J. Korean Math. Soc. **36** (1999), No. 6, pp. 1047–1059

LOCAL CONNECTEDNESS IN FELL TOPOLOGY

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Abstract. Let $C(X)$ ($C_K(X)$) denote the hyperspace of all nonempty closed connected subsets (subcontinua) of a locally compact Hausdorff space X with the Fell topology. We prove that the following statements are equivalent:
(1) X is locally connected. (2) $C(X)$ is locally connected. (3) $C(X)$ is locally connected at each $E \in C_K(X)$. (4) $C_K(X)$ is locally connected.

1. Introduction

In 1962 Fell [2] introduced a topology T_f, now it is called a Fell topology, on the the collection $[2^X]$ of all closed subsets (including the empty set) of a topological spaces X and proved that $([2^X], T_f)$ is compact, and is compact Hausdorff if X is locally compact, no matter how badly unseparated X may be. This topology has proved to be the superior construct in terms of applications, particularly application to optimization, convex analysis, mathematical economics, probability theory, and the theory of capacities (see the references quoted in [1]). While the properties of Vietoris topology parallel those of X closely, the Fell topology is not in general.

The local connectivity of hyperspaces with the Vietoris topology has been extensively studied [3,4,5,8,11,12,13] but it is not with the Fell topology. The purpose of this paper is to explore local properties of some of subspaces of $([2^X], T_f)$, namely local connectedness of the subspace $C(X)$ ($C_K(X)$) of all nonempty closed connected subsets (subcontinua) of X.

Received November 23, 1998.
1991 Mathematics Subject Classification: Primary 54B20, 54B15.
Key words and phrases: Fell topology, Vietoris topology, hyperspace, continuum, local compactness, local connectedness, connected im kleinen.
This paper was supported by Won Kwang University, 1998.

1. Preliminary and fundamental properties in a Fell topology

Let X be a space. Let $[2^X]$ be the collection of all closed subsets of X including the empty set, 2^X the collection of all nonempty closed subsets of X, $\mathcal{K}(X) = \{E \in 2^X : E \text{ is compact}\}$, $C(X) = \{E \in 2^X : E \text{ is connected }\}$, $C_K(X) = C(X) \cap \mathcal{K}(X)$, $\mathcal{F}_n(X) = \{E \in 2^X : E \text{ has at most n elements}\}$ and $\mathcal{F}(X) = \{E \in 2^X : E \text{ is finite}\}$.

For $E \subset X$, let $E^- = \{A \in [2^X] : A \cap E \neq \emptyset\}$, $E^+ = \{A \in [2^X] : A \subset E\}$. The *Fell topology* [2] T_f on $[2^X]$ has as a subbase all sets of the form V^-, where V is an open set of X plus all sets of the form $(K^c)^+$, where K is a compact subset of X and $K^c = X \setminus K$.
Clearly we have three kinds of the form as the basic elements of T_f:
Type 1. $\cap_{i=1}^n V_i^-$, where each V_i is open in X.
Type 2. $(K^c)^+$, where K is compact in X.
Type 3. $(\cap_{i=1}^n V_i^-) \cap (K^c)^+$.

For subsets $E_1, ..., E_n$ of X, let $\langle E_1, ..., E_n\rangle = \{A \in 2^X : A \cap E_i \neq \emptyset$, for each $i = 1, ..., n$ and $A \subset \cup_{i=1}^n E_i\}$. The *Vietoris (finite) topology* [8] T_v on 2^X has as a base all sets of the form $\langle U_1, ..., U_n\rangle$, where $U_1, ..., U_n$ are open sets in X.

In order to distinguish subspaces in different hyperspace topologies, we adopt the following: Let $\mathcal{S} \subset [2^X]$. The subspace $\mathcal{S}$ of $([2^X], T_f)$ is denote by $(\mathcal{S}, T_f)$ and, if $\mathcal{S} \subset 2^X$ then the subspace $\mathcal{S}$ of $(2^X, T_v)$ is denoted by $(\mathcal{S}, T_v)$.

Since the Vietoris Topology on 2^X is finer than the Fell topology on 2^X, some results in Fell topology derived from Vietoris topology [8], for instance, Lemma 1.3 and Proposition 1.4, are included in this section.

In section 2, we investigate connectedness of $(C(X), T_f)$ and (C_K, T_f). Then we proceed to prove that, for a locally compact Hausdorff space X, $(C(X), T_f)$ is locally connected if and only if X is locally connected. In doing so, we first prove that $(C_K(X), T_f)$ is locally connected and dense in $(C(X), T_f)$.

For notational purpose, small letters will denote elements of X, capital letters will denote subsets of X and elements of 2^X, and script letters will denote subsets of 2^X. If $A \subset X$, then $\overline{A}$ (resp., Int(A), Bd(A)) will denote the closure (resp., interior, boundary) of A in X.

PROPOSITION 1.1 [6]. *Let X be a Hausdorff space. Then the Vietoris topology T_v on 2^X is finer than the Fell topology T_f on 2^X.*

Furthermore, if we replace compact sets in the definition of the subbase for T_f by closed sets, then T_f and T_v are equivalent. Hence if X is compact Hausdorff, then $T_f = T_v$.

REMARK. Let X be a Hausdorff space. If $(2^X, T_v)$ is second countable (metrizable, or compact), then $T_v = T_f$. If $(2^X, T_v)$ is second countable (metrizable or compact) then X is compact by [8, Theorem 4.6]. Hence $T_v = T_f$ by Proposition 1.1.

