hölder inequality and the finite sum over Λ_{k_0} independent of k. □

REMARK. In Theorem 4.1 the $\widetilde{\phi}_{k,j}$ does not need to satisfy (3.3)(ii). Theorem 4.1 is valid for $p = \infty$ if $L_p(\Omega)$ is replaced by $\mathcal{C}(\Omega)$ the space of continuous functions on Ω. In particular, for $p = q = \infty$, the theorem provides the following characterization for the Hölder-Zygmund classes $\mathcal{C}^\alpha(\Omega)$ (see Theorem 3.1 and Theorem 3.2 with $p = q = \infty$):

$$\|f\|_{\mathcal{C}^\alpha(\Omega)} \approx \sup_{j\in\Lambda_{k_0}} 2^{k_0 d(\alpha+1/2)} \big|\langle f, \widetilde{\phi}_{k_0,j}\rangle\big| + \sup_{k\geq k_0, j\in\Lambda_k} 2^{kd(\alpha+1/2)} \big|\langle f, \widetilde{\psi}_{k,j}\rangle\big|.$$

We next characterize the space $B^{\alpha}_{q,q}(\Omega)$ with $q = (\alpha/d + 1/\tau)^{-1}$, $1 < \tau \leq \infty$. To this end we recall from [13] the embedding property:

$$B^{\alpha}_{q,q}(\Omega) \subset L_{\tau}(\Omega) \quad \text{or} \quad \|f\|_{L_{\tau}(\Omega)} \leq C\|f\|_{B^{\alpha}_{q,q}(\Omega)} \tag{4.4}$$

with C independent of f. Then any $f \in B^{\alpha}_{q,q}(\Omega)$ enjoys the wavelet decomposition (4.1) with the convergence in L_{τ}.

THEOREM 4.2. *Assume that* $0 < \alpha < \infty$, $1 < \tau \leq \infty$, *and* $0 < q < \infty$. *Also, let* s *and* r *be as in Theorem 4.1. If* $f \in L_{\tau}(\Omega)$ *enjoys the decomposition* (4.1), *then for* $\alpha < \min(r, s)$

$$\|f\|_{B^{\alpha}_{q,q}(\Omega)} \approx 2^{k_0 d(1/2-1/q)} \|(\langle f, \widetilde{\phi}_{k_0,j}\rangle)\|_{l_q(j\in\Lambda_{k_0})} + \|(\langle f, \widetilde{\psi}_{k,j}\rangle)\|_{b^{\alpha}_{q,q}(\Omega)}. \tag{4.5}$$

Proof. A little modification of the proof of Theorem 4.1 completes the proof. The estimate (4.3) becomes

$$2^{k_0 d(1/2-1/p)} \|(\langle f, \widetilde{\phi}_{k_0,j}\rangle)\|_{l_p(j\in\Lambda_{k_0})} \leq C\|f\|_{B^{\alpha}_{q,q}(\Omega)}$$

which can be proved by the embedding property (4.4). □

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