THEOREM 1.2. [10]. *The followings are equivalent:*
(a) $(2^X, T_f)$ *is Hausdorff.*
(b) $(2^X, T_f)$ *is regular.*
(c) X *is locally compact.*

The following lemma is an easy consequence of Proposition 1.1.

LEMMA 1.3. *Let X be a Hausdorff space. Then*
(a) *If $\mathcal{B}$ is a connected (compact) subset of $(2^X, T_v)$, then it is a connected (compact) subset of $(2^X, T_f)$.*
(b) *If $\mathcal{D}$ is dense in $(2^X, T_v)$, it is also dense in $(2^X, T_f)$.*

PROPOSITION 1.4. *Let X be a Hausdorff space.*
(a) *If $f : Z \to (2^X, T_v)$ is a continuous function, then $f : Z \to (2^X, T_f)$ is continuous.*
(b) *The natural map $f : X^n \to (\mathcal{F}_n(X), T_f)$, defined by $f(x_1, ..., x_n) = \{x_1, ..., x_n\}$ is a continuous surjection.*
(c) *Let $f : (2^X, T_f)^n \to (2^X, T_f)$ be defined by $f(A_1, ..., A_n) = \cup_{i=1}^n A_i$. Then f is continuous.*

Proof. (a) This is an easy consequence of Proposition 1.1.
(b) Since the natural map $f : X^n \to (\mathcal{F}_n(X), T_v)$ is a continuous surjection by [8, 2.4.3], hence by (a) it is a continuous surjection.
(c) We give proof for $n = 2$.
Let $f : 2^X \times 2^X \to 2^X$ be the function defined by $f(A_1, A_2) = A_1 \cup A_2$. Let $\mathcal{O} \in T_f$ such that $f(A_1, A_2) = A_1 \cup A_2 \in \mathcal{O}$.

Case 1. Suppose $A_1 \cup A_2 \in \cap_{k=1}^n V_k^- \subset \mathcal{O}$.
Let $\mathcal{F} = \{V_1, .., V_n\}$. Let $\mathcal{F}_1 = \{V_i \in \mathcal{F} : V_i \cap A_1 \neq \emptyset\}$, $\mathcal{F}_2 = \{V_i \in \mathcal{F} : A_2 \cap V_i \neq \emptyset\}$. We reindex the elements of $\mathcal{F}_i$; $\mathcal{F}_1 = \{V_{1_1}, ..., V_{1_p}\}$ and $\mathcal{F}_2 = \{V_{2_1}, ..., V_{2_q}\}$. Let $\mathcal{U}_1 = \cap_{i=1}^p V_{1_i}^-$, and $\mathcal{U}_2 = \cap_{j=1}^q V_{2_j}^-$. Then $\mathcal{U}_1 \times \mathcal{U}_2$ is a neighborhood of (A_1, A_2). Let $(A_1', A_2') \in \mathcal{U}_1 \times \mathcal{U}_2$. Then $A_1' \cap V_{1_i} \neq \emptyset$, $i = 1, .., p$, and $A_2' \cap V_{2_j} \neq \emptyset$ for $j = 1, ..., q$. Thus $(A_1' \cup A_2') \cap V_k \neq \emptyset$ for $k = 1, 2, ..., n$. So $(A_1' \cup A_2') \in \cap_{k=1}^n V_k^-$ and hence $f(\mathcal{U}_1 \times \mathcal{U}_2) \subset (\cap_{k=1}^n V_k^-)$.

Case 2. Suppose $A_1 \cup A_2 \in (\cap_{k=1}^n V_k^-) \cap (K^c)^+ \subset \mathcal{O}$. Take $\mathcal{U}_1 = (\cap_{i=1}^p V_{1_i}^-) \cap (K^c)^+$, $\mathcal{U}_2 = (\cap_{j=1}^q V_{2_j}^-) \cap (K^c)^+$, where $\cap_{i=1}^p V_{1_i}^-$ and $\cap_{j=1}^q V_{2_j}^-$ are defined as in Case 1. Then $f(\mathcal{U}_1 \times \mathcal{U}_2) \subset (\cap_{k=1}^n V_k^-) \cap (K^c)^+$.

Case 3. Suppose $A_1 \cup A_2 \in (K^c)^+ \subset \mathcal{O}$. Take $\mathcal{U}_i = (K^c)^+$ for $i = 1, 2$. Then it is easy to see that $f(\mathcal{U}_1 \times \mathcal{U}_2) \subset (K^c)^+$. □

PROPOSITION 1.5. (a) *Let X be a Hausdorff space. If $\mathcal{O}$ is an open subset of any one of the spaces $(2^X, T_f)$, $(\mathcal{F}(X), T_f)$, and $(\mathcal{K}(X), T_f)$, then $\cup\mathcal{O} = \cup\{E : E \in \mathcal{O}\}$ is open in X.*

(b) *Let X be a locally compact Hausdorff space. Let $\mathcal{O}$ be an open set in $(C_K(X), T_f)$ or $(C(X), T_f)$. If $\{x\} \in \mathcal{O}$ then there is a neighborhood V of x in X such that $V \subset \cup\mathcal{O}$.*

Proof. (a) It suffices to show that if $\mathcal{B}$ is a basic element of $(2^X, T_f)$, then $\cup(\mathcal{B} \cap \mathcal{Y}) = \cup\{E : E \in \mathcal{B} \cap \mathcal{Y}\}$ is open in X, where $\mathcal{Y}$ is any one of the spaces mentioned in (a).

Suppose $\mathcal{B} = (\cap_{i=1}^n V_i^-) \cap (K^c)^+$, where K is compact in X and V_i is open in X for each $i = 1, ..., n$. We show that $\cup(\mathcal{B} \cap \mathcal{Y}) = X \setminus K$ for each $\mathcal{Y}$. Choose a point $x_i \in V_i \setminus K$ for each $i = 1, ..., n$. For each $x \in X \setminus K$, let $E_x = \{x_1, ..., x_n, x\}$. Then E_x is a closed (compact, finite) set and $E_x \in \mathcal{B} \cap \mathcal{Y}$ for $\mathcal{Y} = 2^X$, $\mathcal{F}(X)$ or $\mathcal{K}(X)$. Hence $x \in E_x \subset \cup(\mathcal{B} \cap \mathcal{Y})$ for $\mathcal{Y} = 2^X$, $\mathcal{F}(X)$, or $\mathcal{K}(X)$. Therefore $X \setminus K \subset \cup(\mathcal{B} \cap \mathcal{Y})$. Now let $x \in \cup(\mathcal{B} \cap \mathcal{Y})$. Then $x \in E$ for some $E \in [(\cap_{i=1}^n V_i^-) \cap (K^c)^+] \cap \mathcal{Y}$. Hence $x \in E \subset X \setminus K$.

In a similar manner, one can show that $\cup(\mathcal{B} \cap \mathcal{Y}) = X$ if $\mathcal{B} = \cap_{i=1}^n V_i^-$, and $\cup(\mathcal{B} \cap \mathcal{Y}) = X \setminus K$ if $\mathcal{B} = (K^c)^+$.

(b) There exists a neighborhood V of x in X with compact closure and a compact set K with $V \subset X \setminus K$ such that $\mathcal{V} = (V^- \cap (K^c)^+) \cap C_K(X)$ is a neighborhood of $\{x\}$ with $\mathcal{V} \subset \mathcal{O}$. Let $K' = \overline{V} \setminus V$ and $\mathcal{W} = (V^- \cap ((K \cup K')^c)^+) \cap C_K(X)$. Then $\{x\} \in \mathcal{W} \subset \mathcal{V}$. Let $E \in \mathcal{W}$. Then $E \subset X \setminus K' = V \cup (X \setminus \overline{V})$. Since E is connected and $E \cap V \neq \emptyset$, $E \subset V$. Hence $W = \cup\mathcal{W} \subset V$. On the other hand, for each $y \in V$, we have $\{y\} \in \mathcal{W}$ so that $y \in W$. Hence $\cup\mathcal{W} = V$ is an open set contained in $\cup\mathcal{O}$. □

PROPOSITION 1.5.1. *Let X be a Hausdorff space. If $\mathcal{O}$ is an open set in $(\mathcal{F}_n(X), T_f)$, then $\cup\mathcal{O}$ is open in X. In particular if $\mathcal{O} = [(\cap_{i=1}^n V_i^-) \cap (K^c)^+] \cap \mathcal{F}_n(X)$, where V_i's are pairwise disjoint, then $\cup\mathcal{O} = \cup_{i=1}^n (V_i \backslash K)$.*

Proof. Without loss of generality, let $\mathcal{O} = \mathcal{B} \cap \mathcal{F}_n(X)$, where $\mathcal{B} = (\cap_{i=1}^m V_i^-) \cap (K^c)^+$, $\cap_{i=1}^m V_i^-$, or $(K^c)^+$. Let $U = \cup\mathcal{O}$ and $x \in U$. Then there is an element $E \in \mathcal{O}$ such that $x \in E \in \mathcal{O}$. Let $E = \{x_1, ..., x_p\}$ and we may assume that $x = x_1$. Suppose $\mathcal{B} = (\cap_{i=1}^m V_i^-) \cap (K^c)^+$. Let $\mathcal{S} = \{V_1, ..., V_m\}$ and let $W = \cap\{V_i \in \mathcal{S} : x \in V_i\} \cap (X \setminus K)$. Then for each $y \in W$, the element $E_y = \{y, x_2, ..., x_p\} \in \mathcal{B} \cap \mathcal{F}_n(X)$. Hence $E_y \subset U$. This shows that $x \in W \subset U$. If $\mathcal{B} = \cap_{i=1}^m V_i^-$ or $(K^c)^+$, the proofs are similar.

For the second part, let $W = \cup_{i=1}^n (V_i \setminus K)$ and $U = \cup\mathcal{O}$. We show first that $W \subset U$. Let $x \in W$. Then $x \in V_i \setminus K$ for some i. Let $E \in \mathcal{O}$ and $\{x_i\} = V_i \cap E$ for each $i = 1, ..., n$. Then for each $y \in V_i \setminus K$, $E_y = \{x_1, .., x_{i-1}, y, x_{i+1}, ..., x_n\}$ is an element of $\mathcal{O}$. Hence $V_i \setminus K \subset U$. Thus it follows that $W \subset U$.

Let $x \in U$. Then $x \in E \in \mathcal{O}$ for some E and $\{x\} = E \cap V_i$ for some i. Since $E \subset X \setminus K$, $x \in V_i \setminus K$. Hence $U \subset W$. □

PROPOSITION 1.6. *Let X be a topological space. Then:*

(a) *$\{E \in 2^X : E \subset A\}$ is closed in $(2^X, T_f)$ if $A \subset X$ is closed.*

(b) *$\{E \in 2^X : E \cap A \neq \emptyset\}$ is closed in $(2^X, T_f)$ if $A \subset X$ is compact and X is Hausdorff.*

Proof. (a) Let $\mathcal{A} = \{E \in 2^X : E \subset A\}$ and let B be a limit point of $\mathcal{A}$. Suppose $B \notin \mathcal{A}$. (i.e., $B \not\subset A$). Let $b \in B \setminus A$. Since A is closed and $b \notin A$, there is a neighborhood V of b such that $V \cap A = \emptyset$. Then $B \in V^-$ and $V^- \cap \mathcal{A} = \emptyset$ which contradicts the fact that B is a limit point of $\mathcal{A}$.

(b) Let $\mathcal{A} = \{E \in 2^X : E \cap A \neq \emptyset\}$. Suppose there is an element $B \in 2^X$ which is a limit point of $\mathcal{A}$ but $B \notin \mathcal{A}$. Then there is an open set V containing B such that $V \cap A = \emptyset$. Then $(A^c)^+$ is a neighborhood of B which does not meet $\mathcal{A}$, a contradiction. □

PROPOSITION 1.7. *Let X be a locally compact Hausdorff space. If $\mathcal{B}$ is a compact subset of $(2^X, T_f)$, then $\cup\mathcal{B}$ is closed in X.*

Proof. Let $A = \cup\mathcal{B}$ and $x \in \overline{A}$. Let $\mathcal{F}$ be the collection of neighborhoods V of x such that $\overline{V}$ is compact. For each $V \in \mathcal{F}$, let $\mathcal{C}_V = \mathcal{B} \cap \{E \in 2^X : E \cap \overline{V} \neq \emptyset\}$. Since $\overline{V}$ is compact, $\mathcal{C}_V$ is closed in $(2^X, T_f)$ by Proposition 1.6(b). Also since X is locally compact, $(2^X, T_f)$ is Hausdorff by Theorem 1.2. So that each $\mathcal{C}_V$ is closed subset of the compact set $\mathcal{B}$. Then the collection $\{\mathcal{C}_V : V \in \mathcal{F}\}$ has the finite intersection property. Suppose $\mathcal{C}_{V_i}$, $i = 1, ..., n$ is a finite subcollection. Then $V = \cap_{i=1}^n V_i \in \mathcal{F}$ so that $\mathcal{C}_V \subset \cap_{i=1}^n \mathcal{C}_{V_i}$. Since $\mathcal{B}$ is compact, this collection has a nonempty intersection $\mathcal{D}$. And each element $E \in \mathcal{D}$ contains the point x. (if there is some $E \in \mathcal{D}$ such that $x \notin E$, then there is an element $V \in \mathcal{F}$ such that $\overline{V} \cap E = \emptyset$. This would mean that $E \notin \mathcal{D}$.) Since $\mathcal{D} \subset \mathcal{B}$, $x \in E \in \mathcal{D}$, $x \in A$. Hence A is closed. □

2. Connectedness and Local connectedness in Fell Topology

A compact connected Hausdorff space is called a *continuum.*

A space X is said to be *locally connected at* $x \in X$ if for each neighborhood U of x there is a connected neighborhood V of x such that $V \subset U$. The space X is said to be *locally connected* if X is locally connected at each of its points.

A space X is said to be *connected im kleinen at* $x \in X$ if for each neighborhood U of x there is a component of U which contains x in its interior.

It is known that if X is connected im kleinen at each of its points, then X is locally connected.

PROPOSITION 2.1. *Let X be a Hausdorff space. Then:*

(a) *each $\mathcal{F}_n(X)$ is a closed subset of $(2^X, T_f)$.*

(b) *if X is connected, then each of $(\mathcal{F}_n(X), T_f)$, $(\mathcal{F}(X), T_f)$, $(\mathcal{K}(X), T_f)$, and $(2^X, T_f)$ is connected.*

Proof. (a) Let $E \in 2^X \setminus \mathcal{F}_n(X)$. Since X is Hausdorff, there exist pairwise disjoint open sets, $V_1, ..., V_p$ for some $p > n$ such that $V_i \cap E \neq \emptyset$ for $i = 1, ..., p$. Then $(\cap_{i=1}^p V_i^-) \cap \mathcal{F}_n(X) = \emptyset$ and thus $E \in \cap_{i=1}^p V_i^- \subset 2^X \setminus \mathcal{F}_n(X)$, where $\cap_{i=1}^p V_i^-$ is open in $(2^X, T_f)$. Hence $\mathcal{F}_n(X)$ is closed in $(2^X, T_f)$.

(b) Since X is connected, each of $\mathcal{F}_n(X)$, $\mathcal{F}(X)$, $\mathcal{K}(X)$, and 2^X is connected in $(2^X, T_v)$ by [8, Theorem 4.10]. Hence, by Proposition 1.3(a), each of the hyperspaces is connected in $(2^X, T_f)$. □

LEMMA 2.2 ([8], [3]). (a) *Let* $A_1, ..., A_n$ *be connected subsets of a Hausdorff space* X. *Then* $\langle A_1, ..., A_n\rangle$ *is a connected subset of* $(2^X, T_v)$.

(b) *Let* U *be a connected open subset and* $U_1, ..., U_n$ *be nonempty open subsets of a Hausdorff space* X *such that* $U = \cup_{i=1}^n U_i$. *Then* $\langle U_1, ..., U_n\rangle$ *is connected in* $(2^X, T_v)$.

PROPOSITION 2.3. *If* X *is a connected Hausdorff space, then* $(2^X, T_f)$ *is always locally connected at* X.

Proof. Let $\cap_{i=1}^n V_i^-$ be a basic neighborhood of X in $(2^X, T_f)$. Then $\cap_{i=1}^n V_i^- = \langle V_1, ..., V_n, X\rangle$. Since X is connected, $\langle V_1, ..., V_n, X\rangle$ is connected subset in $(2^X, T_v)$ by Lemma 2.2(b). So $\cap_{i=1}^n V_i^-$ is connected in $(2^X, T_f)$ by Lemma 1.3(b). □

LEMMA 2.4. *Let* X *be a locally compact Hausdorff space. If* X *is connected and locally connected, then each compact subset of* X *is contained in the interior of some subcontinuum of* X.

Proof. Let K be a compact subset of X. Since X is locally connected and locally compact, let $\{U_1, ..., U_n\}$ be a finite open covering of K, where each U_i is open connected and $\overline{U_i}$ compact. Let $a_i \in U_i$ for each i. Then, for each $i = 2, ..., n$, let $\mathcal{U}_i = \{V_{i,i_1}, ..., V_{i,i_k}\}$ be a simple chain from a_1 to a_i, where each V_{i,i_j} is connected, $\overline{V_{i,i_j}}$ compact, and $a_1 \in V_{i,i_1}$ and $a_i \in V_{i,i_k}$. Let $U = \cup_{i=1}^n U_i$, and $V_i = \cup_{j=1}^{i_k} V_{i,i_j}$ for each $i = 2, ..., n$. Then $K \subset (\cup_{i=2}^n V_i) \cup U \subset (\cup_{i=2}^n \overline{V_i}) \cup \overline{U} = N$. It follows that N is compact and connected in X. This completes the proof. □

PROPOSITION 2.5 (a) *Let* X *be a locally connected regular space. If* $\mathcal{O}$ *is an open subset of* $(C(X), T_f)$, *the* $\cup\mathcal{O}$ *is open in* X.

(b) *Let* X *be a locally compact and locally connected Hausdorff space. If* $\mathcal{O}$ *is an open set in* $(C_K(X), T_f)$, *then* $\cup\mathcal{O}$ *is open in* X.

Proof. (a) We give proof for only one type of basic open set. Let $\mathcal{U} = [(\cap_{i=1}^n V_i^-) \cap (K^c)^+] \cap C(X)$ where $(\cap_{i=1}^n V_i^-) \cap (K^c)^+$ is a basis open set in $(2^X, T_f)$. Let $U = \cup\mathcal{U}$ and $x \in U$. Then there is an element $E \in \mathcal{U}$ such that $x \in E$. Since E is contained in the open set $X \setminus K$ and X is locally connected regular, there is a connected neighborhood V of x such that $\overline{V} \subset X \setminus K$. Then $E \cup \overline{V}$ is a closed and connected subset contained in $X \setminus K$, and $(E \cup \overline{V}) \cap V_i \neq \emptyset$ for each $i = 1, ..., n$. Hence $(E \cup \overline{V}) \in \mathcal{U}$. Therefore $V \subset (E \cup \overline{V}) \subset U$.

The proof is similar for other type of basic open set.

(b) Again we give a proof for only one type of basic element. Let $\mathcal{B} = (\cap_{i=1}^{n} V_i^-) \cap (K^c)^+$ be a basic element in 2^X. Let $\mathcal{U} = \mathcal{B} \cap C_K(X)$. Let $U = \cup\mathcal{U}$. We wish to show that U is open in X. Let $x \in U$. Then there is an element $E \in \mathcal{U}$ such that $x \in E$. Since $E \subset X \setminus K$ and $X \setminus K$ is an open subset of the locally compact locally connected Hausdorff space X, there exists a connected neighborhood V in X such that $\overline{V}$ is compact and $\overline{V} \subset X \setminus K$. Then $E \cup \overline{V}^{\circ} \in \mathcal{U}$ and $x \in V \subset \overline{V} \cup E \subset U$. □

We will use the next lemma in several places.

LEMMA 2.6 [7]. *If X is a compact connected Hausdorff space, then $(2^X, T_v)$ and $(C(X), T_v)$ are compact connected Hausdorff. In particular, for each $E \in C(X)$, the set $\mathcal{L}_E = \{F \in C(X) : E \subset F\}$ is a closed and connected subset of $C(X)$.*

PROPOSITION 2.7. *Let X be a compact Hausdorff space. If $\mathcal{B}$ is a connected subset of $(C(X), T_f)$, then $\cup\mathcal{B}$ is connected in X. In particular, if $\mathcal{B}$ is a connected subset of $(C_K(X), T_f)$, then $\cup\mathcal{B}$ is connected in X.*

Proof. Let $A = \cup\mathcal{B}$. Suppose A is not connected. Then there are two nonempty disjoint sets A_1 and A_2 such that $A = A_1 \cup A_2$, $\overline{A}_1 \cap A_2 = \emptyset = A_1 \cap \overline{A}_2$. Let $\mathcal{A}_i = \{E \in \mathcal{B} : E \cap A_i \neq \emptyset\}$ for $i = 1, 2$. It is clear that $\mathcal{A}_i \neq \emptyset$ for each $i = 1, 2$. Since each $E \in \mathcal{B}$ is connected, either $E \subset A_1$ or $E \subset A_2$. Thus $\mathcal{A}_1 \cap \mathcal{A}_2 = \emptyset$ and $\mathcal{B} = \mathcal{A}_1 \cup \mathcal{A}_2$. Let $\overline{\mathcal{A}_i}^f$ denote the closure of $\mathcal{A}_i$ in T_f. Suppose $F \in \mathcal{A}_1 \cap \overline{\mathcal{A}_2}^f$. Then $F \cap A_1 \neq \emptyset$ so that $F \subset A_1$. Let $x_0 \in F$ and V be a neighborhood of x_0. Then $F \in V^-$. Since $F \in \overline{\mathcal{A}_2}^f$, there is an element $E \in \mathcal{A}_2$ such that $E \in V^-$. Since $E \subset A_2$ and $E \cap V \neq \emptyset$, $V \cap A_2 \neq \emptyset$. Thus $x_0 \in \overline{A}_2$. This is a contradiction to the fact that $A_1 \cap \overline{A}_2 = \emptyset$. Hence $\mathcal{A}_1 \cap \overline{\mathcal{A}_2}^f = \emptyset$. Similarly $\overline{A}_1 \cap A_2 = \emptyset$ so that $\overline{\mathcal{A}_1}^f \cap \mathcal{A}_2 = \emptyset$. This means that $\mathcal{B}$ is not connected which is a contradiction.

If $\mathcal{B}$ is a connected subset of $(C_K(X), T_f)$, then it is a connected subset of $(C(X), T_f)$. Hence the conclusion follows. □

PROPOSITION 2.8. *Let X be a compact Hausdorff space. Then X is connected if and only if $(C_K(X), T_f)$ is connected.*

Proof. Suppose X is connected. By Proposition 2.1, $\mathcal{F}_1(X)$ is connected, and contained in $(C_K(X), T_f)$. For each $A \in C_K(X)$, $C(A)$ is a compact and connected subset of $C_K(X)$ in $(2^X, T_v)$ by Lemma 2.6 and hence compact and connected in $(2^X, T_f)$ by Lemma 1.3. Since $\mathcal{F}_1(X) \cap C(A) \neq \emptyset$ for each $A \in C_K(X)$ and $C_K(X) = \cup\{C(A) : A \in C_K(X)\}$, $C_K(X)$ is a connected subset of $(2^X, T_f)$.

Suppose $(C_K(X), T_f)$ is connected. Since $C_K(X) \subset C(X)$, by Proposition 2.6, $X = \cup C_K(X)$ is connected. □

PROPOSITION 2.9. *Let X be a locally compact Hausdorff space and let $x \in X$. Then the followings are equivalent:*

(a) *X is connected im kleinen at x.*

(b) *$(C_K(X), T_f)$ is connected im kleinen at $\{x\}$.*

(c) *$(C(X), T_f)$ is connected im kleinen at $\{x\}$.*

Proof. (*a*) $\Rightarrow$ (*b*). Suppose that X is connected im kleinen at x. Let $\mathcal{U} = [U^- \cap (K^c)^+] \cap C_K(X)$ be a basic neighborhood of $\{x\}$ in $(C_K(X), T_f)$. Without loss of generality, we may assume without loss of generality that $U \subset X \setminus K$. Let V be a neighborhood of x in X such that $\overline{V}$ is compact and $\overline{V} \subset U$. Let $K' = \overline{V} \setminus V$. Since X is connected im kleinen at x, let M be the component of V which contains x in its interior, and let $W = Int(M)$. Then $\{x\} \in \mathcal{W} = [W^- \cap ((K \cup K')^c)^+] \cap C_K(X) \subset [V^- \cap (K^c)^+] \cap C_K(X) \subset \mathcal{U}$. We show that $\mathcal{U}$ has a component which contains $\mathcal{W}$. Let $E \in \mathcal{W}$. Then $E \subset (X \setminus (K \cup K')) = (X \setminus K) \cap (X \setminus K') \subset V \cup (X \setminus \overline{V})$. Since E is connected and $E \cap V \neq \emptyset$, $E \subset V$. Since E and $\overline{M}$ are compact subsets of $\overline{V}$ and $E \cap M \neq \emptyset$, $E \cup \overline{M}$ is a subcontinuum of $\overline{V}$. Hence $E \cup \overline{M} \in \mathcal{U}$. Now let $\mathcal{L}_E = \{F \in C(E \cup \overline{M}) : E \subset F\}$ and $\mathcal{L}_{\overline{M}} = \{F \in C(E \cup \overline{M}) : \overline{M} \subset F\}$. Then by Lemma 2.6 and Lemma 1.3, both $\mathcal{L}_E$ and $\mathcal{L}_{\overline{M}}$ are compact and connected. If $G \in \mathcal{L}_{\overline{M}}$, then $E \subset V \subset \overline{M} \subset G \subset E \cup \overline{M}$. So $G \in \mathcal{U}$. Hence $\mathcal{L}_{\overline{M}} \subset \mathcal{U}$. Similarly $\mathcal{L}_E \subset \mathcal{U}$. Since $E \cup \overline{M} \in \mathcal{L}_E \cap \mathcal{L}_{\overline{M}}$, $\mathcal{L}_E \cup \mathcal{L}_{\overline{M}}$ is a connected subset of $\mathcal{U}$. Also $E \in \mathcal{L}_E$, E and $\overline{M}$ are contained in a connected subset of $\mathcal{U}$. Thus it follows that there is a component of $\mathcal{U}$ which contains $\mathcal{W}$.

(*b*) $\Rightarrow$ (*a*). Suppose that $(C_K(X), T_f)$ is connected im kleinen at $\{x\}$. Let U be a neighborhood of x in X. Let V be a neighborhood of x such that $\overline{V}$ is compact and $\overline{V} \subset U$. Then $\{x\} \in \mathcal{V} = [V^- \cap (K^c)^+] \cap C_K(X) \subset [U^- \cap (K^c)^+] \cap C_K(X)$, where $K = \overline{V} \setminus V$. Let $\mathcal{C}$ be the component of $\mathcal{V}$ which contains $\{x\}$ in its interior. Let $E \in \mathcal{C}$. Then $E \subset V \cup (X \setminus \overline{V})$. Since E is connected and $E \cap V \neq \emptyset$, $E \subset V$. It follows that $\cup \mathcal{C} \subset V$.

Then by Proposition 1.5 (b) there is a neighborhood N of x such that $N \subset \cup Int(\mathcal{C})$. And by Proposition 2.7 $\cup\mathcal{C}$ is connected. Hence it follows that X is connected im kleinen at x.

The proofs of $(a) \Leftrightarrow (c)$ are identical with that of $(a) \Leftrightarrow (b)$. □

PROPOSITION 2.10. *Let X be a locally compact Hausdorff space. If X is locally connected, then $(C_K(X), T_f)$ is dense in $(C(X), T_f)$.*

Proof. Without loss of generality we may suppose that $\mathcal{U} = [(\cap_{i=1}^n V_i^-) \cap (K^c)^+] \cap C(X)$ is an open set in $(C(X), T_f)$. Let $E \in \mathcal{U}$. Let C be the component of $X \setminus K$ which contains E. Also we may assume that $V_i \subset C$ for each $i = 1, ..., n$. Let $x_i \in V_i$ for each i and let $L = \{x_1, ..., x_n\}$. Then, since C is a locally connected and connected open set containing the compact set L, by Lemma 2.4, there exists a continuum M in C containing E in its interior. Then $M \in \mathcal{U} \cap C_K(X)$. Hence $C_K(X)$ is dense in $(C(X), T_f)$. □

COROLLARY 2.10.1. *Let X be a locally compact and locally connected Hausdorff space. Then X is connected if and only if $(C(X), T_f)$ is connected.*

Proof. If X is connected, then $(C(X), T_f)$ is connected by Proposition 2.8 and Proposition 2.10. If $(C(X), T_f)$ is connected, then $X = \cup C(X)$ is connected by Proposition 2.7. □

PROPOSITION 2.11. *Let X be a locally compact Hausdorff space. Then X is locally connected if and only if $(C_K(X), T_f)$ is locally connected.*

Proof. Suppose X is locally connected. Let $E \in C_K(X)$ and let $\mathcal{U} = (\cap_{i=1}^n U_i^-) \cap (K^c)^+$ be a basic open set in $(2^X, T_f)$ such that $E \in \mathcal{U} \cap C_K(X)$. Let C be the component of $X \setminus K$ containing E. Then C is a connected open subset of X. Pick a point $x_i \in U_i \cap C$ for each $i = 1, ..., n$. Let V_i be a neighborhood of x_i such that $V_i \subset U_i \cap C$ for each i. Let $\mathcal{V} = (\cap_{i=1}^n V_i^-) \cap (K^c)^+$. Then $E \in \mathcal{V} \cap C_K(X) \subset \mathcal{U} \cap C_K(X)$. We show that $\mathcal{V} \cap C_K(X)$ is connected by showing that each element $F \in \mathcal{V} \cap C_K(X)$ and E are contained in a connected subset of $\mathcal{V} \cap C_K(X)$.

Let $F \in \mathcal{V} \cap C_K(X)$. Since F is connected, $F \subset C$. Then by Lemma 2.4, there exists a continuum M in C containing the compact set $E \cup F$ in its interior. Now let $\mathcal{L}_E = \{G \in C(M) : E \subset G\}$ and $\mathcal{L}_F = \{G \in C(M) : F \subset G\}$. Then by Lemma 2.6 both $\mathcal{L}_E$ and $\mathcal{L}_F$ are compact and connected with respect to T_v. Hence they are compact and connected with respect to T_f by Lemma 1.3. Clearly $E, F \in \mathcal{L}_E \cup \mathcal{L}_F$. We show that $\mathcal{L}_E \cup \mathcal{L}_F$ is a connected subset of $\mathcal{V} \cap C_K(X)$. Let $G \in \mathcal{L}_F$. Since $F \subset G \subset M \subset C$ and $G \cap V_i \neq \emptyset$ for each i, $G \in \mathcal{V} \cap C_K(X)$. Hence $\mathcal{L}_F \subset \mathcal{V} \cap C_K(X)$. Similarly $\mathcal{L}_E \subset \mathcal{V} \cap C_K(X)$. Since $M \in \mathcal{L}_E \cap \mathcal{L}_F$, $\mathcal{L}_E \cup \mathcal{L}_F$ is a connected subset of $\mathcal{V} \cap C_K(X)$. Hence we conclude that $\mathcal{V} \cap C_K(X)$ is connected.

Suppose $(C_K(X), T_f)$ is locally connected. Let $x \in X$. Let U be a neighborhood of x. Let V be a neighborhood of x such that $\overline{V}$ is compact and $\overline{V} \subset U$. Let $K = \overline{V} \setminus V$. Then $\mathcal{V} = (V^- \cap (K^c)^+) \cap C_K(X)$ is a neighborhood of $\{x\}$ in $(C_K(X), T_f)$. Since $(C_K(X), T_f)$ is locally connected at $\{x\}$, there exists a connected open set $\mathcal{W}$ containing $\{x\}$ in $(C_K(X), T_f)$ such that $\mathcal{W} \subset \mathcal{V}$. Let $E \in \mathcal{W}$. Then $E \in \mathcal{V}$. Since $E \cap V \neq \emptyset$, $E \subset V \cup (X \setminus K)$ and E is connected, we have $E \subset V$. This show that $\cup\mathcal{W} \subset V$. Also by Proposition 1.5 (b), there exists a neighborhood N of x such that $N \subset \cup\mathcal{W}$ and by Proposition 2.7, $\cup\mathcal{W}$ is connected. This shows that X is connected im kleinen at x. Since X is connected im kleinen at each of its points, it follows that X is locally connected. □

PROPOSITION 2.12. *Let X be a locally compact Hausdorff space. The following statements are equivalent:*

(a) *X is locally connected.*

(b) *$(C(X), T_f)$ is locally connected at each $E \in C(X)$.*

(c) *$(C(X), T_f)$ is locally connected at each $E \in C_K(X)$.*

Proof. (a) $\Rightarrow$ (b). Let $E \in C(X)$ and $E \neq X$. Let $\mathcal{U} = (\cap_{i=1}^n U_i^-) \cap (K^c)^+$ be a basic neighborhood of E in $(2^X, T_f)$. Then $\mathcal{U} \cap C(X)$ is a neighborhood of E in $(C(X), T_f)$. Let $x_i \in E \cap U_i$ for each $i = 1, ..., n$. Since $E \subset X \setminus K$, there exists a connected neighborhood V_i of x_i such that $\overline{V_i} \subset U_i \cap (X \setminus K)$ for each i. Let $\mathcal{V} = (\cap_{i=1}^n V_i^-) \cap (K^c)^+$. Then $\mathcal{V} \subset \mathcal{U}$ and $\mathcal{V} \cap C(X)$ is a neighborhood of E in $(C(X), T_f)$. Since $C_K(X)$ is dense in $(C(X), T_f)$, $\mathcal{V} \cap C_K(X)$ is dense in $\mathcal{V} \cap C(X)$.

We show that $\mathcal{V} \cap C(X)$ is connected in $(C(X), T_f)$ by showing that any two elements of $\mathcal{V} \cap C_K(X)$ are contained in a connected subset of $\mathcal{V} \cap C_K(X)$. Let W be the component of $X \setminus K$ containing E. Let $A_1, A_2 \in \mathcal{V} \cap C_K(X)$. Then $A_1 \cup A_2 \subset W$. By Lemma 2.4 there exists a continuum $Q \subset W$ containing $A_1 \cup A_2$ in its interior. Let $\mathcal{L}_{A_i} = \{F \in C(Q) : A_i \subset F\}$ for each $i = 1, 2$. Then each $\mathcal{L}_{A_i}$ is connected by Lemma 2.6. For each $F \in \mathcal{L}_{A_i}$, $A_i \subset F \subset W$ so that $F \cap V_i \neq \emptyset$ and

$F \subset X \setminus K$. Hence $F \in \mathcal{V} \cap C_K(X)$. Since $A_i, Q \in \mathcal{L}_{A_i}$, $\mathcal{L}_{A_1} \cup \mathcal{L}_{A_2}$ is a connected subset of $\mathcal{V} \cap C_K(X)$. Thus $\mathcal{V} \cap C_K(X)$ is connected. Therefore, $\mathcal{V} \cap C(X)$ is connected.

Suppose X is connected. Since an arbitrary neighborhood of X in $(2^X, T_f)$ has the form V^-, we choose V to be a connected open set such that $\overline{V}$ is compact. Then one can show in a similar manner as before that $V^- \cap C_K(X)$ is connected. Since $V^- \cap C_K(X)$ is dense in $V^- \cap C(X)$, $V^- \cap C(X)$ is connected. Hence $(C(X), T_f)$ is locally connected at X.

$(b) \Rightarrow (c)$ is obvious.

$(c) \Rightarrow (a)$. Let $x \in X$ and U be a neighborhood of x in X. Let V be a neighborhood of x such that $\overline{V} \subset U$ and $\overline{V}$ is compact. Let $K = \overline{V} \setminus V$. Then $\{x\} \in (V^- \cap (K^c)^+) \cap C(X)$. Since $(C(X), T_f)$ is locally connected at $\{x\}$, there exists a connected open set $\mathcal{U}$ in $(C(X), T_f)$ containing $\{x\}$ such that $\mathcal{U} \subset (V^- \cap (K^c)^+) \cap C(X)$. By Proposition 2.7 and Proposition 1.5 (b), $\cup\mathcal{U}$ is connected and contains a neighborhood N of x. Let $E \in \mathcal{U}$. Then $E \in V^- \cap (K^c)^+$ so that $E \subset X \setminus K = V \cup (X \setminus \overline{V})$. Since $E \cap V \neq \emptyset$ and E is connected, $E \subset V$. It follows that $\cup\mathcal{U} \subset V$. This shows that X is connected im kleinen at x. It follows that X is locally connected. □

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